CSE 521
Algorithms

Depth First Search and
Strongly Connected Components

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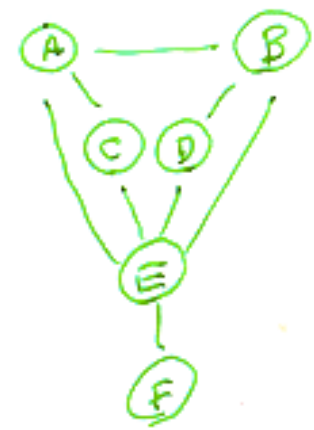
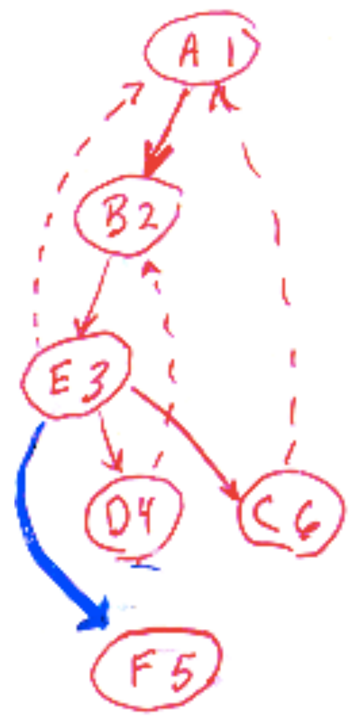
Undirected Depth-First Search



- Key Properties:
  1. No "cross-edges"; only tree- or back-edges
  2. Before returning, DFS(v) visits all vertices reachable from v via paths through previously unvisited vertices



Directed Depth First Search

- Algorithm: Unchanged
- Key Properties:

  2. Unchanged

  1'. Edge (v,w) is:

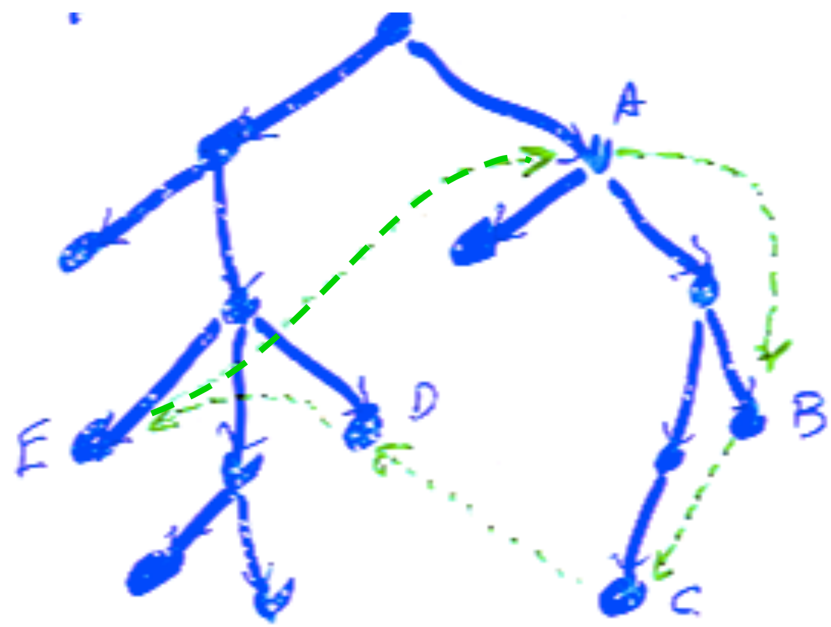
As before	Tree-edge	if w unvisited
	Back-edge	if w visited, #w<#v, on stack
New	Cross-edge	if w visited, #w<#v, not on stack
	Forward-edge	if w visited, #w>#v

Note: Cross edges *only* go "Right" to "Left"

An Application:

G has a cycle ⇔ DFS finds a back edge

- ⇐ Easy - back edge (x,y) plus tree edges y, …, x form a cycle.
- ⇒ Why can't we have something like this?:



Lemma 1

Before returning, dfs(v) visits w iff

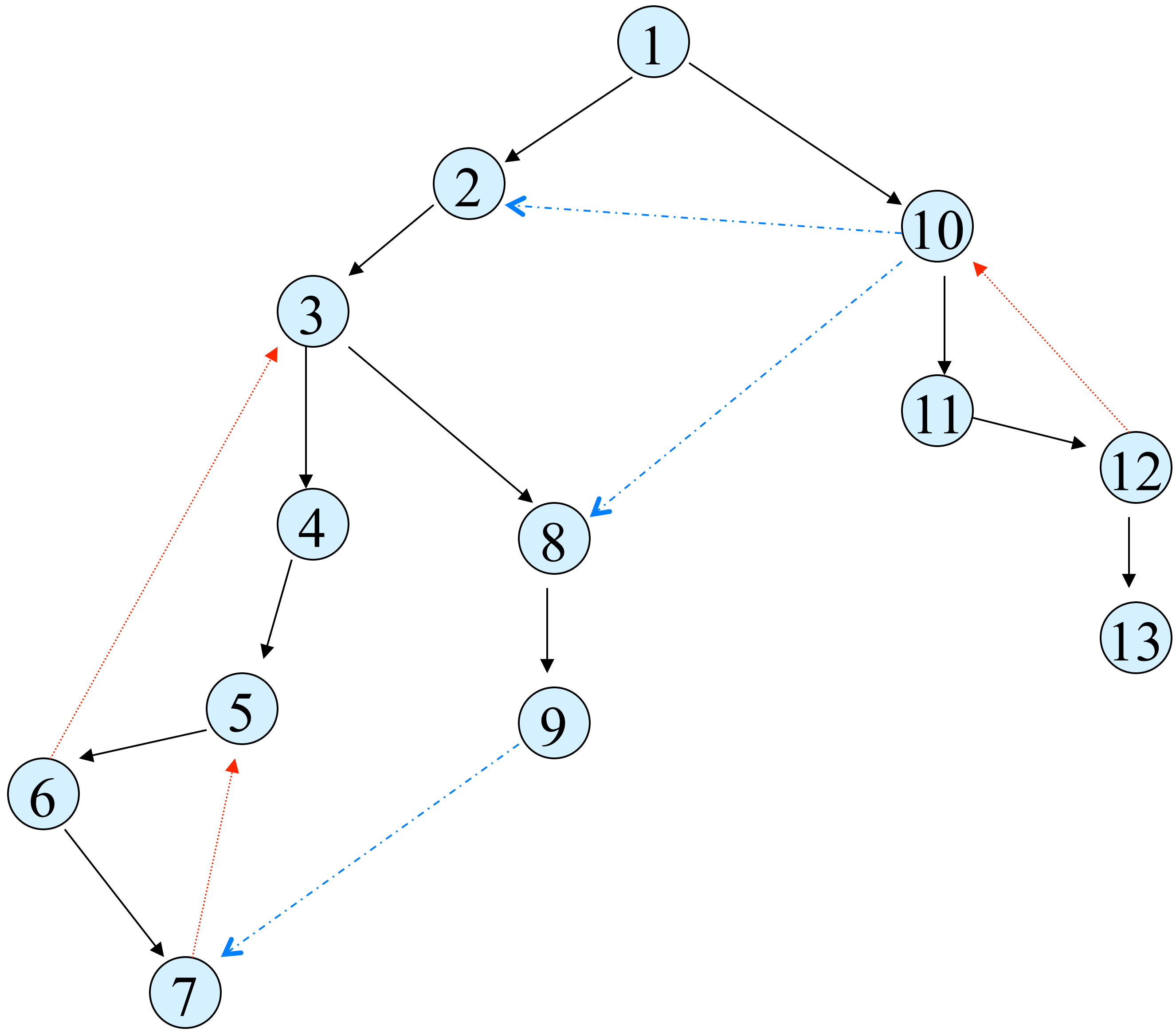
- w is unvisited
- w is reachable from v via a path through unvisited vertices

Proof sketch:

- dfs follows all direct out-edges
- call dfs recursively at each unvisited one
- use induction on # of such w

Strongly Connected Components

- Defn: G is *strongly connected* if for all u,v there is a (directed) path from u to v and from v to u.
  [Equivalently:
  there is a circuit through u and v.]
- Defn: a *strongly connected component* of G is a maximal strongly connected (vertex-induced) subgraph.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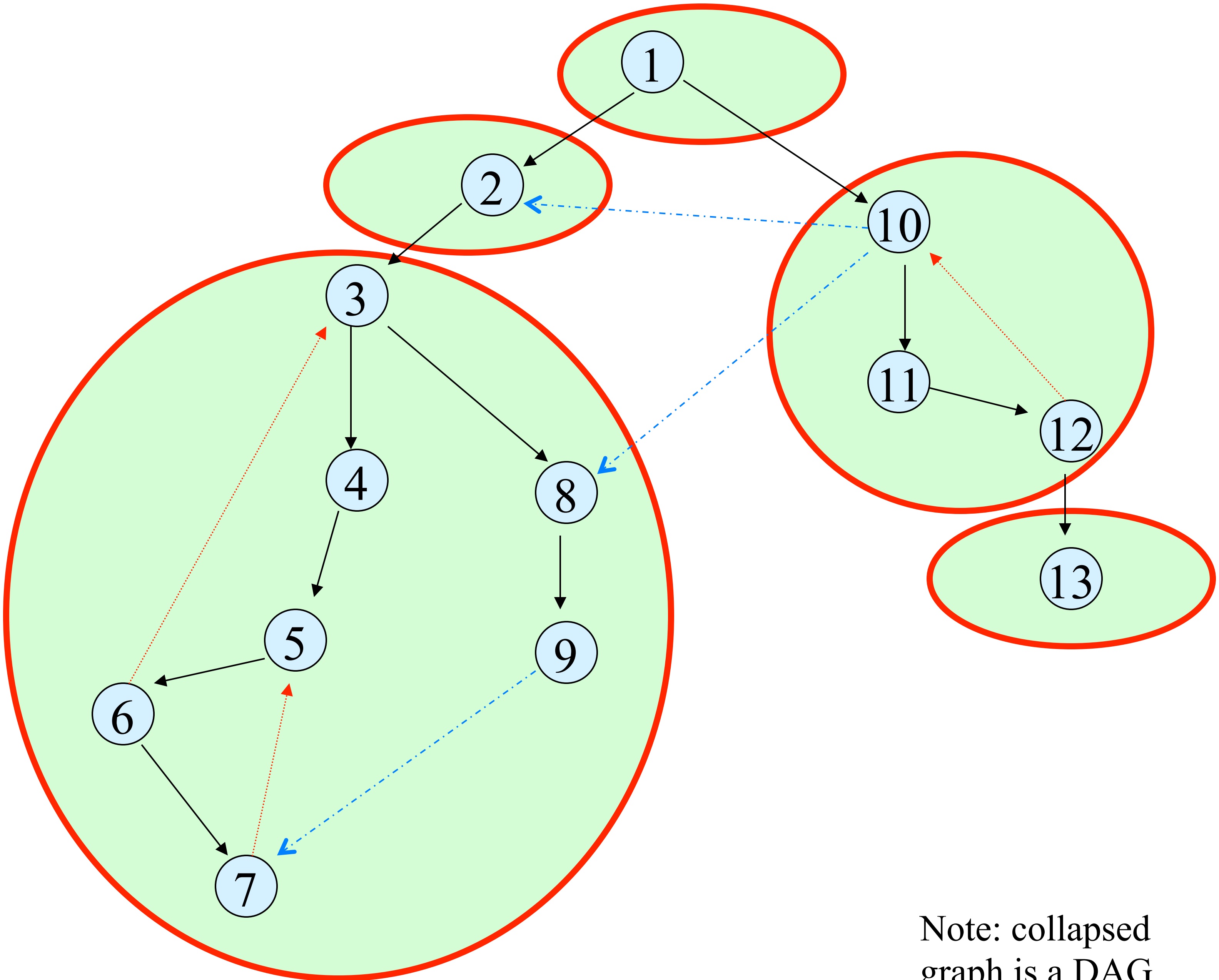


