CSE 521
Algorithms
Spring 2003

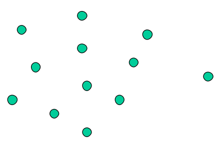
Computational Geometry
Convex Hull
Line Segment Intersection
Voronoi Diagram

Geometric Algorithms

- Algorithms about points, lines, planes, polygons, triangles, rectangles and other geometric objects.
- Applications in many fields
  - robotics, graphics, CAD/CAM, geographic systems

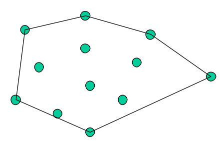
Convex Hull in 2-dimension

- Given n points on the plane find the smallest enclosing curve.



Convex Hull in 2-dimension

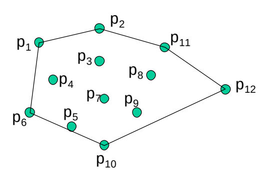
- The convex hull is a polygon whose vertices are some of the points.



Definition of Convex Hull Problem

- Input:
  Set of points $p_1, p_2, \ldots, p_n$ in 2 space. (Each point is an ordered pair $p = (x,y)$ of reals.)
- Output:
  A sequence of points $p_{i1}, p_{i2}, \ldots, p_{ik}$ such that traversing these points in order gives the convex hull.

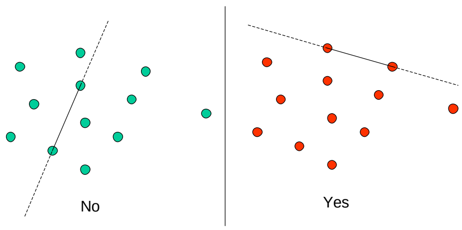
Example



Input: $p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_8, p_9, p_{10}, p_{11}, p_{12}$
Output: $p_6, p_1, p_2, p_{11}, p_{12}, p_{10}$

Slow Convex Hull Algorithm

- For each pair of points p, q determine if the line from p to q is on the convex hull.



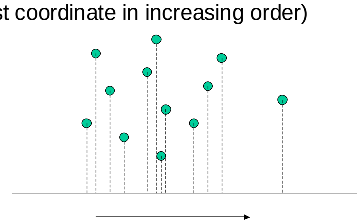
Slow Convex Hull Algorithm

- For each pair of points p, q, form the line that passes through p and q and determine if all the other points are on one side of the line.
  - If so the line from p to q is on the convex hull
  - Otherwise not
- Time Complexity is $O(n^3)$
  - Constant time to test if point is on one side of the line from (p_1,p_2) to (q_1,q_2).

$$0 = (q_2 - p_2)x + (p_1 - q_1)y + p_2q_1 - p_1q_2$$

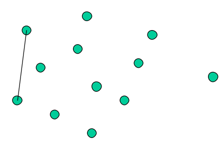
Graham's Scan Convex Hull Algorithm

- Sort the points from left to right (sort on the first coordinate in increasing order)



