CSE 521: Design and Analysis of Algorithms I **Spring 2016**

Lecture 16: Duality and the Minimax theorem

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Last lecture we introduce linear programs (LP) and saw how to model problems with LPs. In this lecture we study duality of LPs. Roughly speaking, the dual of a minimization LP is a maximization LP which its solutions provide lower bounds for the optimum of the original LP. First we show how to construct dual of an LP. Then we talk about relation between the optimum of an LP and its dual. In particular, we describe the weak and strong duality theorems. Finally using the LP duality, we prove the Minimax theorem which is an important result in the game theory.

16.1 LP Duality

Before formally define dual of an LP, let's see an easy example. Let $\mathcal{P}_1$ be the following LP and try to find some lower bounds on its optimal value, OPT.

$$\begin{aligned} \min \quad & 2x_1 + 3x_2 \\ \text{s.t.} \quad & 2x_1 + 4x_2 \geq 5 \\ & 2x_1 + x_2 \geq 3 \\ & 3x_1 + 2x_2 \geq 5 \\ & x_1, x_2 \geq 0 \end{aligned} \tag{16.1}$$

The most trivial lower bound is 0, as $x_1, x_2 \geq 0$. If we use other constraints of the LP, we can produce more accurate lower bounds as follows:

- **Attempt 1**: Using the second constraint, we have $2x_1 + 3x_2 \geq 2x_1 + x_2 \geq 3$, So OPT ≥ 3.
- **Attempt 2**: Combining the first two constraints, we get

  $$2x_1 + 3x_2 = 2/3(2x_1 + 4x_2) + 1/3(2x_1 + x_2) \geq 2/3 \cdot 5 + 1/3 \cdot 3 = 11/3$$

  , So OPT $\geq 11/3$.
- **Attempt 3**: If we choose coefficient more carefully and involve the last constraint, we can say

  $$2x_1 + 3x_2 = 5/8(2x_1 + 4x_2) + 1/4(3x_1 + 2x_2) \geq 5/8 \cdot 5 + 1/4 \cdot 5 = 35/16.$$

  So OPT $\geq 35/8$.

In each of the above attempts, we take a positive linear combination of the constraints, which is less than the objective function. This can be formalized by letting $y_1, y_2, y_3 \geq 0$ be the coefficients of our positive linear combination which satisfy

$$2x_1 + 3x_2 \geq y_1(2x_1 + 4x_2) + y_2(2x_1 + x_2) + y_3(3x_1 + 2x_2). \tag{16.2}$$

Then $5y_1 + 3y_2 + 5y_3$ is a lower bound on OPT and we seek to maximize it. One can observe that (16.2) holds if and only if the coeffiiceints x_1 and x_2 in the LHS are greater than or equal to their corresponding on the RHS, or equivalently

$$2y_1 + 2y_2 + 3y_2 \le 2 \text{ and } 4y_1 + y_2 + 2y_3 \le 3.$$

Therefore, every solution of the following LP, say $\mathcal{D}_1$, has value at most OPT($\mathcal{P}_1$).

$$\begin{aligned} \max \quad & 5y_1 + 3y_2 + 5y_3 \\ \text{s.t.} \quad & 2y_1 + 2y_2 + 3y_2 \le 2 \\ & 4y_1 + y_2 + 2y_3 \le 3 \\ & y_1, y_2, y_3 \ge 0 \end{aligned} \tag{16.3}$$

We refer to this LP as the dual and the original LP ($\mathcal{P}_1$) as the primal. One property of the dual which can be verified easily is that the dual of the dual is equal to the primal.

In our example, if we let y_1, y_2, y_3 be equal to $1/4, 0, 5/8$ which are the values we used in the last attempt, then we will see the value of the dual is exactly equal to the value of the primal. It simply implies this value is the optimum for both LPs, as the value of the any solution of the dual is a lower bound on that of the primal. In general we are gonna see the *Strong Duality* theorem says the optimum of an LP is equal to the optimum of its dual if both are finite values.