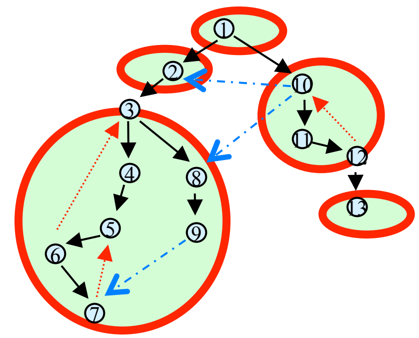
Note: collapsed
graph is a DAG

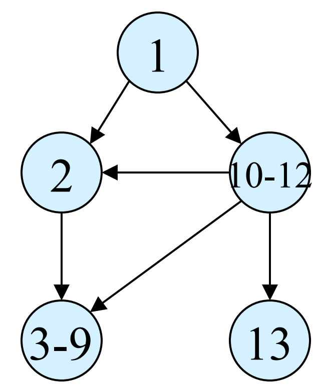
Uses for SCC's

- Optimizing compilers:
  - SCC's in program flow graph = loops
  - SCC's in call graph = mutual recursion
- Operating Systems: If (u,v) means process u is waiting for process v, SCC's show deadlocks.
- Spreadsheet eval: circular dependencies
- Econometrics: SCC's might show highly interdependent sectors of the economy.
- Etc.

Directed Acyclic Graphs

- If we collapse each SCC to a single vertex we get a directed graph with no cycles
  - a **directed acyclic graph** or **DAG**
- Many problems on directed graphs can be solved as follows:
  - Compute SCC's and resulting DAG
  - Do one computation on each SCC
  - Do another on the overall DAG
  - Example: Spreadsheet evaluation





Two Simple SCC Algorithms

- u,v in same SCC iff there are paths u $\rightarrow$ v & v $\rightarrow$ u

- Transitive closure: $O(n^3)$

- DFS from every u, v: $O(ne) = O(n^3)$

Goal:

- Find all Strongly Connected Components in linear time, i.e., time O(n+e)

  (Tarjan, 1972)

Definition

The *root* of an SCC is the first vertex in it visited by DFS.

Equivalently, the root is the vertex in the SCC with the smallest DFS number.

Lemma 2

Exercise: show that each SCC is a *contiguous* subtree.

All members of an SCC are descendants of its root.

Proof:

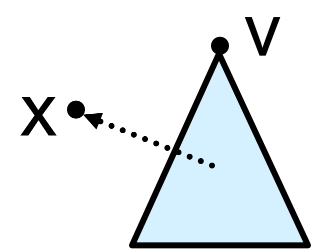
- all members are reachable from all others
- so, all are reachable from its root
- all are unvisited when root is visited
- so, all are descendants of its root (Lemma 1)

Subgoal

- Can we identify some root?
- How about the root of the first SCC completely explored (returned from) by DFS?

- Key idea: no exit from first SCC
  (first SCC is leftmost “leaf” in collapsed DAG)