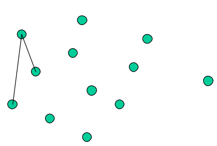
Convex Hull Algorithm

- Process the points in left to right order



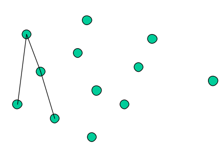
Convex Hull Algorithm

- Right Turn



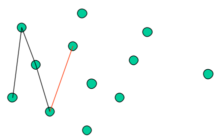
Convex Hull Algorithm

- Right Turn



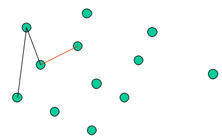
Convex Hull Algorithm

- Left Turn – back up

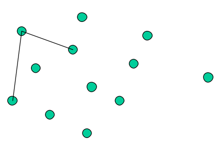


Convex Hull Algorithm

- Left Turn – back up

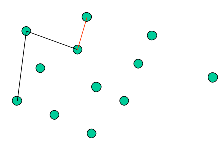


Convex Hull Algorithm

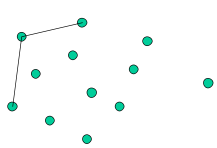


Convex Hull Algorithm

- Left Turn – back up

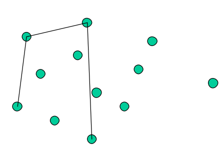


Convex Hull Algorithm



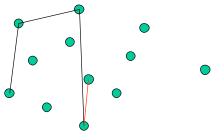
Convex Hull Algorithm

- Right Turn

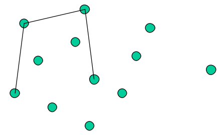


Convex Hull Algorithm

- Left Turn – back up

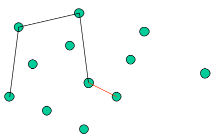


Convex Hull Algorithm

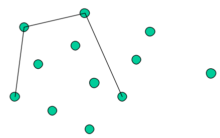


Convex Hull Algorithm

- Left Turn – back up

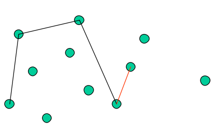


Convex Hull Algorithm

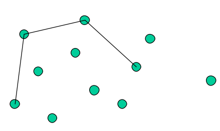


Convex Hull Algorithm

- Left Turn – back up

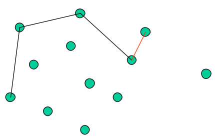


Convex Hull Algorithm

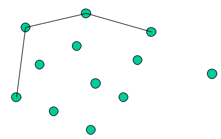


Convex Hull Algorithm

- Left Turn – back up

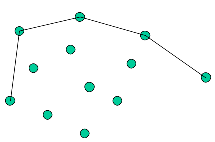


Convex Hull Algorithm



Convex Hull Algorithm

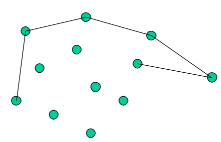
- Upper convex hull is complete



Continue the process in reverse order to get the lower convex hull

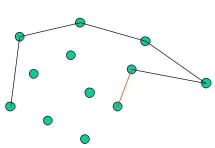
Convex Hull Algorithm

- Right Turn

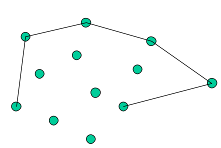


Convex Hull Algorithm

- Left Turn – back up

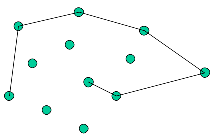


Convex Hull Algorithm



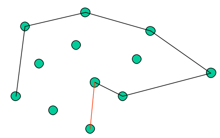
Convex Hull Algorithm

- Right Turn



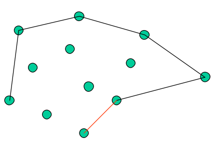
Convex Hull Algorithm

- Left Turn – back up

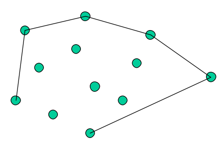


Convex Hull Algorithm

- Left Turn – back up

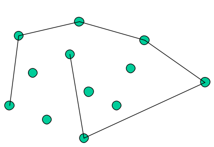


Convex Hull Algorithm



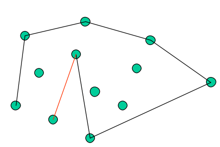
Convex Hull Algorithm

- Right Turn

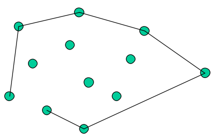


Convex Hull Algorithm

- Left Turn – back up

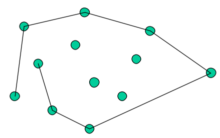


Convex Hull Algorithm



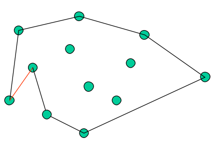
Convex Hull Algorithm

- Right Turn



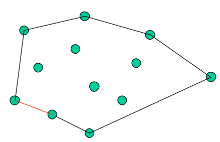
Convex Hull Algorithm

- Left Turn – back up



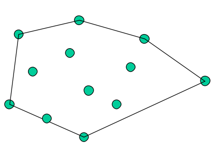
Convex Hull Algorithm

- Left Turn – back up



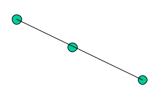
Convex Hull Algorithm

- Done!



