Tensors and low rank tensor decompositions

S_{ij} = Score of person i on test j

	Classics	Math	Music	...
Alice	19	26	17	...
Bob	8	17	9	...
Carol	7	12	7	...
⋮	⋮	⋮	⋮	

$$\overbrace{S = \vec{x}_{quant}\vec{y}_{quant}^{\,T} + \vec{x}_{verb}\vec{y}_{verb}^{\,T}}^{\text{rank - 2}}$$

$$S_{ij} \approx \underbrace{\vec{x}_{quant,i}}_{\text{person } i \text{ quant. reason}} \overbrace{\vec{y}_{quant,j}}^{\text{test } j \text{ quant. meas.}} + \vec{x}_{verb,i} \cdot \vec{y}_{verb,j}$$

	Classics	Math	Music	...
Alice	$\boxed{19}$	26	17	...
Bob	8	17	9	...
Carol	7	12	7	...
⋮	⋮	⋮	⋮	

S

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	Quantitative	Verbal
Alice	4	3
Bob	3	1
Carol	2	1
⋮	⋮	⋮

$\vec{x}_{quant}$

$\vec{y}_{quant}$

	Classics	Math	Music	...
Quantitative	1	5	2	...
Verbal	5	2	3	...

$$S = \vec{x}_1\vec{y}_1^{\,T} + \vec{x}_2 y_2^T$$

$$19 = 4 \cdot 1 + 3 \cdot 5$$

$$S = \begin{bmatrix} | & | \\ x_1 & x_2 \\ | & | \end{bmatrix} \begin{bmatrix} - & y_1^T & - \\ - & y_2^T & - \end{bmatrix}$$

	Classics	Math	Music	...
Alice	$\boxed{19}$	26	17	...
Bob	8	17	9	...
Carol	7	12	7	...
⋮	⋮	⋮	⋮	

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	Quantitative	Verbal
Alice	1	3
Bob	2	1
Carol	1	1
⋮	⋮	⋮

	Classics	Math	Music	...
Quantitative	1	5	2	...
Verbal	6	7	5	...

$$S = \vec{x}_{quant} \otimes \vec{y}_{quant} \otimes \vec{z}_{quant} + \vec{x}_{verb} \otimes \vec{y}_{verb} \otimes \vec{z}_{verb}$$

$\vec{x} \in \mathbb{R}^{n_1}$ $\qquad$ $\vec{y} \in \mathbb{R}^{n_2}$ $\qquad$ $\vec{z} \in \mathbb{R}^{n_3}$

$n_1 =$ # people $\qquad$ $n_2 =$ # subjects $\qquad$ $n_3 = 2$ (# times of day)

	Quantitative	Verbal
Alice	4	3
Bob	3	1
Carol	2	1
⋮		⋮

$\vec{x}_{quant,1}$

$\otimes$

	Quantitative	Verbal
Classics	1	5
Math	5	2
Music	2	3
⋮	⋮	⋮

$\vec{y}_{quant,1}$

$\otimes$

	Quantitative	Verbal
Day	1	1
Night	2	1
⋮	⋮	⋮

$\vec{z}_{quant,1}$

day $\qquad$ night

S_{ijk} = score of person i on test j at time $k \in \{1, 2\}$

$n_1 \times n_2 \times n_3$

$$S_{ijk} = \vec{x}_{quant,i} \cdot \vec{y}_{quant,j} \cdot \vec{z}_{quant,k} + \vec{x}_{verb,i} \cdot \vec{y}_{verb,j} \cdot \vec{z}_{verb,k} \qquad \forall i,j,k \quad (*)$$

Cor: If $\underbrace{\{\vec{x}_{quant}, \vec{x}_{verb}\}}_{\text{lin indep}}$, $\underbrace{\{\vec{y}_{quant}, \vec{y}_{verb}\}}_{\text{lin indep}}$, $\underbrace{\{\vec{z}_{quant}, \vec{z}_{verb}\}}_{\text{lin indep}}$,

then (*) is unique (up to scaling)

Def: A $n_1 \times n_2 \times \cdots \times n_k$ array $A \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_k}$ is called a <u>k-tensor</u>.

vector = 1-tensor

matrix = 2-tensor

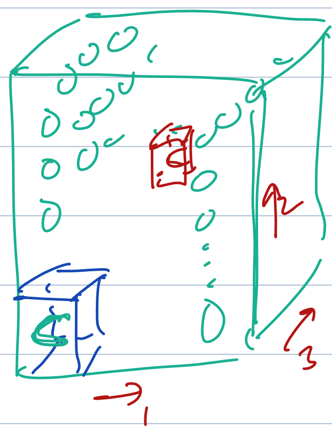
$$(\vec{u}_1 \otimes \vec{u}_2 \otimes \vec{u}_3 \otimes \cdots \otimes \vec{u}_k)_{i_1 i_2 \cdots i_k} := (\vec{u}_1)_{i_1} (\vec{u}_2)_{i_2} \cdots (\vec{u}_k)_{i_k}$$

<u>Rank:</u> A <u>rank-1</u> k-tensor is of the form

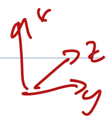
$$\boxed{\vec{u}_1 \otimes \vec{u}_2 \otimes \vec{u}_3 \otimes \cdots \otimes \vec{u}_k}$$

