CSE 521
Algorithms

Huffman and Arithmetic Codes: Optimal Data Compression Methods

Compression Example

a	45%
b	13%
c	12%
d	16%
e	9%
f	5%

100k file, 6 letter alphabet:

File Size:

- ASCII, 8 bits/char: 800kbits
- $2^3 > 6$; 3 bits/char: 300kbits

Why?

- Storage, transmission vs 5 Ghz cpu

Compression Example

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100k file, 6 letter alphabet:

File Size:

ASCII, 8 bits/char: 800kbits

$2^3 > 6$; 3 bits/char: 300kbits

better: →

2.52 bits/char 74%*2 +26%*4: 252kbits

Optimal?

E.g.:		Why not:
a	00	00
b	01	01
d	10	10
c	1100	110
e	1101	1101
f	1110	1110

1101110 = cf or ec?

Data Compression

Binary character code ("code")

- each k-bit *source string* maps to unique *code word* (e.g. k=8)
- "compression" alg: concatenate code words for successive k-bit "characters" of source

Fixed/variable length codes

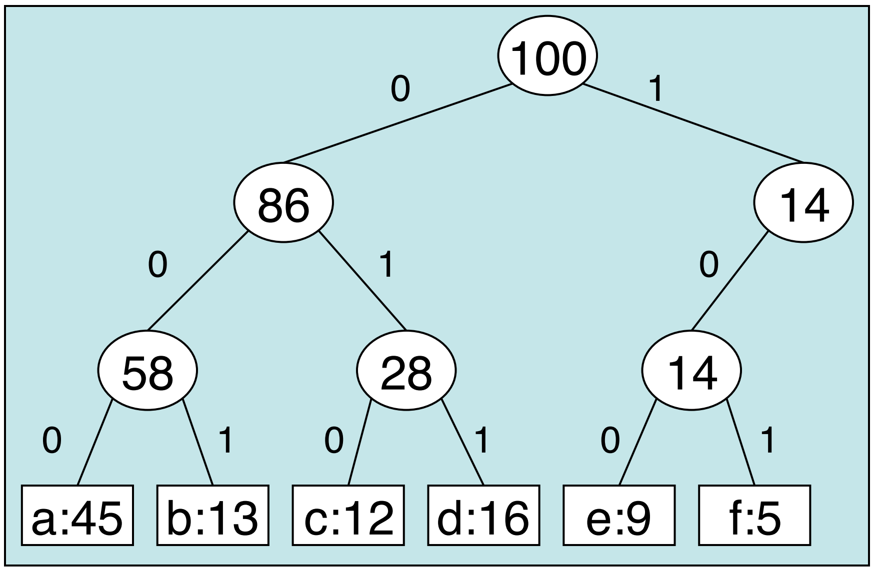
- all code words equal length?

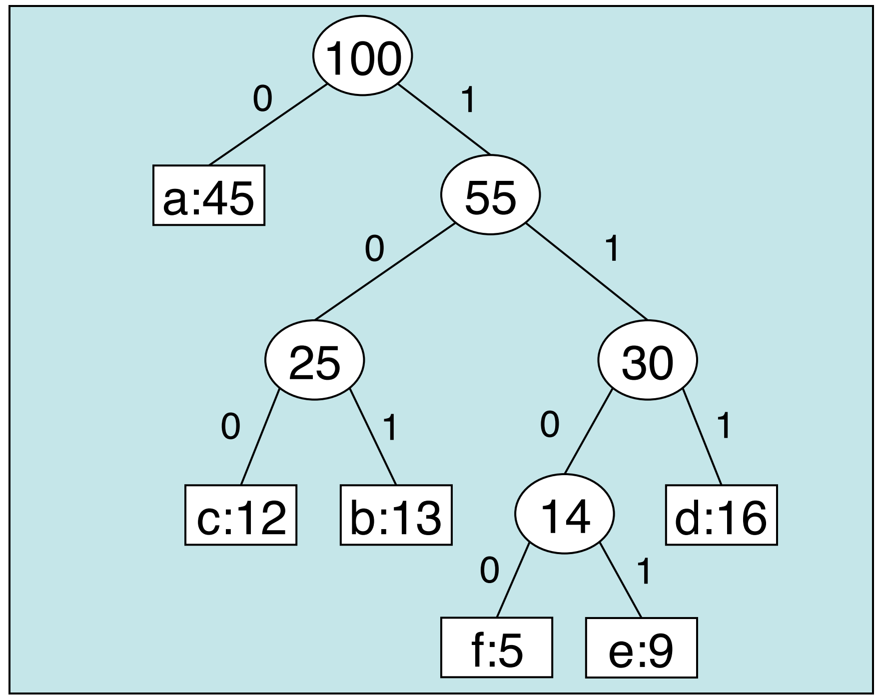
Prefix codes

- no code word is prefix of another (unique decoding)