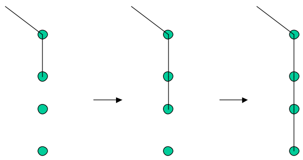
Co-linear Points

- Not a left turn
  - Middle point is included in the convex hull



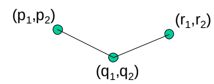
Vertical Points

- Sort
  - First increasing in x
  - Second decreasing in y



Testing For Left Turn

- Slope increases from one segment to next



left turn $\dfrac{q_2 - p_2}{q_1 - p_1} < \dfrac{r_2 - q_2}{r_1 - q_1}$

$(q_2 - p_2)(r_1 - q_1) < (r_2 - q_2)(q_1 - p_1)$ to avoid dividing by zero

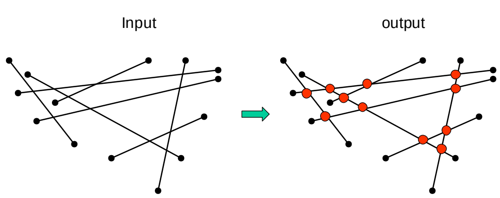
Time Complexity of Graham's Scan

- Sorting – O(n log n)
- During the scan each point is "visited" at most twice
  - Initial visit
  - back up visit (happens at most once)
- Scan - O(n)
- Total time O(n log n)
- This is best possible because sorting is reducible to finding convex hull.

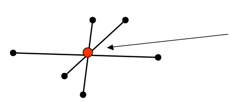
Notes on Convex Hull

- O(n log n)
  - Graham (1972)
- O(n h) algorithm where h is the size of hull
  - Jarvis' March, "Gift wrapping" (1973)
  - Output sensitive algorithm
- O(n log h) algorithm where h is size of hull
  - Kirkpatrick and Seidel (1986)
- d-dimensional Convex Hull
  - $\Omega(n^{d/2})$ in the worst case because the output can be this large.

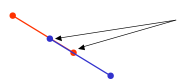
Line Segment Intersection Problem



Special cases



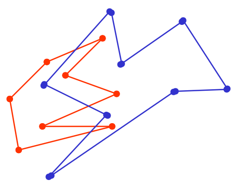
report the point and all the lines that meet there.



Report the segment and all the lines that meet on it.

Polygon Intersection

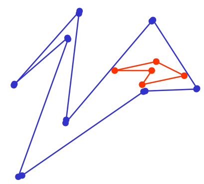
- Polygons have no self intersections



Use line segment intersection to solve polygon intersection

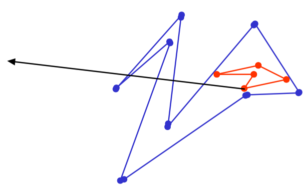
Polygon Intersection

- What if no line segment intersections?



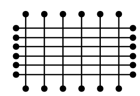
Polygon Intersection

- Intersect a ray from each polygon with the other
  - Inside, if ray has an odd number of intersections, otherwise outside. Jordan curve theorem (1887).



Issues

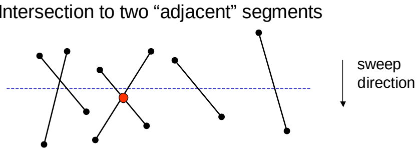
- With n line segments there may be $O(n^2)$ intersections.



- Goal: Good output sensitive algorithm
  - $O(n \log n + s)$ would be ideal where s is the number of intersections.

