CSE 521: Algorithms

2: Analysis

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Our correct TSP algorithm was incredibly slow

Basically slow no matter what computer you have

We want a general theory of "efficiency" that is

- Simple
- Objective
- Relatively independent of changing technology
- But still predictive – "theoretically bad" algorithms should be bad in practice and vice versa (usually)

"Runs fast on typical real problem instances"

Pro:

sensible, bottom-line-oriented

Con:

moving target (diff computers, compilers, Moore's law)

highly subjective (how fast is "fast"? What's "typical"?)

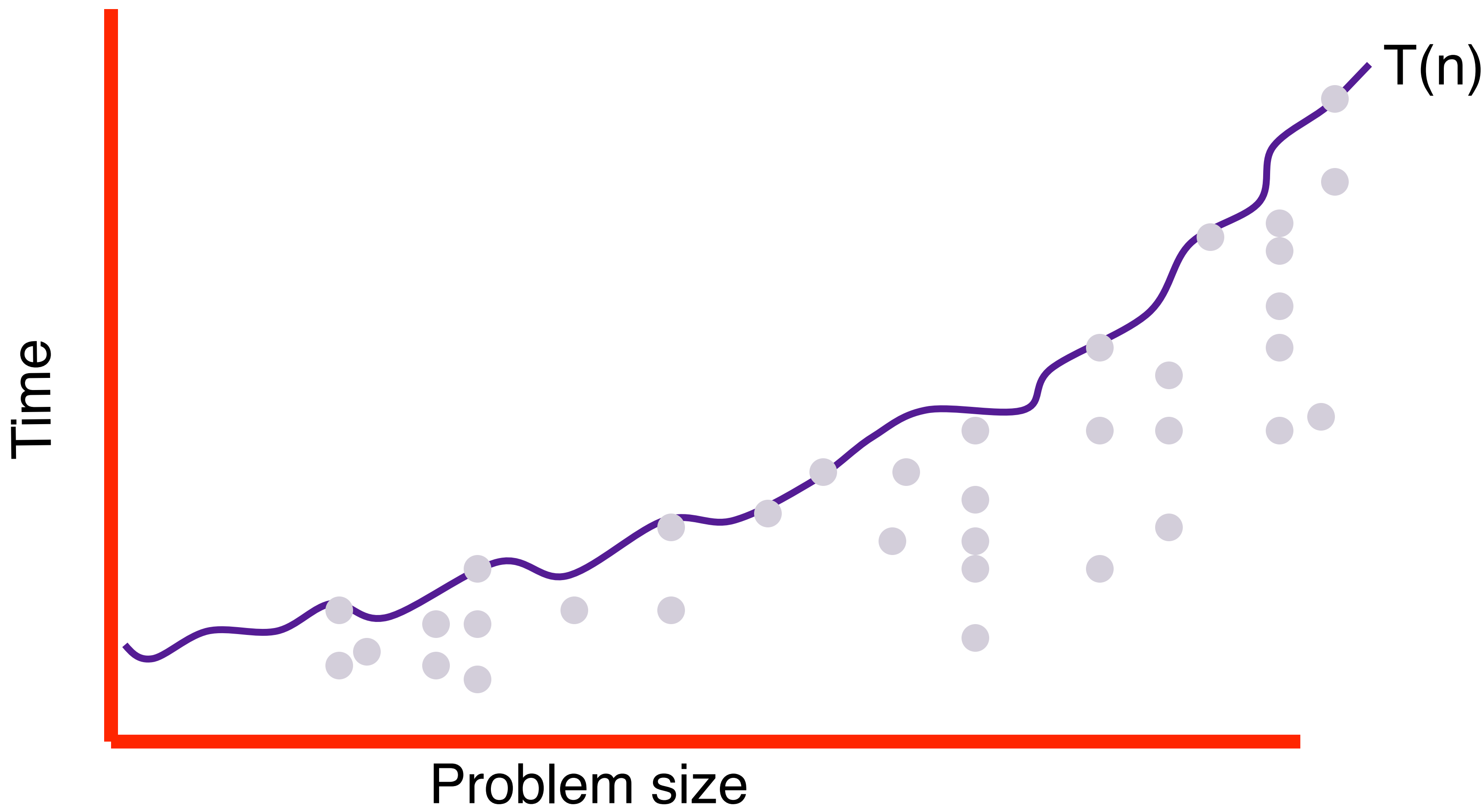
The *time complexity* of an algorithm associates a number T(n), the worst-case time the algorithm takes, with each problem size n.