Definition

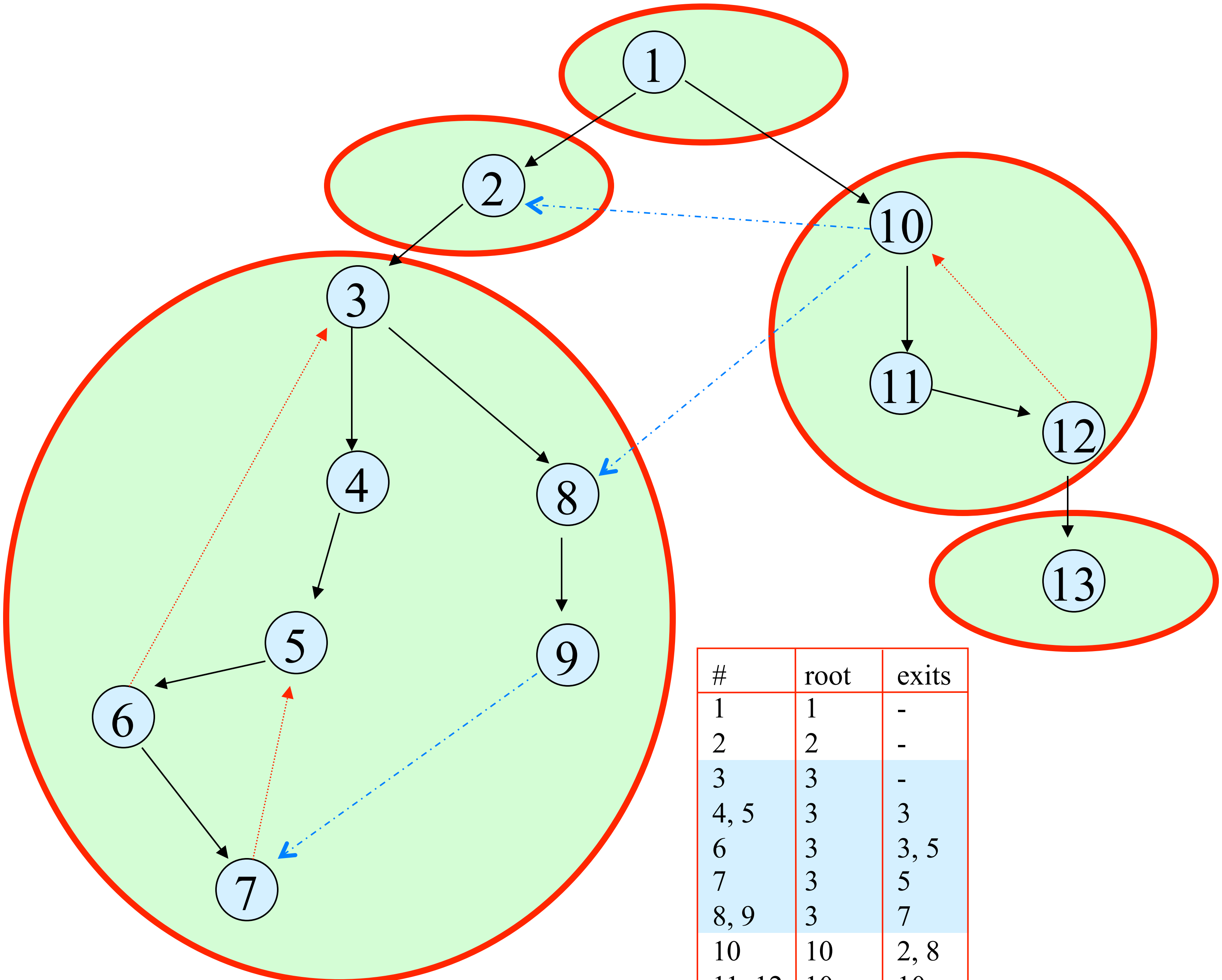


x is an *exit* from v (from v's subtree) if

- x is not a descendant of v, but
- x is the head of a (cross- or back-) edge from a descendant of v (including v itself)

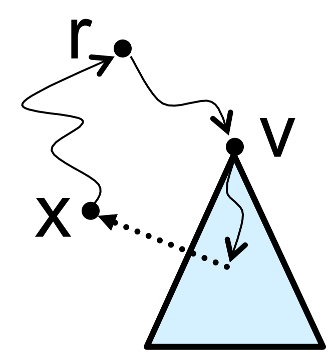
NOTE: #x < #v

Ex: node #1 cannot have an exit.



#	root	exits
1	1	-
2	2	-
3	3	-
4, 5	3	3
6	3	3, 5
7	3	5
8, 9	3	7
10	10	2, 8
11, 12	10	10
13	13	-

Lemma 3: Nonroots have exits



Idea: Follow cycle to root

If v is not a root, then v has an exit.

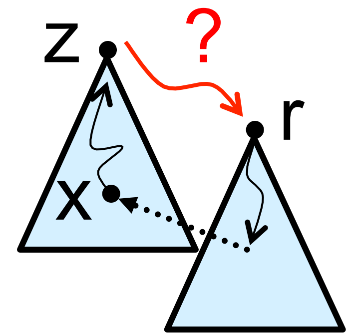
Proof:

- let r be root of v's SCC
- r is a proper ancestor of v (Lemma 2)
- let x be the first vertex that is not a descendant of v on a path v → r .
- x is an exit

Cor (contrapositive): If v has no exit, then v is a root.

NB: converse not true; some roots do have exits

Lemma 4: No Escaping 1st Root



Idea: Exit ⇒ Bigger Cycle

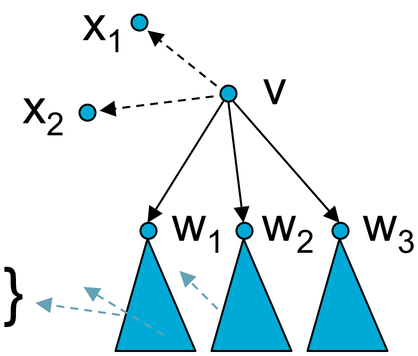
If r is the first root from which dfs returns, then r has no exit

Proof (by contradiction):

