Name: ______________________________

1	/ 9
2	/10
3	/16
4	/39
total	/74

CSE 521

Final Exam

March 15, 2004

Instructions:

- You have 1 hour and 50 minutes to complete the exam.
- The exam is closed book, closed notes.
- Please do not turn the page until you are instructed to do so.
- Feel free to ask me to clarify anything.
- Good luck!

1. (9 points) In all of the following, you are given an s-t flow network G, where $c(u, v)$ is the (integer) capacity of edge (u, v), and f is an integer flow in G (not necessarily maximum, unless explicitly stated.).

   - (2 points) *True or False:* Let (A', B') be an s-t cut. If $\nu(f) = \sum_{u \in A', v \in B'} c(u, v)$, then (A', B') is a min cut.
   - (2 points) *True or False:* Let f be a maximum flow in G. Let (A'', B'') be an s-t cut (not necessarily minimum). Then $\sum_{u \in B'', v \in A''} f(u, v) = 0$.
   - (5 points) Suppose that (A, B) is a min s-t cut with k edges from A to B and ℓ edges from B to A. Suppose that in the residual graph with respect to the flow f, all edges have residual capacity at most r. Give the best upper bound you can on the number of augmentations that would be needed to arrive at a maximum flow (starting from the flow f).

2. (10 points) Consider the following scheduling problem:

   Input: A set of n jobs, such that job i has weight w_i and processing time p_i.

   Output: A schedule (an order in which to process the jobs) that minimizes $\sum_j w_j C_j$, where C_j is the completion time of job j (i.e., $\sum_{i \in S_j} p_i$, where S_j is the set of jobs that are scheduled before job j, including job j itself.)

   Consider the greedy algorithm that schedules the jobs in nonincreasing order of $\frac{w_j}{p_j}$.

   Give a *brief* exchange argument to prove that this algorithm is optimal.

3. (16 points)

- (10 points) Consider the linear partition problem:

  Input: A sequence S of n nonnegative integers $\{s_1, \ldots, s_n\}$ and an integer k.

  Output: A partition of S into k ranges, so as to minimize the maximum sum over ranges.

  Give the recurrence you would use find the optimal value of this quantity. Do not explain your recurrence, however, make sure that all quantities that appear in your recurrence are clearly defined. Be sure to include all base cases as well.

  (For example, if the input sequence is

  $$\{100, 200, 300, 400, 500, 600, 700, 800, 900\},$$

  then the partition into 3 parts

  $$\{100, 200, 300 \mid 400, 500, 600 \mid 700, 800, 900\},$$

  results in the last part having a sum of 2400. For this input, the optimal partition is

  $$\{100, 200, 300, 400, 500 \mid 600, 700 \mid 800, 900\},$$

  which gives a value of 1700 for the objective function – the maximum sum of elements in any part is for the last part with a sum of 1700.)

- (6 points) What would be the running time of the corresponding dynamic programming algorithm as a function of k and n? (You do not need to give the algorithm.)

4. (39 points) Consider the problem of scheduling on unrelated parallel machines:

   Input: A set J of jobs, a set M of machines, and for each $j \in J$ and $i \in M$, an integer p_{ij}, the number of time units required to execute job j on machine i.

   Output: An assignment of the jobs to the machines so as to minimize the *makespan*, i.e., the maximum processing time on any machine.

   - (10 points) Give an integer linear programming formulation of this scheduling problem. Your program should have a variable x_{ij} for each $j \in J$ and $i \in M$ which is 1 if job j is assigned to machine i and 0 otherwise, and a variable T for the makespan.

- (3 points) What changes would you make to get the linear programming relaxation of this program?

- (3 points) What is the integrality gap for this program? (Hint: consider the special case where there is only one job with processing time M.)

- (4 points) Give the values of the variables at some corner of the polytope defining the feasible region for the linear programming relaxation.

- (5 points) Suppose you tried randomly rounding the solution of the linear programming relaxation, by treating the values x^*_{ij} of the optimal fractional variables as a probability distribution that determines which machine a given job gets assigned to, i.e., assign job j to machine i with probability x^*_{ij}. This gives a solution Let X_i be the random variable whose value is the completion time of all the jobs assigned to machine i. What is the expected value of X_i in terms of the optimal solution to the linear programming relaxation?

- (10 points) What is the dual of the linear programming relaxation?

- (4 points) Let Z_d be value of the objective function of the dual for some feasible solution to the dual, let Z^*_{p-rel} be the value of the optimal solution to the primal (the linear programming relaxation), and let Z^*_{p-int} be the value of the optimal solution to the integer linear programming formulation for this scheduling problem. How are these three values Z_d, Z^*_{p-int}, and Z^*_{p-rel} ordered relative to one another?