Mathematically,

$T: N \to R$

i.e., T is a function mapping non-negative integers (problem sizes) to real numbers (number of steps).

"Reals" so we can say, e.g., sqrt(n) instead of $\lceil \text{sqrt}(n) \rceil$


T(n)
Time
Problem size

Asymptotic growth rate, i.e., characterize growth rate of worst-case run time as a function of problem size, up to a constant factor, e.g. $T(n) = O(n^2)$

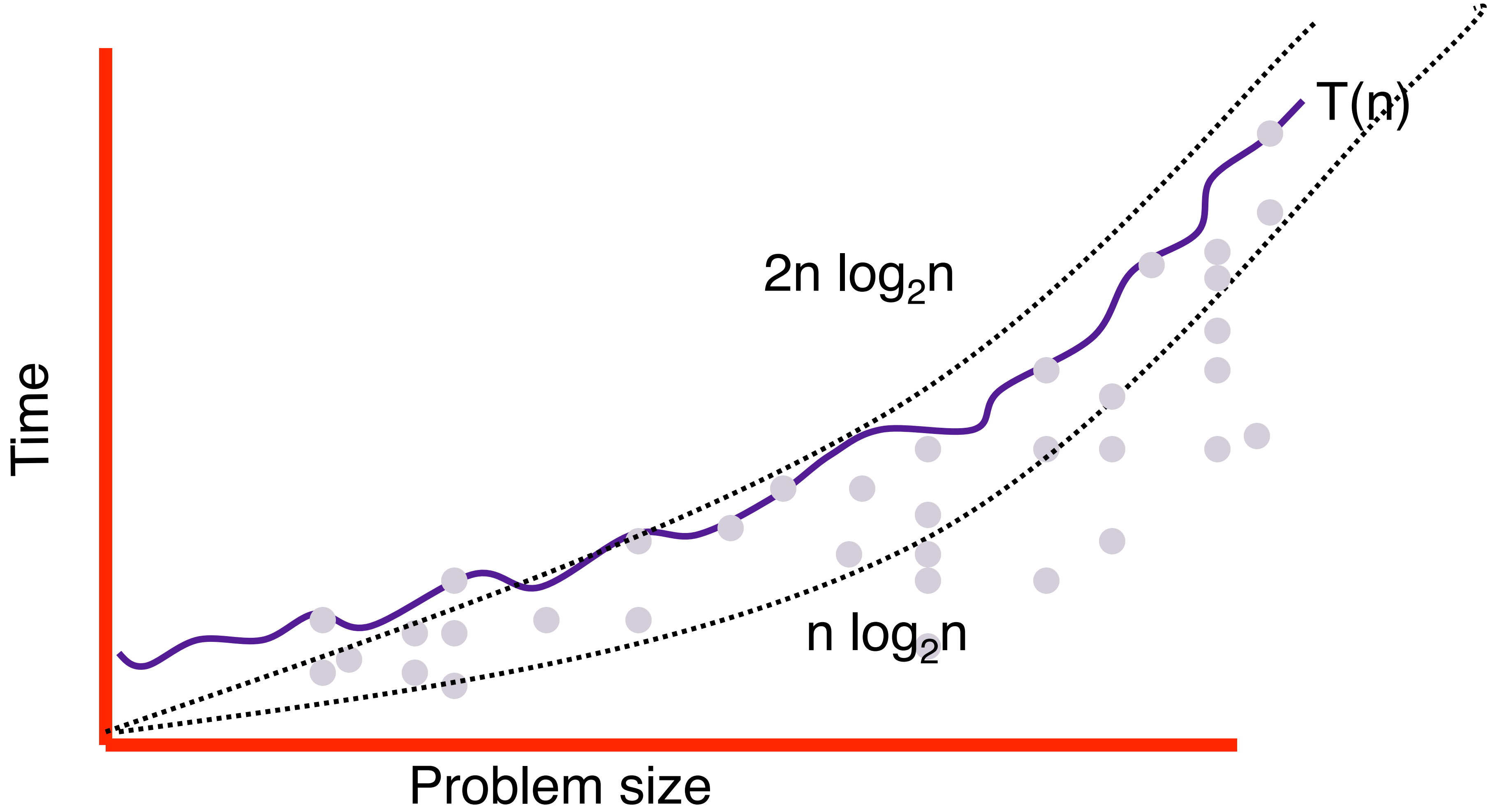
Why not try to be more precise?