16.1.1 Duality for General LPs

Now, we generalize what we did for $\mathcal{P}_1$ to any LP of the following form which is called the *standard form* of LPs.

$$\begin{aligned} \min \quad & \langle c, x\rangle \\ \text{s.t.} \quad & Ax \ge b \\ & x \ge 0 \end{aligned} \tag{16.4}$$

In the above $A \in \mathbb{R}^{m\times n}, c \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$. So we have n variables and m constraints other than the constraint $x \ge 0$. Similar to $\mathcal{P}_1$, we can define the dual of the above LP as follows:

$$\begin{aligned} \max \quad & \langle b, y\rangle \\ \text{s.t.} \quad & A^T y \le c \quad , \\ & y \ge 0 \quad . \end{aligned} \tag{16.5}$$

where $y \in \mathbb{R}^m$. Note, for every constraint in the primal we have one variable in the dual and every constraint in the dual corresponds to a variable in the primal.

If we have an LP which is not in the standard form, we can simply convert it to a standard form LP. However, we can also directly define the dual of LPs in the general form. If we have an LP

$$\min\{\langle c, x\rangle | Ax \ge b,\ x \in \mathbb{R}^n\},$$

then its dual is

$$\max\{\langle b, y\rangle | A^T y = c, y \ge 0,\ y \in \mathbb{R}^m\}.$$

16.2 Duality Theorems

There are two duality theorems called the weak duality theorem and the strong duality theorem which demonstrate the connection between the primal and the dual. Roughly speaking, the weak duality theorem says that the optimum of the dual is a lower bound for the optimum of the primal (if the primal is a minimization problem). The strong duality theorem states these are equal if they are bounded.

Theorem 16.1 (weak duality)**.** *if x is a feasible solution of $\mathcal{P} = \min\{\langle c, x\rangle | Ax \geq b\}$ and y is a feasible solution of $\mathcal{D} = \max\{\langle b, y\rangle | A^T y = c, y \geq 0\}$, then $\langle c, x\rangle \geq \langle y, b\rangle$.*

Proof. Since $y \geq 0$ and $Ax \geq b$, we get

$$\langle b, y\rangle \leq \langle Ax, y\rangle. \tag{16.6}$$

We also know $A^T y = c$, so

$$\langle Ax, y\rangle = \langle A^T y, x\rangle = \langle c, x\rangle.$$

Therefore, combining these two we have $\langle y, b\rangle \leq \langle c, x\rangle$ and we are done. □

Theorem 16.2 (strong duality)**.** *For any LP and its dual, one of the following holds:*

1. *The primal is infeasible and the dual has unbounded optimum.*
2. *The dual is infeasible and the primal has unbounded optimum.*
3. *Both of them are infeasible.*
4. *Both of them are feasible and their optimum value is equal.*

Here we don't include a proof of the strong duality theorem.

16.2.1 Complementary Slackness

Similar to before, let $\mathcal{P} = \min\{\langle c, x\rangle | Ax \geq b\}$ be the primal and let $\mathcal{D} = \max\{\langle b, y\rangle | A^T y = c, y \geq c\}$ be its dual. we investigated the relation between $\mathcal{P}$ and $\mathcal{D}$ by the duality theorems. In particular, we saw according to the strong duality theorem a pair of feasible solutions (x, y) to $(\mathcal{P}, \mathcal{D})$ are optimum solutions if and only if $\langle c, x\rangle = \langle b, y\rangle$. Now we introduce another necessary and sufficient condition for (x, y) to be optimum solutions, known as the complementary slackness condition.

Theorem 16.3 (Complementary Slackness)**.** *Let x, y be primal/dual feasible solutions respectively. Then, x, y are optimum solutions to $\mathcal{P}, \mathcal{D}$ if and only if for each constraint $\langle a_i, x\rangle \geq b_i$ of the primal we have*

$$\begin{aligned} \langle a_i, x\rangle > b_i \quad &\Rightarrow \quad y_i = 0, \\ y_i > 0 \quad &\Rightarrow \quad \langle a_i, x\rangle = b_i. \end{aligned}$$

