CSE 521: Design and Analysis of Algorithms **Fall 2019**

Assignment 3

Deadline: Nov 10th (at 6:00PM) in Canvas

Remember that you can always use the approximation $1 - x \approx e^{-x}$.

1) Let a, b be arbitrary real numbers. Fix $w > 0$ and let $s \in [0, w)$ chosen uniformly at random. Show that

$$\mathbb{P}\left[\left\lfloor \frac{a-s}{w} \right\rfloor = \left\lfloor \frac{b-s}{w} \right\rfloor\right] = \max\left\{0, 1 - \frac{|a-b|}{w}\right\}.$$

Recall that for any real number c, $\lfloor c \rfloor$ is the largest integer which is at most c.

Hint: Start with the case where $a = 0$.

2) In this problem we design an LSH for points in $\mathbb{R}^d$ with the ℓ_2 distance function,

$$d(p, q) = \|p - q\|_2 = \sqrt{\sum_i (p_i - q_i)^2}.$$

Let $w \gg r$, and let s be uniformly distributed in $[0, w)$. Let g be a d-dimensional Gaussian vector, i.e., for all $1 \leq i \leq d$, $g_i \sim \mathcal{N}(0, 1)$ and all coordinates of g are chosen independently. Consider the hash function

$$h(p) = \left\lfloor \frac{\langle g, p \rangle - s}{w} \right\rfloor$$

a) **Optional:** For a random variable $X \sim \mathcal{N}(0, \sigma^2)$ prove that $\mathbb{E}[|X|] = \sigma\sqrt{2/\pi}$.

b) Show that for any two points p, q, $\langle g, p \rangle - \langle g, q \rangle$ is distributed as a normal random variable. What is the mean and variance of this random variable? In this part you can use the fact that any linear combination of independent normal random variables is also a normal random variable.

c) Use Problem (1) to estimate the probability that $h(p) = h(q)$. Note that this probability is over the randomness of g and s. To make calculations simple, assume that w is large enough such that $\mathbb{P}[|\langle g, p \rangle - \langle g, q \rangle| > w] = 0$.

d) Use the statement of part (b) to determine for what values of p_1, p_2, is this family $(r, c \cdot r, p_1, p_2)$ sensitive?

3) A matrix A is skew symmetric if $A = -A^T$. For example,

$$A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 1 \\ 3 & -1 & 0 \end{bmatrix}$$

is skew symmetric. Let $A \in \mathbb{R}^{(2n+1)\times(2n+1)}$ be a skew symmetric matrix. Prove that $\det(A) = 0$.

4) Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of A.

a) If there exists $k \geq 0$ such that for every $1 \leq i \leq n$, $\sum_j |A_{i,j}| \leq k$. prove that $\lambda_1(A) \leq k$.

b) Now, suppose there is a $k \geq 0$ such that for every $1 \leq i \leq n$, $\sum_j |A_{i,j}| \geq k$, prove that $\rho(A) \geq k/\sqrt{n}$ where $\rho(A) = \max\{|\lambda_1|, |\lambda_n|\}$ is the largest eigenvalue of A in absolute value.