Average-case, e.g., is hard to define, analyze

Technological variations (computer, compiler, OS, …) easily 10x or more

Being more precise is a ton of work

A key question is "scale up": if I can afford this today, how much longer will it take when my business is 2x larger? (E.g. today: cn^2, next year: $c(2n)^2 = 4cn^2$: 4 x longer.) Big-O analysis is adequate to address this.



Time
Problem size
2n $\log_2$n
n $\log_2$n
T(n)

Given two functions f and $g:N\to R$

f(n) is O(g(n)) iff there is a constant c>0 so that
f(n) is eventually always $\leq$ c g(n)

f(n) is Ω (g(n)) iff there is a constant c>0 so that
f(n) is eventually always $\geq$ c g(n)

f(n) is Θ (g(n)) iff there is are constants c_1, c_2>0 so that
eventually always $c_1 g(n) \leq f(n) \leq c_2 g(n)$

$10n^2-16n+100$ is $O(n^2)$ also $O(n^3)$

$10n^2-16n+100 \le 11n^2$ for all $n \ge 10$

$10n^2-16n+100$ is $\Omega(n^2)$ also $\Omega(n)$

$10n^2-16n+100 \ge 9n^2$ for all $n \ge 16$

Therefore also $10n^2-16n+100$ is $\Theta(n^2)$

$10n^2-16n+100$ is not $O(n)$ also not $\Omega(n^3)$

Transitivity.

If $f = O(g)$ and $g = O(h)$ then $f = O(h)$.

If $f = \Omega(g)$ and $g = \Omega(h)$ then $f = \Omega(h)$.

If $f = \Theta(g)$ and $g = \Theta(h)$ then $f = \Theta(h)$.

Additivity.

If $f = O(h)$ and $g = O(h)$ then $f + g = O(h)$.

If $f = \Omega(h)$ and $g = \Omega(h)$ then $f + g = \Omega(h)$.

If $f = \Theta(h)$ and $g = O(h)$ then $f + g = \Theta(h)$.

Claim: For any a, and any b>0, $(n+a)^b$ is $\Theta(n^b)$

$(n+a)^b \le (2n)^b$ for $n \ge |a|$

$= 2^b n^b$

$= cn^b$ for $c = 2^b$

so $(n+a)^b$ is $O(n^b)$

$(n+a)^b \ge (n/2)^b$ for $n \ge 2|a|$ (even if $a < 0$)

