CSE 521: The Design and Analysis of Algorithms
Assignment #4
May 9, 1997
due: Friday, May 16

1. In this problem you will prove a converse (of sorts) to Theorem 17.10. A *subset system* is a pair $M = (S, I)$ such that S is a finite nonempty set and I is a nonempty family of subsets of S that is hereditary. (Thus, a matroid is a subset system that also has the exchange property.) For a fixed subset system M, suppose that Greedy(M, w) of Section 17.4.2 returns an optimal set for every positive weight function $w : S \rightarrow \mathcal{R}^+$. Prove that M is a matroid.

   Hint: Let $n = |S|$. If $A \in I$ and $B \in I$ violate the exchange property, let
   $$w(x) = \begin{cases} n+1, & \text{if } x \in A \\ n, & \text{if } x \in B - A \\ 1, & \text{if } x \notin A \cup B \end{cases} .$$

2. Exercise 17.5-2 on page 353. You should not need to assume anything special about A, such as that this test is being used in conjunction with Greedy.

3. Problem 17-3 on page 354. This problem improves the $O(n^2)$ running time of Section 17.5.

4. Let $G = (V, E)$ be an undirected graph. A *matching* in G is a subset $F \subseteq E$ such that, for all $v \in V$, at most one edge in F is incident to v.

   (a) A common optimization problem is that of "maximum edge-weighted matching". In this problem, each edge $e \in E$ has positive real weight $w(e)$, and you are looking for a matching in G whose total edge weight is maximized. Consider the subset system $M = (S, I)$, where $S = E$ and $A \in I$ if and only if A is a matching. Give a counterexample showing that Greedy(M, w) does not return a maximum edge-weighted matching.

   (b) A variant is the problem of finding a "maximum vertex-weighted matching". In this problem, each vertex $v \in V$ has a positive real weight $w(v)$, and you are looking for a matching F in G that maximizes the total weight of all vertices incident to edges in F. Show that there is a matroid $M = (S, I)$ and weight function $w : S \rightarrow \mathcal{R}^+$ such that an optimal subset of M corresponds to a maximum vertex-weighted matching. (Therefore, by Theorem 17.10, Greedy(M, w) computes a maximum vertex-weighted matching.)

Hint: Let $S = V$. For the exchange property, call the edges in the matching corresponding to A "A-edges", and the edges in the matching corresponding to B "B-edges". You can assume that no A-edge goes from A to B-A. (Why?) Starting with some vertex in B-A, create as long a path P as possible that alternates between B-edges and A-edges, and consider the various cases of how this path ends.

(c) What is the running time of Greedy for the maximum vertex-weighted matching problem? Justify your answer. Don't forget that your algorithm should construct a matching, not just an optimal subset of M.

5. In class we saw that $n-1$ Weighted-Unions applied to n singleton sets always resulted in a tree of height at most $\lfloor \log_2 n \rfloor$. In this problem, you will show that that bound is tight.

   Let $n = 2^k$. Specify $n-1$ Weighted-Union operations on n singleton sets that result in a tree of height exactly k. Is there anything interesting about the number of nodes of depth i in the resulting tree, where $0 \leq i \leq k$?