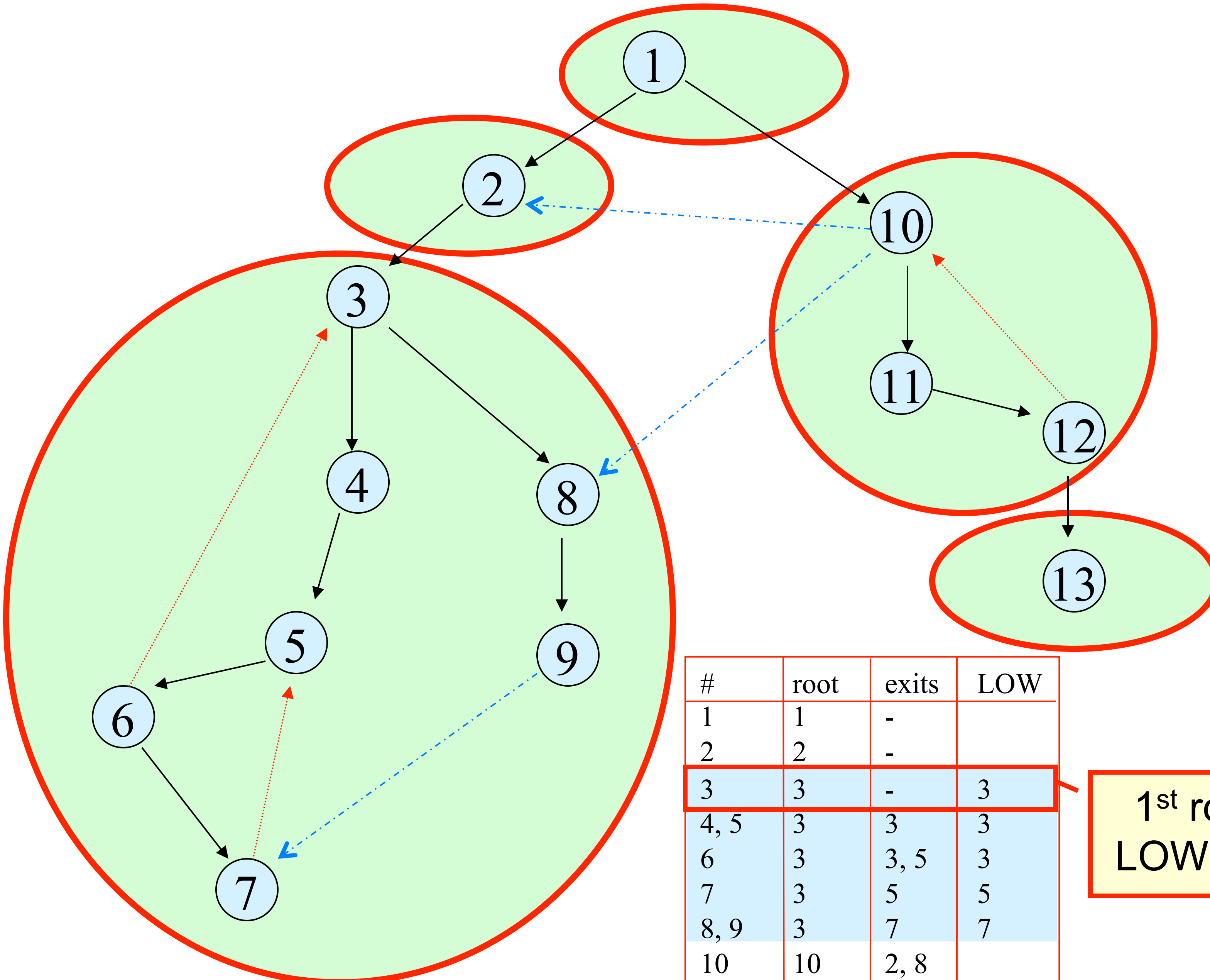
- Suppose x is an exit
- let z be root of x's SCC
- r not reachable from z, else in same SCC
- $\#z \leq \#x$ (z ancestor of x; Lemma 2)
- $\#x < \#r$ (x is an exit from r)
- $\#z < \#r$, no $z \rightarrow r$ path, so return from z first
- Contradiction

How to Find Exits (in 1st component)

- All exits x from v have #x < #v
- Suffices to find any of them, e.g. min #
- Defn:
  LOW(v) = min({ #x | x an exit from v} ∪ {#v})
- Calculate inductively:
  LOW(v) = min of:
  - #v
  - { LOW(w) | w a child of v }
  - { #x | (v,x) is a back- or cross-edge }



- 1st root : LOW(v)=v



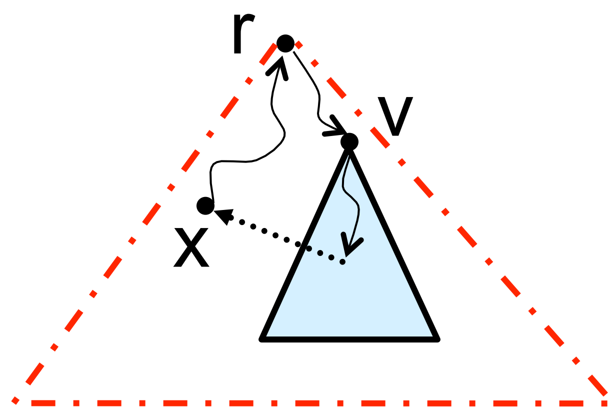
#	root	exits	LOW
1	1	-	
2	2	-	
3	3	-	3
4, 5	3	3	3
6	3	3, 5	3
7	3	5	5
8, 9	3	7	7
10	10	2, 8	
11, 12	10	10	
13	13	-	

1st root: LOW(v)=v

Finding Other Components

- Key idea: No exit from
  - 1^{st} SCC
  - 2^{nd} SCC, except maybe to 1^{st}
  - 3^{rd} SCC, except maybe to 1^{st} and/or 2^{nd}
  - ...

Lemma 3'



If v is not a root, then v has an exit .

in v's SCC

Proof:

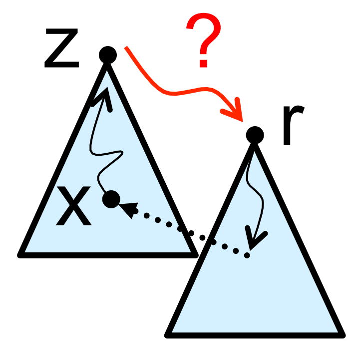
- let r be root of v's SCC
- r is a proper ancestor of v (Lemma 2)
- let x be the first vertex that is not a descendant of v on a path v → r .
- x is an exit

in v's SCC

Cor: If v has no exit , then v is a root.

in v's SCC

Lemma 4'



kth

If r is the ~~first~~ root from which dfs returns, then r has no exit

except possibly to the first (k-1) components

Proof:

- Suppose x is an exit
- let z be root of x's SCC
- r not reachable from z, else in same SCC
- #z ≤ #x (z ancestor of x; Lemma 2)
- #x < #r (x is an exit from r)
- #z < #r, no z → r path, so return from z first
- ~~Contradiction~~

i.e., x in first (k-1)