$= 2^{-b} n^b$

$= c'n$ for $c' = 2^{-b}$

so $(n+a)^b$ is $\Omega(n^b)$

Claim: For any a, b>1 $\log_a n$ is $\Theta(\log_b n)$

$$\log_a b = x \text{ means } a^x = b$$

$$a^{\log_a b} = b$$

$$(a^{\log_a b})^{\log_b n} = b^{\log_b n} = n$$

$$(\log_a b)(\log_b n) = \log_a n$$

$$c \log_b n = \log_a n \text{ for the constant } c = \log_a b$$

$$\text{So}:$$

$$\log_b n = \Theta(\log_a n) = \Theta(\log n)$$

Asymptotic Bounds for Some Common Functions

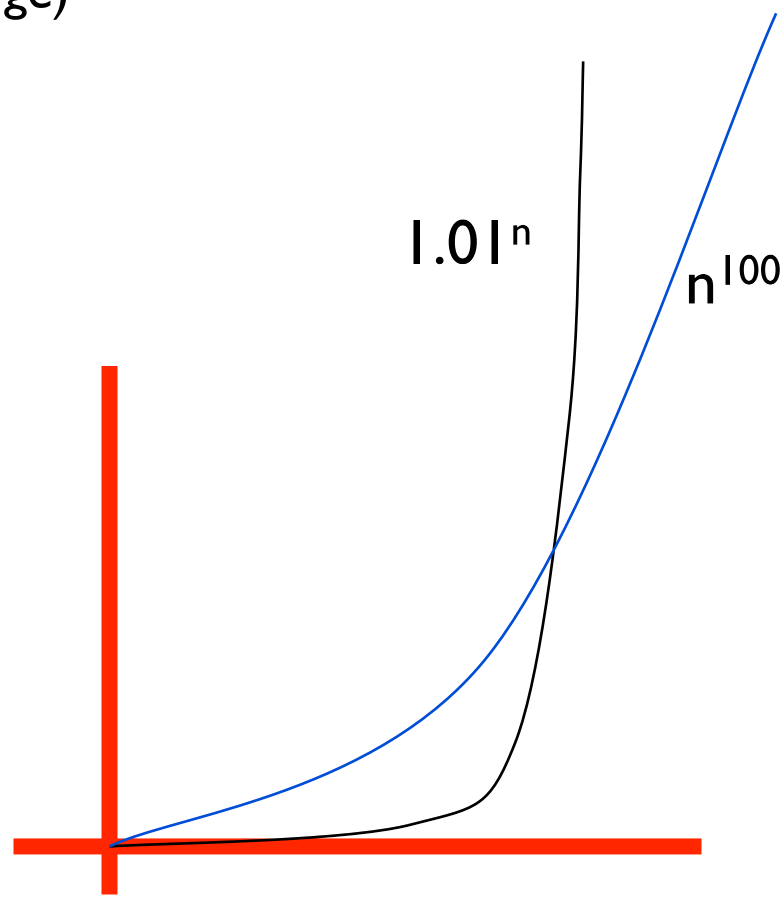
Polynomials:

$a_0 + a_1 n + \ldots + a_d n^d$ is $\Theta(n^d)$ if $a_d > 0$

Logarithms:

$O(\log_a n) = O(\log_b n)$ for any constants $a,b > 0$

For all r > 1 (no matter how small)
and all d > 0, (no matter how large)
$n^d = O(r^n)$