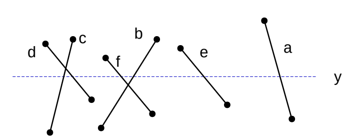
Plane Sweep Algorithm

- Sweep a plane vertically from top to bottom maintaining the set of known future events.
- Events
  - Beginning of a segment
  - End of a segment
  - Intersection to two "adjacent" segments



Segment List

- We maintain ordered list of segments



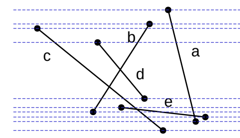
segment ordering at y = c, d, f, b, e, a

Key Idea in the Algorithm

- Just before an intersection event the two line segments must be adjacent in the segment order.
- When a new adjacency occurs between two lines we must check for a possible new intersection event.

Initialization

- Event Queue
  – contains all the beginning points and all the end points of segments ordered by decreasing y value.



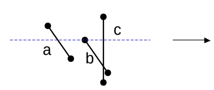
Event Queue
$b_a, b_b, b_c, b_d, e_d, b_e, e_b, e_e, e_a, e_c$

- Segment List
  – Empty

Algorithm

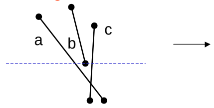
- Remove the next event from the event queue

begin segment event



1. Insert b into the segment list between a and c
2. Check for intersections with adjacent segments (a,b) and (b,c), and add any to event queue

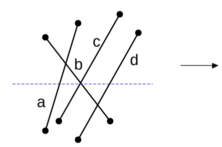
end segment event



1. Delete b from the segment list
2. Check for intersections with adjacent segments (a,c), and add any to event queue

Algorithm

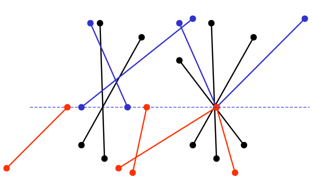
intersection event



1. Reverse the order of b and c on the segment list
2. Check for intersections with adjacent segments (a,c) and (b,d) and add any to event queue

Complications

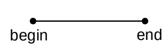
- Several events can coincide.



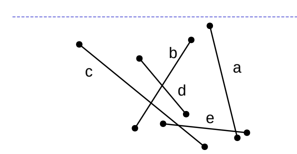
- begin segment event
- end segment event
- intersection event