How to Find Exits (in ~~1st~~ kth component)

- All exits x from v have #x < #v
- Suffices to find any of them, e.g. min #
- Defn:
  LOW(v) = min({ #x | x an exit from v } ∪ {#v})
- Calculate inductively:
  LOW(v) = min of:
  - #v
  - { LOW(w) | w a child of v }
  - { #x | (v,x) is a back- or cross-edge }

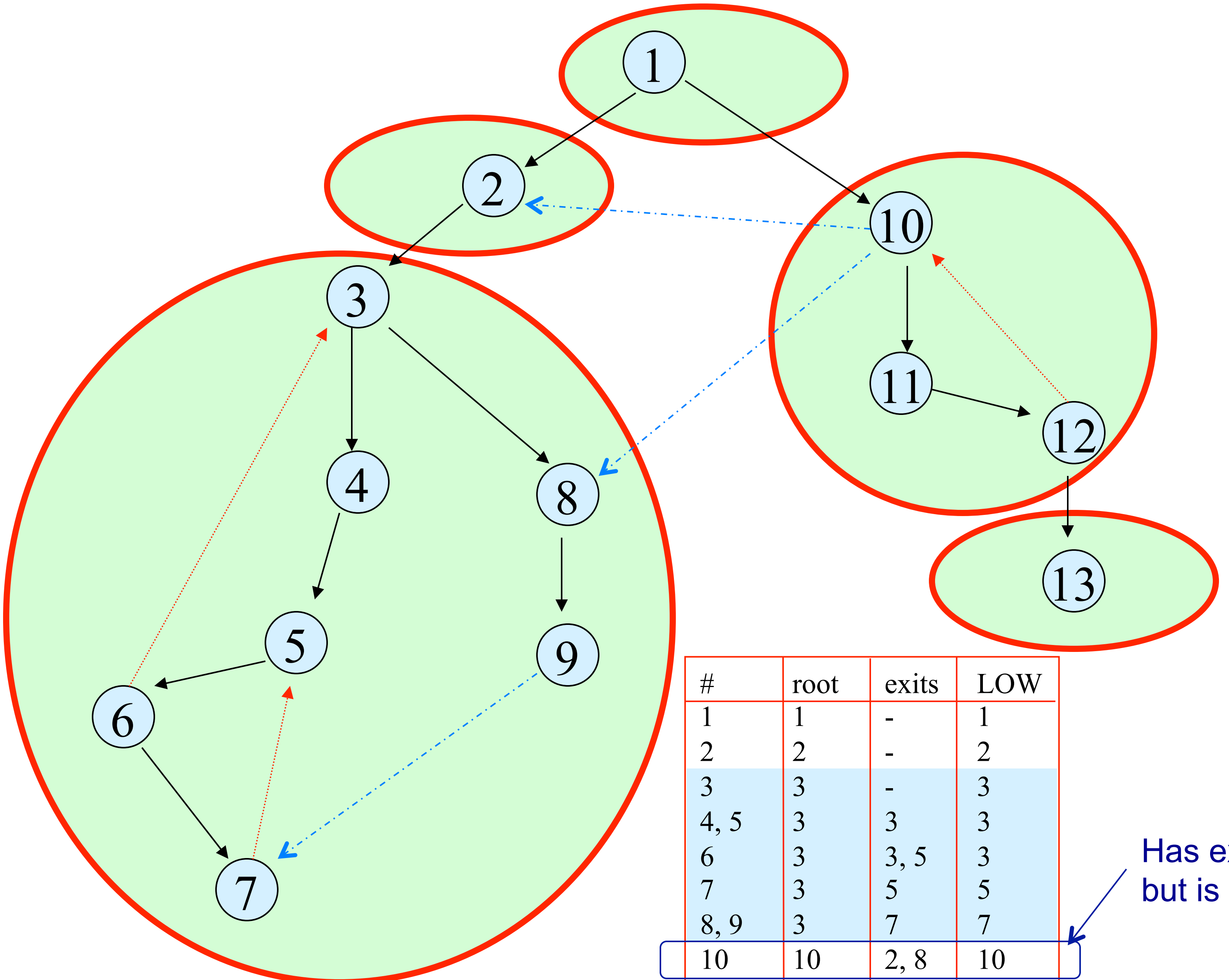
and x not in first (k-1) components

NB: defn of "exit" has not changed, but we're not interested in exits ***into*** previous SCCs

SCC Algorithm

#v = DFS number
v.low = LOW(v)
v.scc = component #

```
SCC(v)
   #v = vertex_number++; v.low = #v; push(v)
   for all edges (v,w)
     if #w == 0 then
         SCC(w); v.low = min(v.low, w.low) // tree edge
     else if #w < #v && w.scc == 0 then
         v.low = min(v.low, #w)       // cross- or back-edge
   if #v == v.low then                // v is root of new scc
     scc#++;
     repeat
         w = pop(); w.scc = scc#;    // mark SCC members
     until w==v
```



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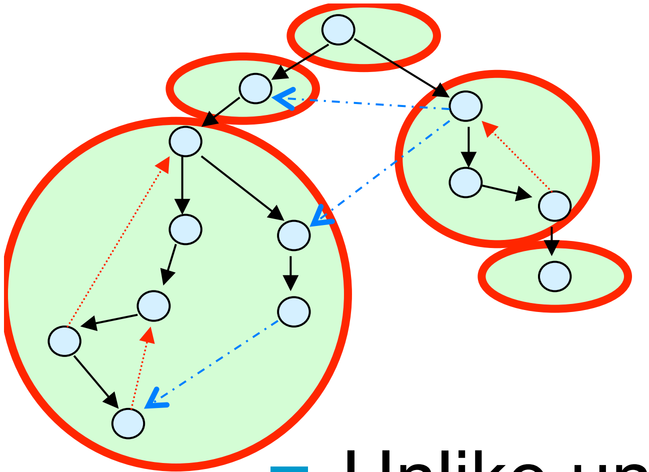
1	1	-	1
2	2	-	2
3	3	-	3
4, 5	3	3	3
6	3	3, 5	3
7	3	5	5
8, 9	3	7	7
10	10	2, 8	10
11, 12	10	10	10
13	13	-	13