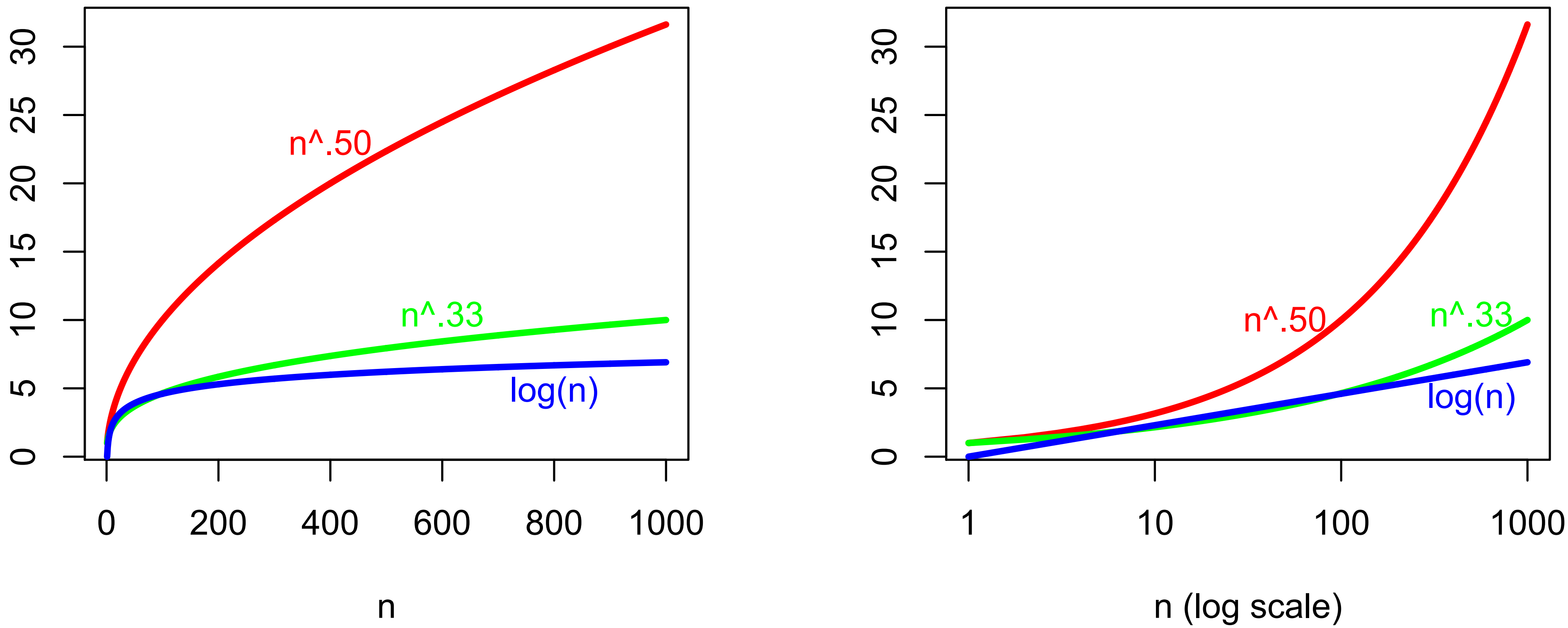


In short, every exponential grows faster than every polynomial!

Logarithms:

For all $x > 0$, *(no matter how small)* $\log n = O(n^x)$

log grows slower than every polynomial



f(n) is o(g(n)) iff $\lim_{n\to\infty} f(n)/g(n)=0$

that is g(n) *dominates* f(n)

If $a \leq b$ then n^a is $O(n^b)$

If $a < b$ then n^a is $o(n^b)$

Note:
if f(n) is Θ (g(n)) then it cannot be o(g(n))

$n^2 = o(n^3)$ [Use algebra]:

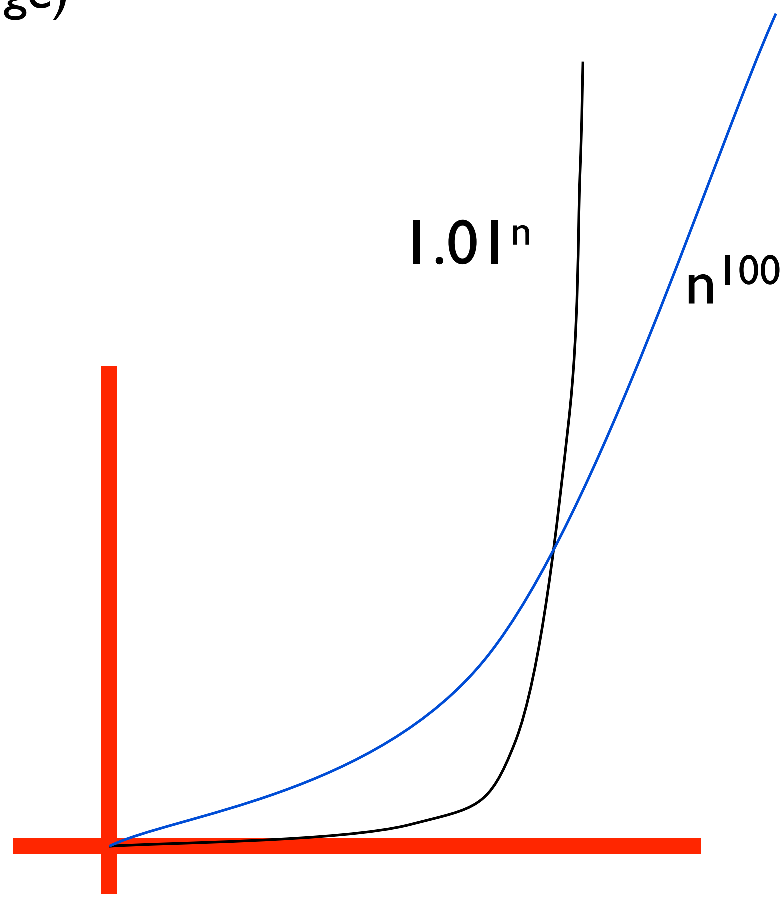
$$\lim_{n \to \infty} \frac{n^2}{n^3} = \lim_{n \to \infty} \frac{1}{n} = 0$$