do in left to right order

- Horizontal lines



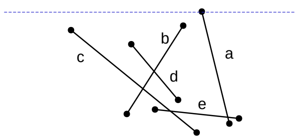
Example



Segment List

Event Queue
b_a, b_b, b_c, b_d, e_d, b_e, e_b, e_e, e_a, e_c

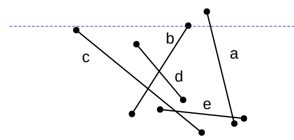
Example



Segment List
a

Event Queue
b_b, b_c, b_d, e_d, b_e, e_b, e_e, e_a, e_c

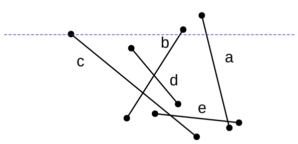
Example



Segment List
b, a

Event Queue
b_c, b_d, e_d, b_e, e_b, e_e, e_a, e_c

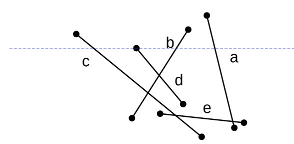
Example



Segment List
c, b, a

Event Queue
b_d, $i_{(c,b)}$, e_d, b_e, e_b, e_e, e_a, e_c

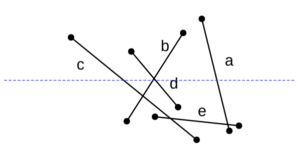
Example



Segment List
c, d, b, a

Event Queue
$i_{(d,b)}$, $i_{(c,b)}$, e_d, b_e, e_b, e_e, e_a, e_c

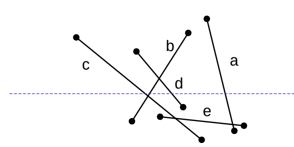
Example



Segment List
c, b, d, a

Event Queue
$i_{(c,b)}$, e_d, b_e, e_b, e_e, e_a, e_c

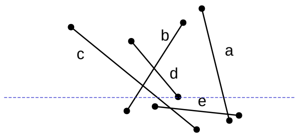
Example



Segment List
b, c, d, a

Event Queue
e_d, b_e, e_b, e_e, e_a, e_c

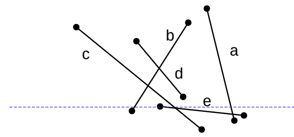
Example



Segment List
b, c, a

Event Queue
b_e, e_b, e_e, e_a, e_c

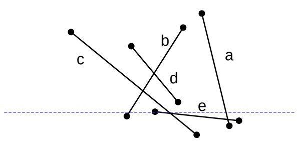
Example



Segment List
b, e, c, a

Event Queue
$i_{(e,c)}, e_b, e_e, e_a, e_c$

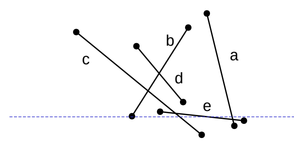
Example



Segment List
b, c, e, a

Event Queue
$e_b, i_{(e,a)} e_e, e_a, e_c$

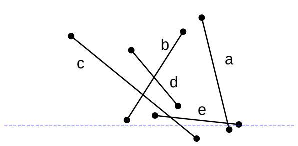
Example



Segment List
c, e, a

Event Queue
$i_{(e,a)}, e_e, e_a, e_c$

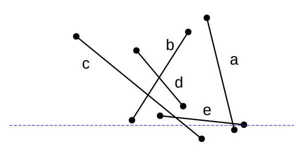
Example



Segment List
c, a, e

Event Queue
e_e, e_a, e_c

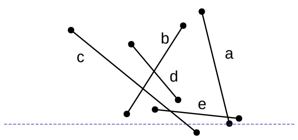
Example



Segment List
c, a

Event Queue
e_a, e_c

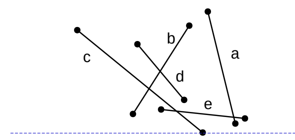
Example



Segment List
c

Event Queue
e_c

Example



Segment List

Event Queue

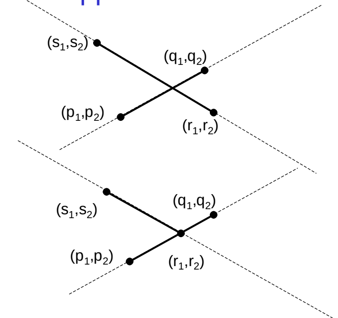
Data Structures

- Event List
  - Priority queue ordered by decreasing y, then by increasing x
  - Delete minimum, Insertion
- Segment List
  - Balanced binary tree search tree
  - Insertion, Deletion
  - Reversal can be done by deletions and insertions
- Time per event is O(log n)

Finding Line Segment Intersections

- Given line segments $(p_1,p_2),(q_1,q_2)$ and $(r_1,r_2),(s_1,s_2)$ do they intersect, and if so where.
- Where? Solve
  - $0 = (q_2 – p_2)x + (p_1 – q_1)y + p_2q_1 – p_1q_2$
  - $0 = (s_2 – r_2)x + (r_1 – s_1)y + r_2s_1 – r_1s_2$
- If?
  - (p_1,p_2) and (q_1,q_2) on opposite sides of line $(r_1,r_2),(s_1,s_2)$ and
  - (r_1,r_2) and (s_1,s_2) on opposite sides of line $(p_1,p_2),(q_1,q_2)$