Prefix Codes = Trees

a	45%
b	13%
c	12%
d	16%
e	9%
f	5%





1 0 1 0 0 0 0 0 1

f a b

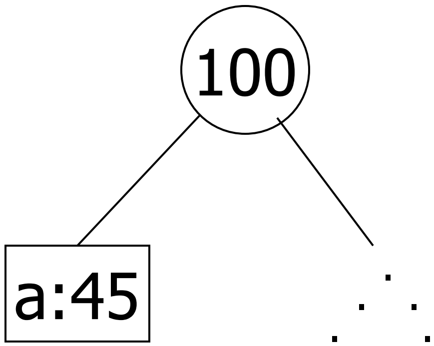
1 1 0 0 0 1 0 1

f a b

Greedy Idea #1

a	45%
b	13%
c	12%
d	16%
e	9%
f	5%

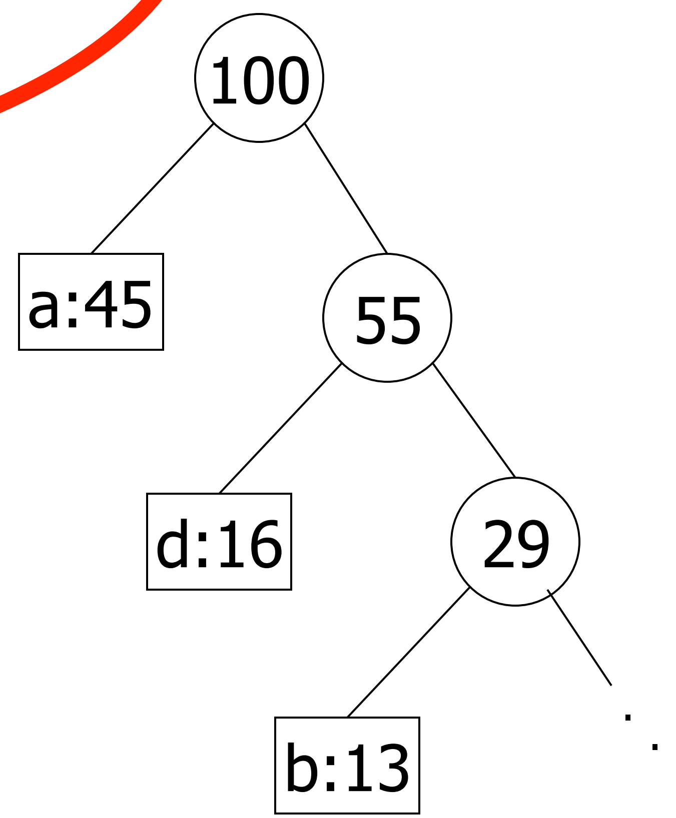
Put most frequent under root, then recurse …



Greedy Idea #1

a	45%
b	13%
c	12%
d	16%
e	9%
f	5%

Top down: Put *most* frequent under root, then recurse



Too greedy: unbalanced tree

.45*1 + .16*2 + .13*3 … = 2.34

not too bad, but imagine if all freqs were ~1/6:

(1+2+3+4+5+5)/6=3.33

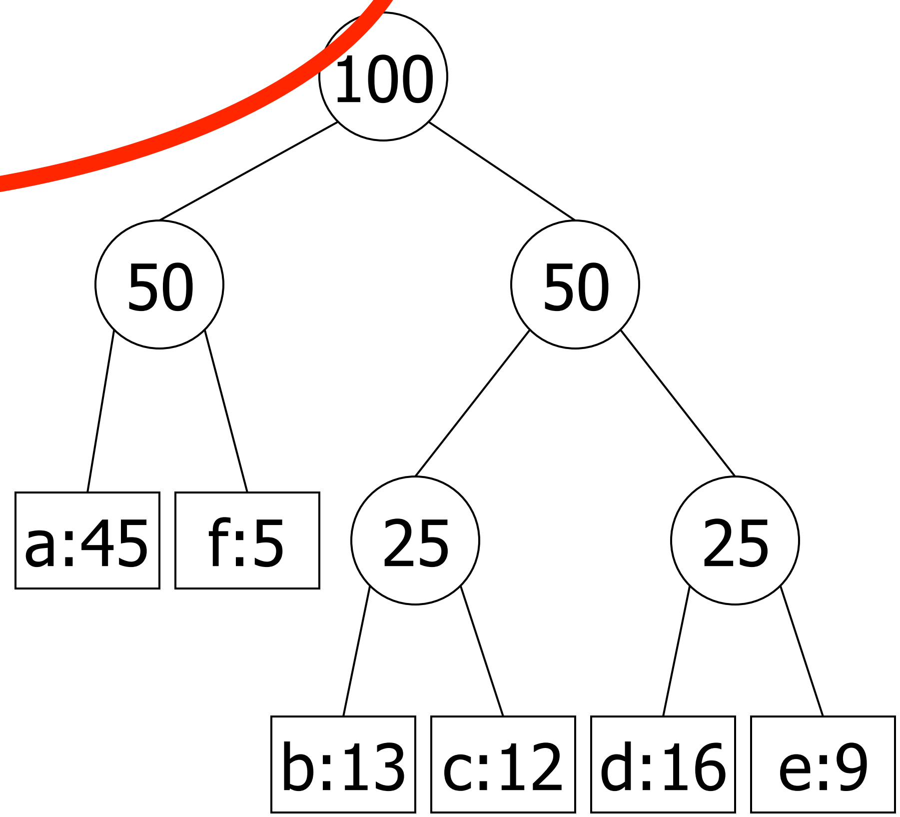
Greedy Idea #2

a	45%
b	13%
c	12%
d	16%
e	9%
f	5%

Top down: Divide letters into 2 groups, with ~50% weight in each; recurse
(Shannon-Fano code)

Again, not terrible
2*.5+3*.5 = 2.5

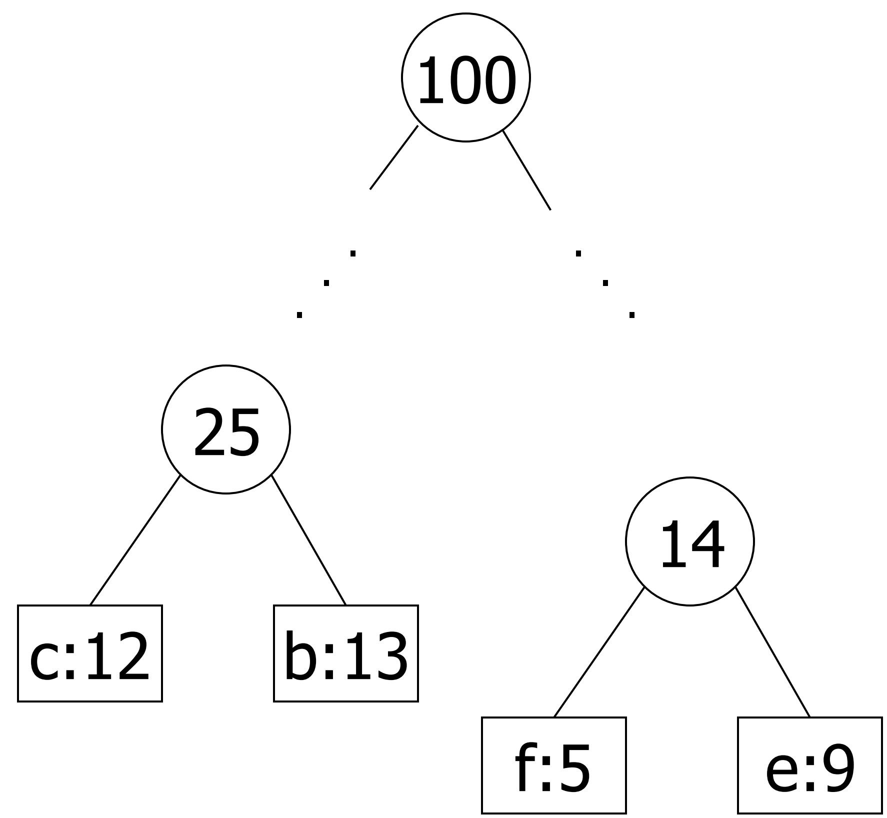
But this tree can easily be improved! (How?)