$\vec{u}_i \in \mathbb{R}^{n_i}$, $i = 1, \dots, k$

Rank of a k-tensor A = smallest r s.t. we can write A as a sum of rank-1 tensors

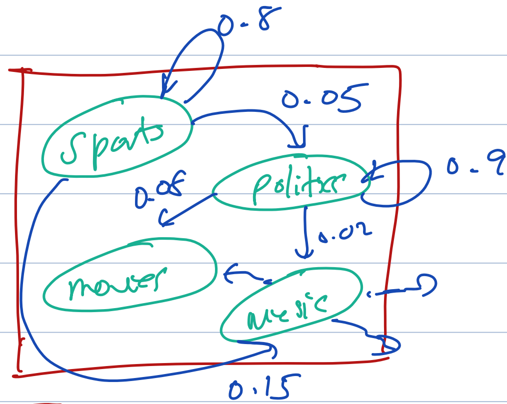
A 

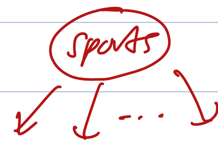
$$= \left((C, 0, \dots, 0) \otimes (1, 0, \dots, 0) \otimes (1, 0, \dots, 0) \right)$$

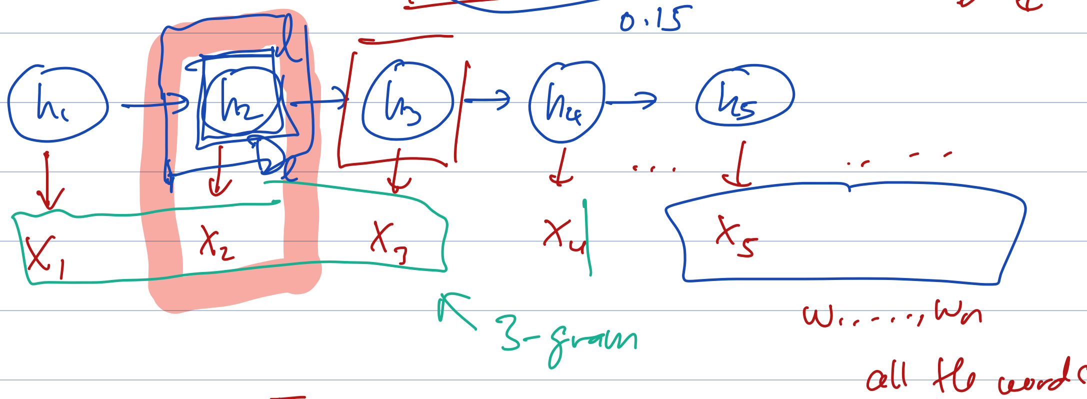


HMM: Topics

$T = \#$ topics







$$A_{ijk} = \mathbb{P}[X_1 X_2 X_3 = w_i w_j w_k]$$

$$= \sum_{\ell=1}^{T} \mathbb{P}[h_2 = \ell] \cdot \Big[\mathbb{P}[X_2 = w_j \mid h_2 = \ell] \; \mathbb{P}[X_3 = w_k \mid h_2 = \ell] \; \mathbb{P}[X_1 = w_i \mid h_2 = \ell] \Big]$$

$$A = \sum_{\ell=1}^{T} \lambda_\ell (g_\ell \otimes x_\ell \otimes y_\ell)$$

$$(g_\ell)_j = \mathbb{P}[X_2 = w_j \mid h_2 = \ell]$$

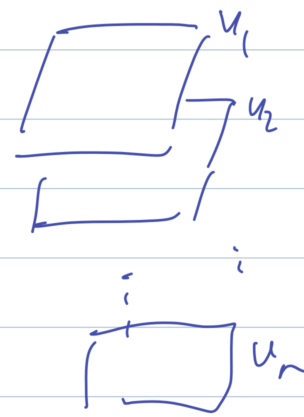
$$(x_\ell)_k = \mathbb{P}[X_3 = w_k \mid h_2 = \ell] \quad \dots$$

$\boxed{n \times n \times n \text{ tensor as input}}$

Alg: Look at random slices of the tensor.

$$T = \sum_{i=1}^{r} \lambda_i (\vec{x}_i \otimes \vec{y}_i \otimes \vec{z}_i)$$

Pick two random vectors $\vec{u}, \vec{v}$



(1) $T_{\vec{u}} = \sum_{i=1}^{n} u_i T[:,:,i]$

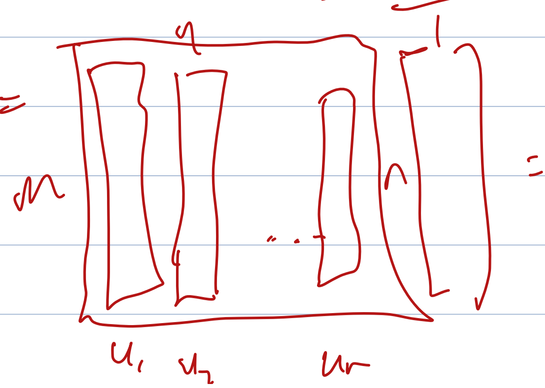
(2) $T_{\vec{v}} = \sum_{i=1}^{n} v_i T[:,:,i]$

(3) $\vec{x}_i$ = eigenvectors of $T_{\vec{u}} (T_{\vec{v}})^{-1}$

$\vec{y}_i$ = eigenvectors of $T_{\vec{v}} (T_{\vec{u}})^{-1}$

Note that matrix-vec mult is also a slice

$Mu =$



$$\boxed{T_{\vec{u}} = X D_{\vec{u}} Y^T \qquad T_{\vec{v}} = X D_{\vec{v}} Y^T}$$

"simultaneous" SVD is unique (!)

$$T_u (T_v)^{-1} = X D_{\vec{u}} Y^T Y D_{\vec{v}}^{-1} X^{-1}$$

$$= \underbrace{X D_{\vec{u}} D_{\vec{v}}^{-1} X^{-1}}_{\text{eigenvectors give the } X\text{'s}}$$