Proof. Suppose x, y are optimum solutions of $\mathcal{P}, \mathcal{D}$. We show that the above two relations hold. By duality theorem, $c^T x = b^T y$. Therefore, the inequality (16.6) must be tight, i.e.,

$$\langle b, y\rangle = \langle Ax, y\rangle$$

In other words,

$$\sum_i y_i(\langle a_i, x\rangle - b_i) = 0.$$

Since x is a feasible solution of the primal, for each $\langle a_i, x\rangle \geq b_i$, and $y_i \geq 0$. So, the LHS of the above is a sum of nonnegative numbers. Since the sum is equal to zero, all of the terms of this summation must be zero, i.e., for each i

$$y_i(\langle a_i, x\rangle - b_i) = 0.$$

This means that for each i either $y_i = 0$ or $\langle a_i, x\rangle = b_i$. This completes the proof of one direction. To prove the converse just note that all of the steps of the above proof are biconditional. □

16.3 Applications of LP Duality

In this section we discuss one important application of duality. It is the *Minimax* theorem which proves existence of Mixed Nash equilibrium for two-person zero-sum games and proposes an LP to find it. Before stating this, we need a couple of definitions. A two-person game is defined by four sets (X, Y, A, B) where

1. X and Y are the set of strategies of the first and second player,respectively.
2. A, B are real-valued functions defined on $X \times Y$.

The game is played as follows. Simultaneously, Player (I) chooses $x \in X$ and Player (II) chooses $y \in Y$, each unaware of the choice of the other. Then their choices are made known and (I) wins $A(x, y)$ and (II) wins $B(x, y)$. A and B are called utility function for player (I) and (II), and obviously the goal of both players is to maximize their utility. The game is called a *zero-sum* game if $A = -B$.

A mixed strategy for a player is just a distribution over his/her strategies. The last thing we need to define is mixed Nash equilibrium.

Definition 16.4. *Suppose that we a have zero-sum game (X, Y, A) and let p, q be two mixed strategies for player (I) and (II), respectively. Then (p, q) is a mixed Nash equilibrium if no player can increase his/her expected utility by choosing another mixed strategy after knowing the other player's mixed strategy, or equivalently*

$$\mathbb{E}_{p,q}[A(x,y)] = \max_{p' \in \textit{mixed strategies on } X} \mathbb{E}_{p',q}[A(x,y)] = \min_{q' \in \textit{mixed strategies on } Y} \mathbb{E}_{p,q'}[A(x,y)],$$

i.e., p is the best response to q and q is the best response to p.

It is proved by Nash [Nas+50] that every n-person game has one Nash equilibrium. In general finding Nash equilibrium is a very hard problem [DGP09]. However, in the case of two-player zero-sum games there is a polynomial time algorithm to find it. In particular let (X, Y, A) represents a two-player zero-sum game. If x, y are two mixed strategies for (I) and (II), then one can see the expected utility of I is $x^T Ay$ and for (II) it is $-x^T Ay$. So player (I) wants to maximize $x^T Ay$ and (II) wants to minimize it. Then we can see, if there are mixed strategies x^*, y^* for (I) and (II) satisfying

$$\max_x \min_y x^T Ay = \min_y \max_x x^T Ay = x^{*T} Ay^*,$$

then (x^*, y^*) is a mixed Nash equilibrium. The following nice result by Neumann guarantees such x^*, y^* exists and gives an LP such that its optimum solution is x^* and the optimum solution of its dual is y^*. The proof is an application of the strong duality theorem.

Theorem 16.5 (The Minimax Theorem [Neu28])**.** *For every two-person zero-sum game (X, Y, A) there is a mixed strategy x^* for player I and a mixed strategy y^* for player (II) such that,*

$$\max_x \min_y x^T Ay = \min_y \max_x x^T Ay = x^{*T} Ay^*, \tag{16.7}$$

where in the above x and y represent mixed strategies for (I) and (II), respectively. Moreover, x^*, y^* can be found by an LP.