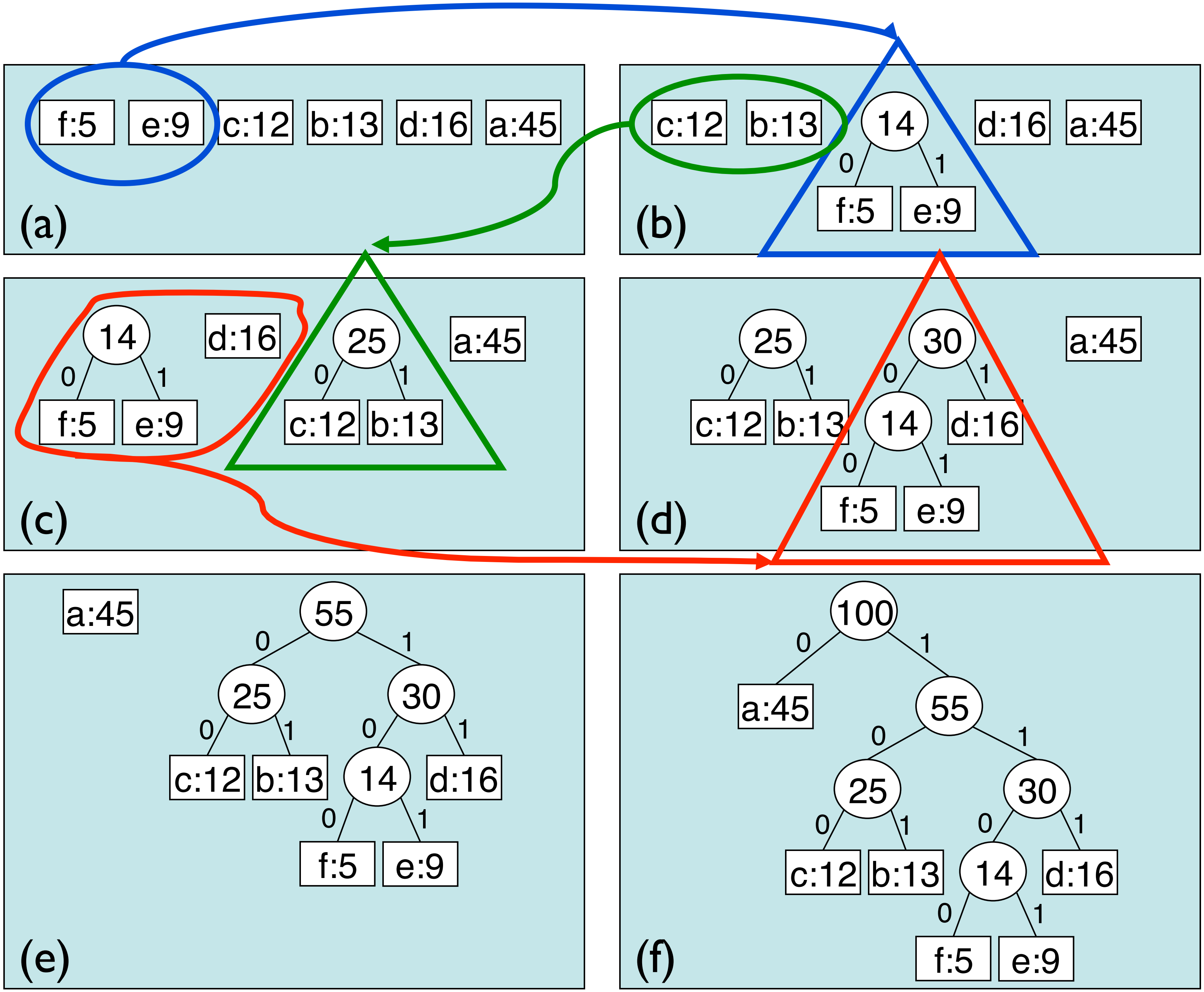


Greedy idea #3

a	45%
b	13%
c	12%
d	16%
e	9%
f	5%

Bottom up: Group *least* frequent letters near bottom





.45*1 + .41*3 + .14*4 = 2.24 bits per char

Huffman's Algorithm (1952)

Algorithm:

insert node for each letter into priority queue by freq
while queue length > 1 do
    remove smallest 2; call them x, y
    make new node z from them, with f(z) = f(x) + f(y)
    insert z into queue

Analysis: O(n) heap ops: O(n log n)

Goal: Minimize $B(T) = \sum_{c \in C} \text{freq}(c) * \text{depth}(c)$

Correctness: ???

Correctness Strategy

Optimal solution may not be unique, so cannot prove that greedy gives the *only* possible answer.

Instead, show that greedy's solution is as good as any.