Has exits,
but is a root

Complexity

- Look at every edge once
- Look at every vertex (except via in-edge) at most once

- Time = O(n+e)



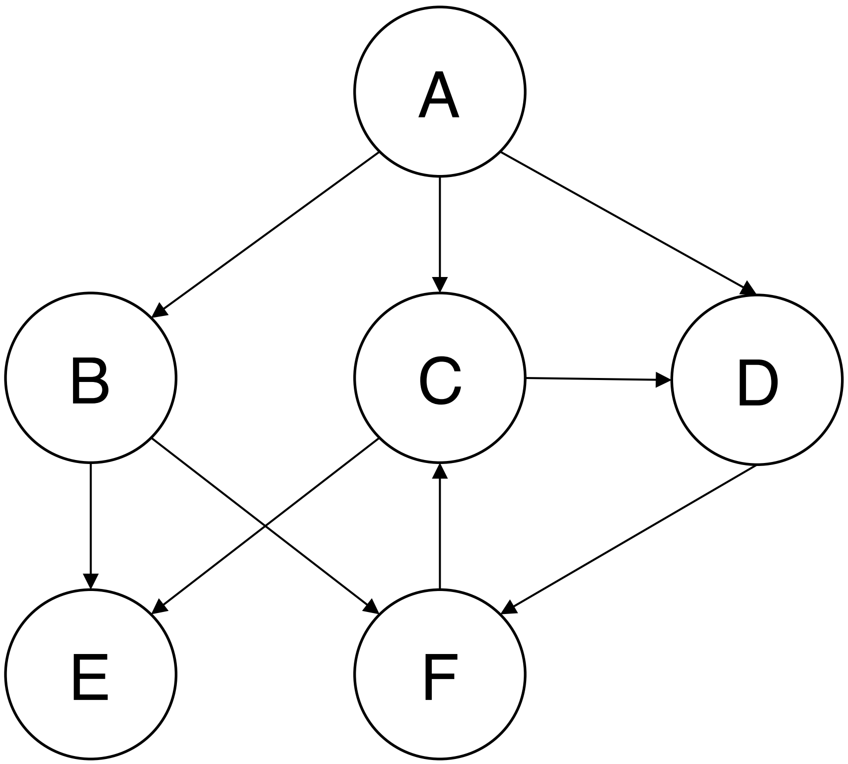
Where to start

- Unlike undirected DFS, start vertex matters
- Add "outer loop":

  mark all vertices unvisited
  while there is unvisited vertex v do
      scc(v)

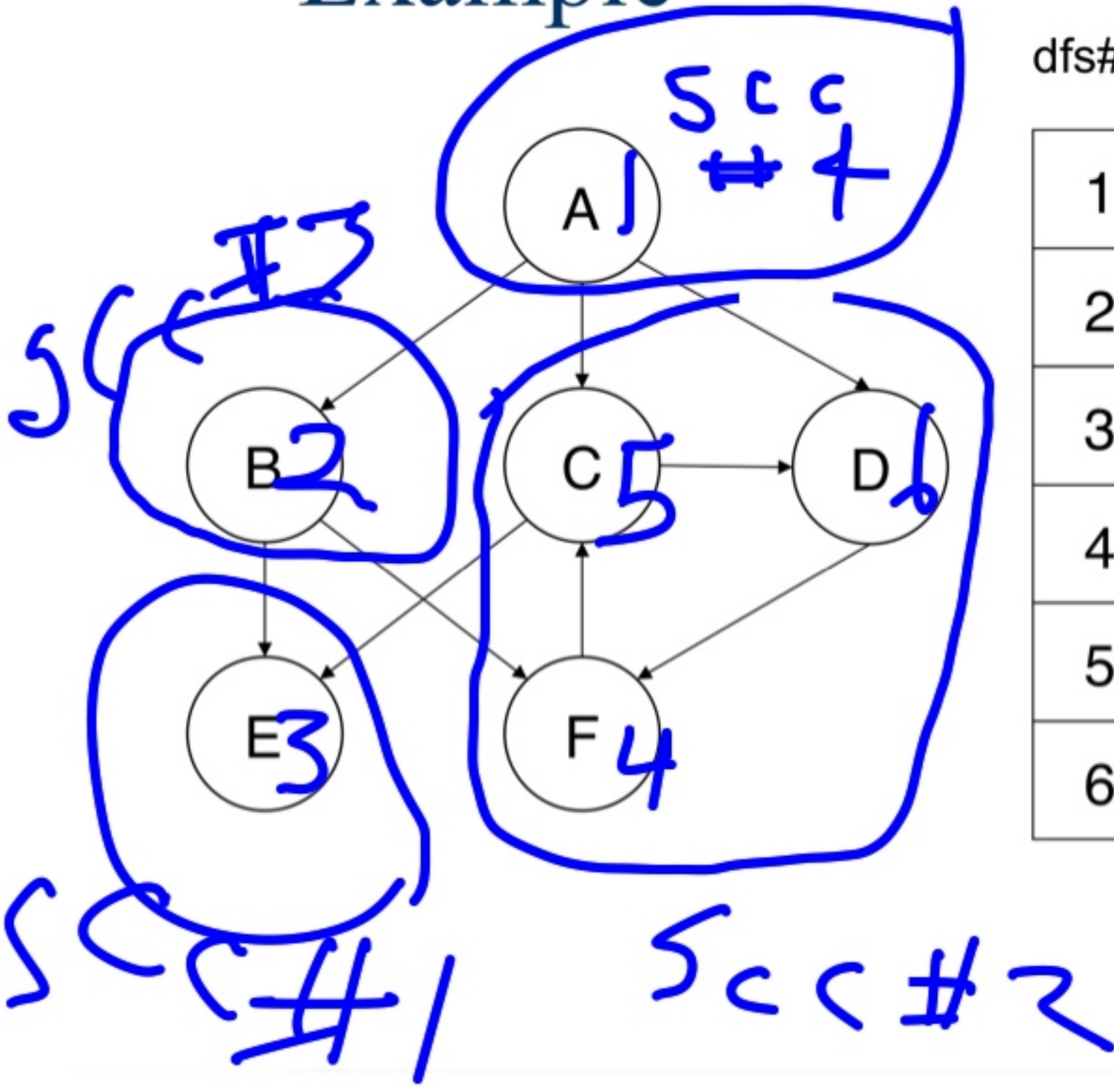
- Exercise: redo example starting from another vertex, e.g. #11 or #13 (which become #1)