Proof. Let $a_1, \ldots, a_n$ and $a^1, \ldots, a^m$ be columns and rows of A, respectively. Firstly, observe that for a vector x,

$$\min_y x^T Ay = \min_i x^T A \mathbf{1}_i = \min_i \langle x, a_i \rangle,$$

because Ay is a distribution over $a_1, \dots, a_n$. Taking the max over all distributions x, we have

$$\max_x \min_y x^T Ay = \max_x \min_i \langle x, a_i \rangle$$

Similarly we can see that

$$\min_y \max_x x^T Ay = \min_y \max_i \langle a^i, y \rangle.$$

Therefore (16.7) is equivalent to

$$\max_x \min_i \langle x, a_i \rangle = \min_y \max_i \langle a^i, y \rangle = x^* A y^*. \tag{16.8}$$

Both $\max_x \min_i \langle x, a_i \rangle$ and $\min_y \max_i \langle a^i, y \rangle$ can be formulated by LPs. Then the idea is to show the corresponding LP's are dual of each other and feasible, so they are equal by the strong duality theorem. First note that $\max_x \min_i \langle x, a_i \rangle$ is equivalent to

$$\begin{aligned} \max \quad & t \\ \text{s.t.} \quad & \langle x, a_i \rangle \geq t \quad \forall 1 \leq i \leq n \\ & \sum_{i=1}^{m} x_i = 1 \\ & x_i \geq 0 \qquad \forall 1 \leq i \leq m \end{aligned} \tag{16.9}$$

where the last two inequalities guarantee x is a distribution. It is better to write the above LP in the standard form as follows:

$$\begin{aligned} \max \quad & t \\ \text{s.t.} \quad & t - \langle x, a_i \rangle \leq 0 \quad \forall 1 \leq i \leq n \\ & \sum_{i=1}^{m} x_i = 1 \\ & x_i \geq 0 \qquad \forall 1 \leq i \leq m \end{aligned} \tag{16.10}$$

We can write the dual of the above LP as follows: We have a dual variable y_i corresponding to each primal constraint $\langle x, a_i \rangle \geq t$ and a dual variable w corresponding to the constraint $\sum_{i=1}^{m} x_i = 1$. Since y_i's correspond to the inequality constraints in the primal, we need nonegativity constraints on y_i's. Since w corresponds to an equality constraint, it will be a free variable. The objective function must be $\min w$, because only the primal constraint corresponding to w has a constant term. In the dual we need to have $m+1$ constraints, one for the primal variable t and the other m constraints are for the x_i's. Since only t appears in the objective of the primal, the constraint corresponding to t has a constant term 1. The dual constraint corresponding to x_i will be as follows:

$$\sum_{j=1}^{m} -a_{i,j} y_i + w \geq 0,$$

or equivalently, $w - \langle y, a^i \rangle \geq 0$. This gives the following dual formulation:

$$\begin{aligned} \min \quad & w \\ \text{s.t.} \quad & w - \langle y, a^i \rangle \geq 0 \quad \forall 1 \leq i \leq m \\ & \sum_{i=1}^{m} y_i = 1 \\ & y_i \geq 0 \qquad \forall 1 \leq i \leq n \end{aligned} \tag{16.11}$$

Now, observe that this is exactly the LP corresponding to $\min_y \max_i \langle a^i, y \rangle$. Moreover, letting x, y be arbitrary distributions and $w = \max_i \langle y, a^i \rangle$ and $t = \min_i \langle x, a_i \rangle$, shows that both are feasible. So, by the duality theorem,

$$\max_x \min_i \langle x, a_i \rangle = \min_y \max_i \langle a^i, y \rangle.$$

Let x^*, y^* be the optimal solutions of the primal and dual respectively. Then, we have

$$\min_i \langle x^*, a_i \rangle = \max_i \langle a^i, y^* \rangle.$$

So, by (16.8),

$$\min_y x^* Ay = \min_i \langle x^*, a_i \rangle = \max_i \langle a^i, y^* \rangle = \max_x x^T Ay^*.$$

But, this means that

$$x^* Ay^* \leq \max_x x^T Ay^* = \min_y x^* Ay \leq x^* Ay^*.$$

So, all of the above inequalities must be equalities. □

References

[DGP09] C. Daskalakis, P. W. Goldberg, and C. H. Papadimitriou. "The complexity of computing a Nash equilibrium". In: *SIAM Journal on Computing* 39.1 (2009), pp. 195–259 (cit. on p. 16-4).

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