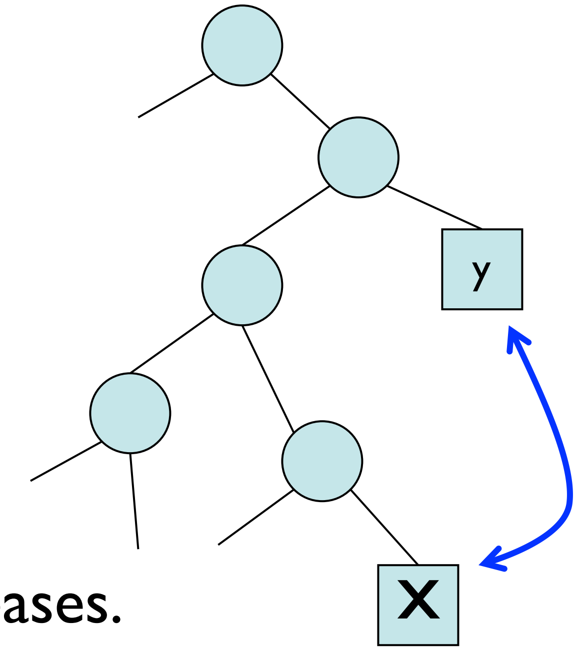
How: an exchange argument

Defn: A pair of leaves x,y is an inversion if

depth(x) ≥ depth(y)

and

freq(x) ≥ freq(y)



Claim: If we flip an inversion, cost never increases.

Why? All other things being equal, better to give more frequent letter the shorter code.

$$\underbrace{(d(x)*f(x) + d(y)*f(y))}_{\text{before}} - \underbrace{(d(x)*f(y) + d(y)*f(x))}_{\text{after}} =$$

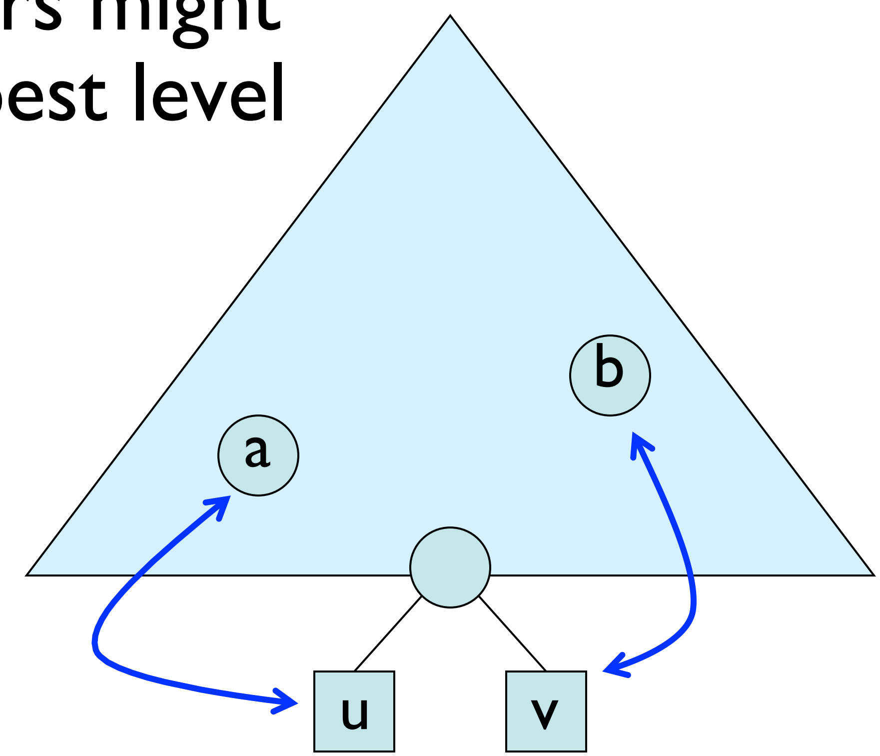
$$(d(x) - d(y)) * (f(x) - f(y)) \geq 0$$

I.e., non-negative cost savings.

Lemma 1:
"Greedy Choice Property"

The 2 least frequent letters might as well be siblings at deepest level

- Let a be least freq, b 2^{nd}
- Let u, v be siblings at max depth, $f(u) \leq f(v)$ (why must they exist?)
- Then (a,u) and (b,v) are inversions. Swap them.



Lemma 2

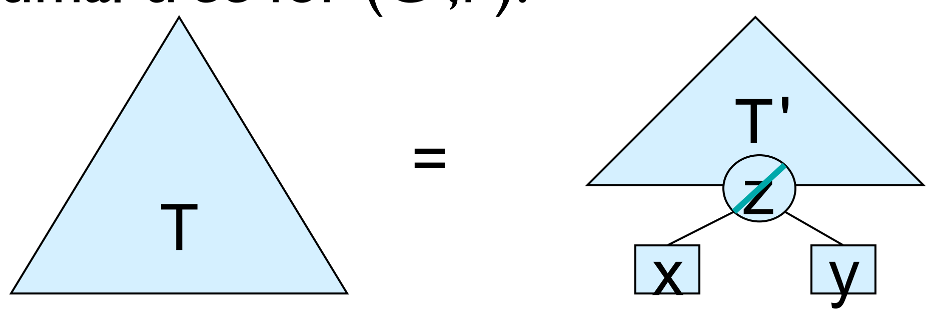
Let (C, f) be a problem instance: C an n-letter alphabet with letter frequencies f(c) for c in C.