Example



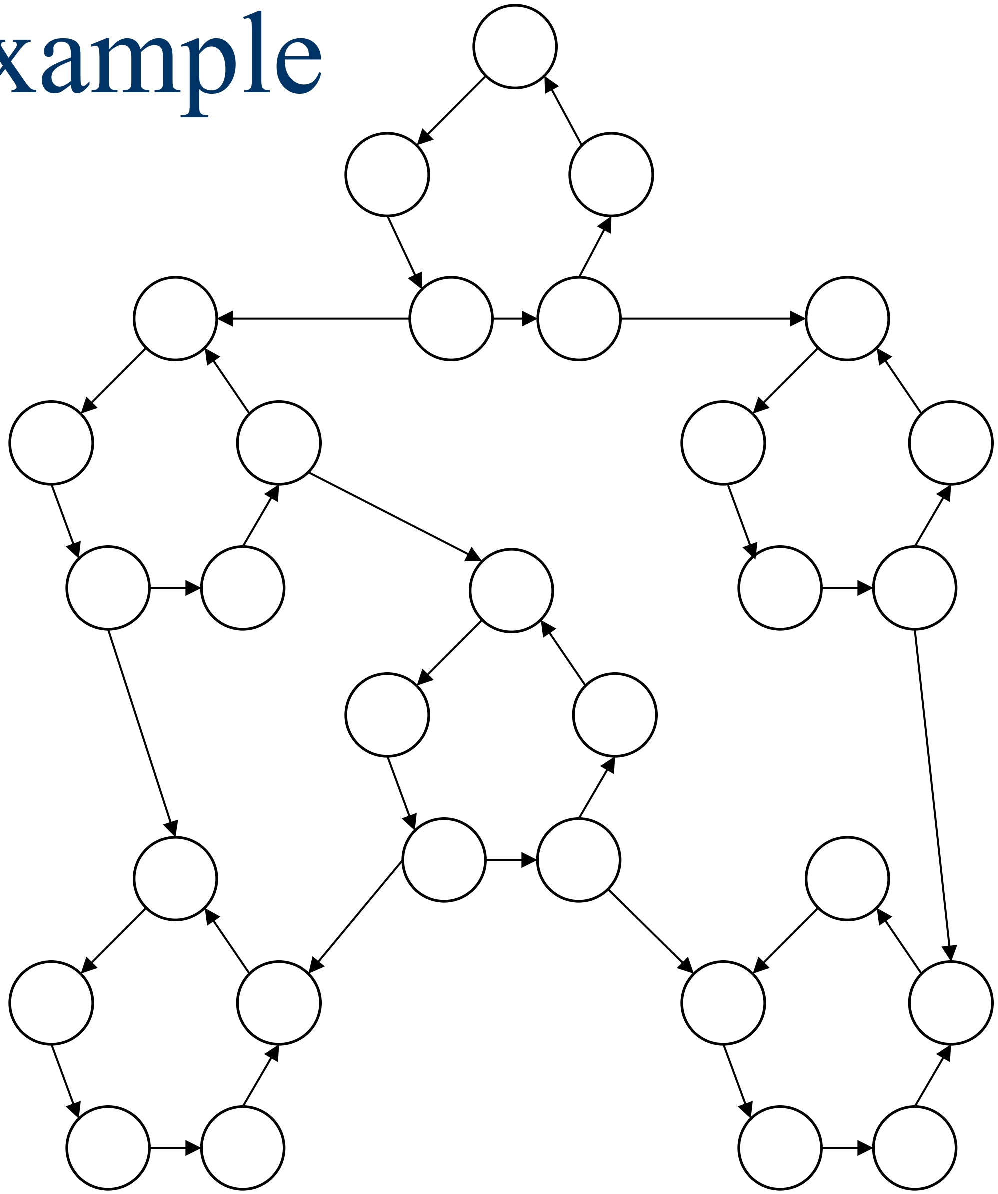
dfs#	v	root	exits	low(v)
1				
2				
3				
4				
5				
6				

Example



dfs#	v	root	exits	low(v)
1	A	1	–	1
2	B	2	–	2
3	E	3	–	3
4	F	4	–	4
5	C	4	3,4	~~3~~ 4
6	D	4	4	~~6~~ 4

Example



v	Low(v)

v	Low(v)