CSE 531: Computability and Complexity — Autumn 2005

Problem Set #4 — Instructor: Paul Beame

Due on Tuesday, **November 29, 2005** in class.

Instructions: Same as Problem Set 1.

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1. (10 points) A nondeterministic *linear bounded automaton* or NLBA is a variant of a 1-tape *nondeterministic* Turing machine in which the tape head is not allowed to move off the input: if it attempts to move off the right end of the input then it syas where it is just as it did off the left end of the input. (Note that the definition in Sipser's text of LBA's does not allow nondeterminism but the standard definition of LBA's is the one we are calling NLBA's here.) Show that
$$A_{NLBA} = \{\langle M, w\rangle \mid M \text{ is an NLBA that accepts } w\}$$
is PSPACE-complete.

2. (5 points) Show that $TQBF$ restricted to formulas where the part following the quantifiers is in conjunctive normal form is still PSPACE-complete.

3. (5 points) Let A be the language of properly nested parentheses,i.e. $(()) \in A$ and $((()())() \in A$ but $)(\notin A$. Show that $A \in \mathsf{L}$.

4. (15 points) This problem concerns ***branching programs*** which are described in Section 10.2 of Sipser's book. We briefly repeat the definition here. A (Boolean) *branching program* is a directed acyclic graph where all nodes are labeled by variables, except for two output nodes labeled 0 or 1. The nodes that are labeled by variables are called query nodes, each of which has two outgoing edges, one labeled 0 and the other labeled 1. Both output nodes have no outgoing edges, and one of the nodes of the branching program is designated the start node. A branching program determines a Boolean function as follows. Take any assignment to the variables appearing on its query nodes and, beginning at the start node, follow the path determined by taking the outgoing edge from each query node according to the value assigned to the indicated variable (i.e. take the 0-edge is the variable is 0 and 1-node if it is 1). Do this until one of the output nodes is reached. The label of this output node is the output of the branching program on that input.

   Define the *size* of a branching program to be the number of nodes in it and the *length* of a branching program to be the length of a longest path from an input node to an output node.

   A *family of branching programs* is a sequence $\mathcal{B} = B_1, B_2, \ldots, B_n, \ldots$ of branching programs such that each B_n only queries Boolean variables $x_1, \ldots, x_n$. A family of branching programs decides a language A iff for all $x \in \{0,1\}^*$, $x \in A \Leftrightarrow B_{|x|}$ outputs 1 on input x. The *size* of a family of branching programs is a function $size : \mathbb{N} \to \mathbb{N}$ such that $size(n) = size(B_n)$. The *length* of a family of branching programs is a function $length : \mathbb{N} \to \mathbb{N}$ such that $length(n) = length(B_n)$.

   (a) Describe a branching program family of size $O(n^2)$ that recognizes the language
   $$\text{MAJORITY} = \{w \in \{0,1\}^* \mid w \text{ contains at least } |w|/2 \text{ 1's}\}.$$

(b) Prove that if an (offline) Turing machine decides a language $A \subseteq \{0,1\}^*$ using $S(n)$ space and $T(n)$ time then there is a family of branching programs $\mathcal{B}$ deciding A such that

- the length of $\mathcal{B}$ is at most $T(n)$, and
- the size of $\mathcal{B}$ is at most $n2^{O(S(n))}$

and use this to conclude that every language in L is decided by a polynomial size family of branching programs.

5. (10 points) Recall that a directed graph is *strong connected* iff every two nodes are connected by a directed path in each direction. Let

$$\text{STRONGLY-CONNECTED} = \{\langle G \rangle \mid G \text{ is a strongly connected graph}\}.$$

Show that STRONGLY-CONNECTED is NL-complete.

6. (5 points) Show that $2COLOR \leq_L UPATH$.

7. (10 points) Let $DISTANCE = \{\langle G, s, t, k \rangle \mid$ the distance from s to t in unweighted directed graph G is precisely $k\}$. Show that $DISTANCE \in \mathsf{NL}$.

8. **(Extra Credit)** Define the language

$$\text{UCYCLE} = \{\langle G \rangle \mid G \text{ is an undirected graph that has a cycle}\} .$$

Prove that UCYCLE $\in \mathsf{L}$.