For any x, y in C, z not in C, let C' be the (n-1) letter alphabet C - {x,y} ∪ {z} and for all c in C' define

$$f'(c) = \begin{cases} f(c), & \text{if } c \neq x,y,z \\ f(x) + f(y), & \text{if } c = z \end{cases}$$

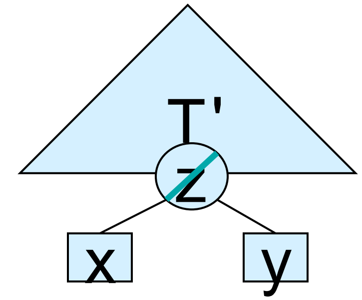
Let T' be an optimal tree for (C',f').

Then



is optimal for (C,f) among all trees having x,y as siblings

Proof:



$$B(T) = \sum_{c \in C} d_T(c) \cdot f(c)$$

$$B(T) - B(T') = d_T(x) \cdot (f(x) + f(y)) - d_{T'}(z) \cdot f'(z)$$
$$= (d_{T'}(z) + 1) \cdot f'(z) - d_{T'}(z) \cdot f'(z)$$
$$= f'(z)$$

Suppose $\hat{T}$ (having x & y as siblings) is better than T, i.e.

$B(\hat{T}) < B(T)$. Collapse x & y to z, forming $\hat{T}'$; as above:

$$B(\hat{T}) - B(\hat{T}') = f'(z)$$

Then:

$$B(\hat{T}') = B(\hat{T}) - f'(z) < B(T) - f'(z) = B(T')$$

Contradicting optimality of T'

Theorem: Huffman gives optimal codes

Proof: induction on |C|

Basis: n=1,2 – immediate

Induction: n>2

Let x,y be least frequent

Form C′, f′, & z, as above

By induction, T′ is opt for (C′,f′)

By lemma 2, T′ →T is opt for (C,f) among trees with x,y as siblings

By lemma 1, some opt tree has x, y as siblings

Therefore, T is optimal.