$n^3 = o(e^n)$ [Use L'Hospital's rule 3 times]:

$$\lim_{n \to \infty} \frac{n^3}{e^n} = \lim_{n \to \infty} \frac{3n^2}{e^n} = \lim_{n \to \infty} \frac{6n}{e^n} = \lim_{n \to \infty} \frac{6}{e^n} = 0$$

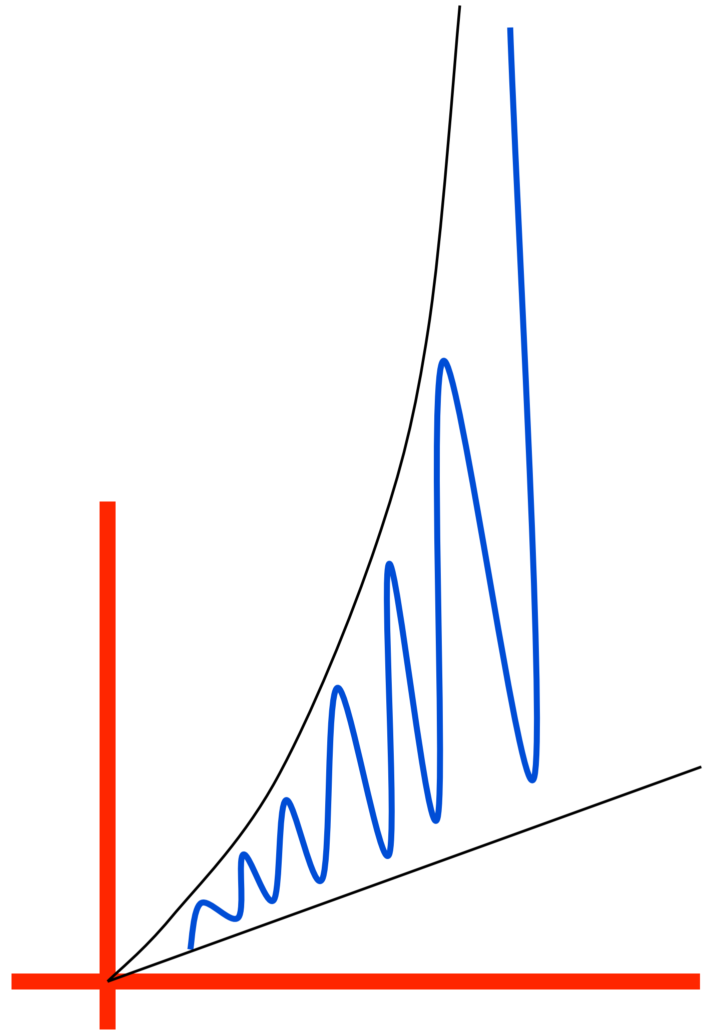
For all $r > 1$ (no matter how small)
and all $d > 0$, (no matter how large)
$n^d = O(r^n)$

$n^d = o(r^n)$, even



In short, every exponential grows faster than every polynomial!

$$f(n) = \begin{Bmatrix} n^2, & n\ even \\ n, & n\ odd \end{Bmatrix}$$



$f(n) \neq \Theta(n^a)$ for any a.

Fortunately, such nasty cases are rare

$f(n \log n) \neq \Theta(n^a)$ for any *a, either, but at least it's simpler*.

P: Running time $O(n^d)$ for some constant d

(d is independent of the input size n)

Nice scaling property: there is a constant c s.t. *doubling* n, time increases only by a factor of c.

(E.g., $c \sim 2^d$)

Contrast with exponential: For any constant c, there is a d such that $n \rightarrow n+d$ increases time by a factor of more than c.

(E.g., c = 100 and d = 7 for 2^n vs 2^{n+7})

Typical initial goal for algorithm analysis is to find an

- asymptotic
- upper bound on
- worst case running time
- as a function of problem size

This is rarely the last word, but often helps separate good algorithms from blatantly poor ones - concentrate on the good ones!