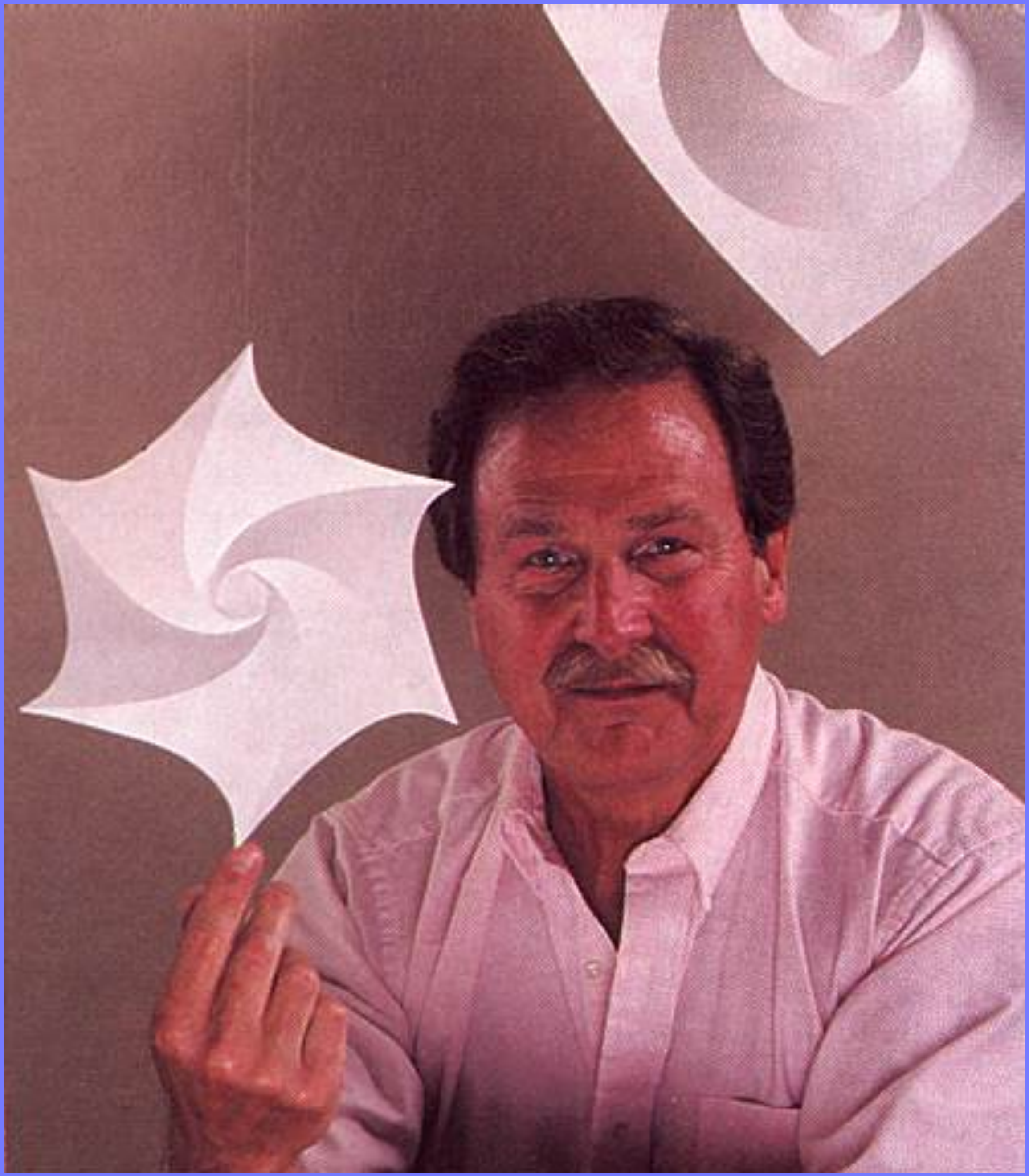
Data Compression

Huffman is optimal.

BUT still might do better!

- Huffman encodes fixed length blocks. What if we vary them?
- Huffman uses one encoding throughout a file. What if characteristics change?
- What if data has structure? E.g. raster images, video,…
- Huffman is lossless. Necessary?

LZW, MPEG, …



David A. Huffman, 1925-1999

Arithmetic Coding

In some ways a generalization of Huffman coding

Can provide better compression (by relaxing some of the Huffman assumptions) approaching theoretical limit

Algorithmically very different

Arithmetic Coding

Shannon Bound

letter i, prob p_i

Indp

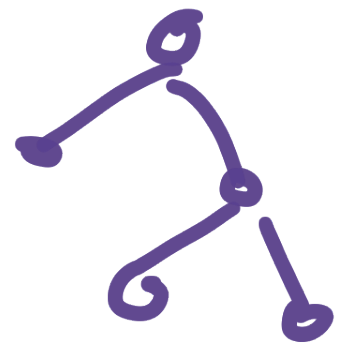
need $-\sum p_i \log_2 p_i$

on average (bits per character)

Ex

$\{a, b, c\}$, $p = 1/3$

Huffman



$\frac{1}{3} \cdot 1 + \frac{2}{3} \cdot 2 = 1.67$

bits/char

Shannon

$$-\sum \frac{1}{3} \log_2 \frac{1}{3} = \log_2 3$$

$$= 1.585 < 1.666$$

An Idea:

message: abach...

View as $(.01021\ldots)_3$ (base 3)

In more detail

msg .01021

find interval for

.01021x for all x

Send some

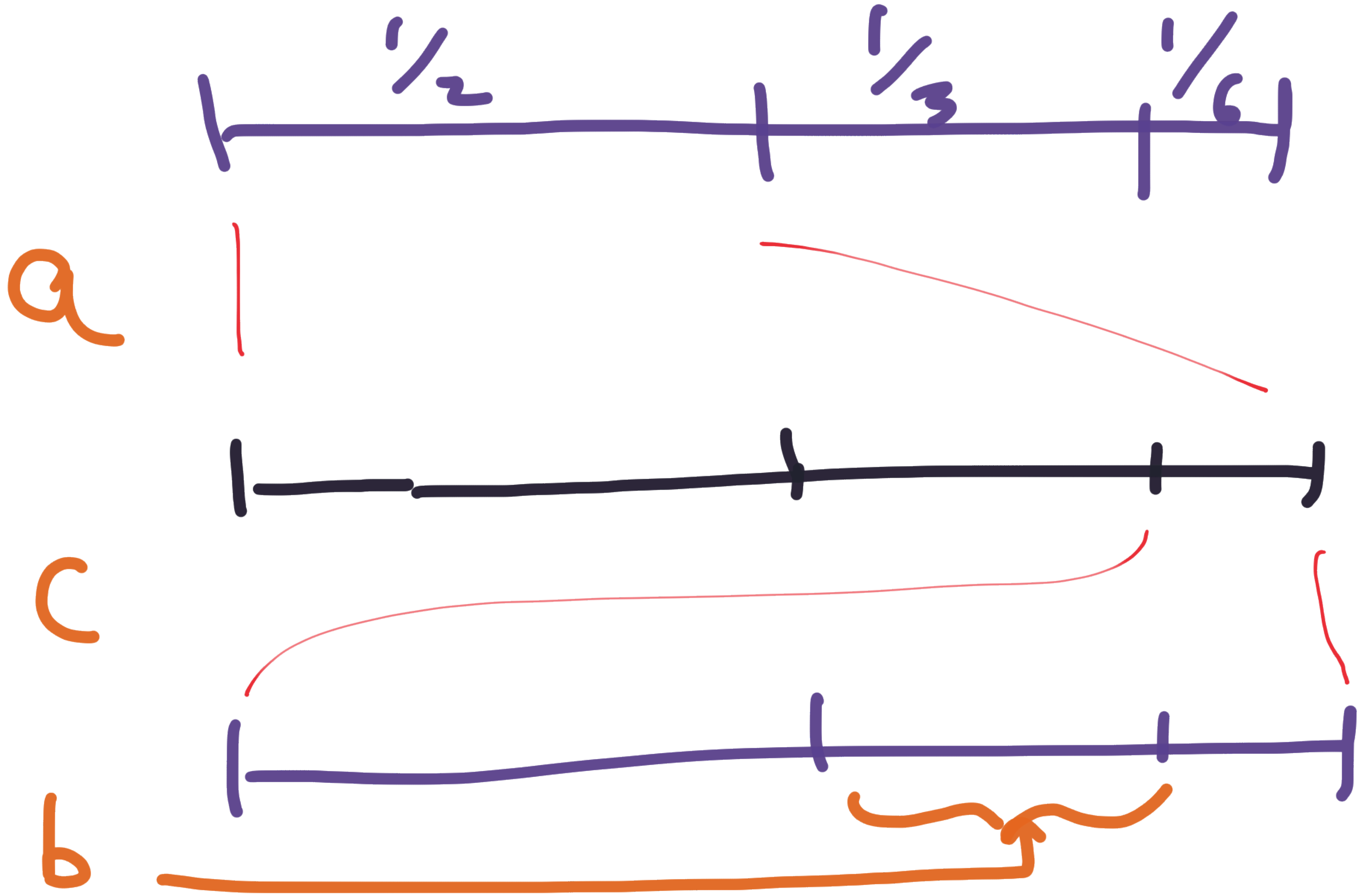
$v \in [.01021, .01021222\ldots)$

(any such v will do; might as well be the shortest one in binary)

What about ≠ frequency?

Ex: $P_a = 1/2$, $P_b = 1/3$, $P_c = 1/6$

Same idea, but unequal intervals. E.g. "a" maps to 1st half; "ac" to last sixth of 1st half.



$$abc \rightarrow 0 + \frac{5}{6} \cdot \frac{1}{2} + \left[\frac{1}{2} + \frac{5}{6}\right) \cdot \frac{1}{2} \cdot \frac{1}{6}$$

In general, if i^{th} letter of the alphabet a_i has frequency p_i, and $q_i = sum_{j<i}\, p_i$

Associate an interval $(b, l) = \{ x \mid b <= x < b+l \}$ with a string as follows:

empty string => interval (0, 1)

if string s => interval (b, l) then

string sa_i => interval $(b+q_i, l*p_i)$

How many bits?

msg .01021

find interval for

.01021x for all x

Send some

$v \in [.01021, .0102122222\ldots)$

(any such v will do; might as well be the shortest one in binary)

Fact

interval of width $\frac{1}{4} \le \varepsilon < \frac{1}{2}$ contains $k/4$ for exactly one integer k

More generally need $\lceil -\log_2 \varepsilon \rceil$ to encode a point in an interval of width ε.

Arithmetic Coding

i^{th} letter, p_i

Shannon: $-\sum p_i \log_2 p_i$

msg length n, expect

np_i of letter i

So interval length

$$\approx \prod p_i^{n p_i}$$

$$\left\lceil -\lg_2 \prod p_i^{n p_i} \right\rceil$$

$$\approx -n \sum p_i \log_2 p_i$$

= Shannon

More:

non-independent (patients)

adaptive

(But must be careful about arithmetic, # bits)