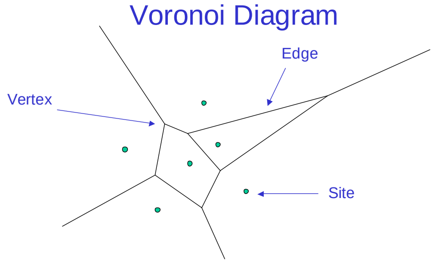
Opposite Sides



Notes on Line Segment Intersection

- Total time for plane sweep algorithm is O(n log n + s log n) where s is the number of intersections.
  - n log n for the initial sorting
  - log n per event
- Plane sweep algorithms were pioneered by Shamos and Hoey (1975).
- Intersection Reporting - Bentley and Ottmann (1979)

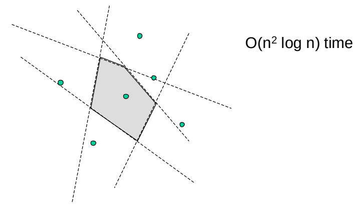
Voronoi Diagram



Each site defines an area of points nearest to it.
Boundaries are perpendicular bisectors.
http://www.cs.cornell.edu/Info/People/chew/Delaunay.

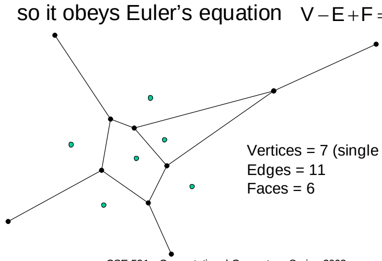
Brute Force

- Each Voronoi area is the intersection of half spaces defined by perpendicular bisectors.



Linear Size of Voronoi Diagram

- The Voronoi Diagram is a planar embedding so it obeys Euler's equation $V - E + F = 2$



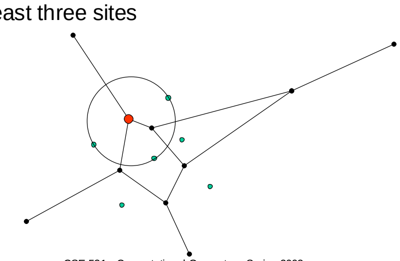
Vertices = 7 (single vertex at infinity)
Edges = 11
Faces = 6

Linear Size of Voronoi Diagram

- F = E - V + 2 (Euler's equation)
- n = F (one site per face)
- 2E $\geq$ 3V because each vertex is of degree at least 3 and each edge has 2 vertices.
- n $\geq$ 3V/2 – V + 2 = V/2 + 2
- 2n – 2 $\geq$ V
- n > E – (2n – 2) + 2
- 3n – 4 $\geq$ E

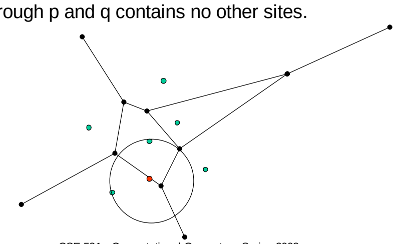
Properties Voronoi Diagram

1. A vertex is the center of a circle through at least three sites



Properties Voronoi Diagram

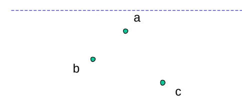
2. A point on a perpendicular bisector of sites p and q is on an edge if the circle centered at the point through p and q contains no other sites.



Fortune's Sweep

- We maintain a "beach line," a sequence of parabolic segments that is the set of point equidistant from a site and the sweep line.
- Events
  - Site event – new site is encountered by the sweep line
  - Circle event – new vertex is inserted into the Voronoi diagram

Example

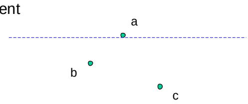


beach line

event queue
a, b, c

Example

site point event

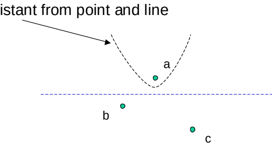


beach line
a

event queue
b, c

Example

points equidistant from point and line

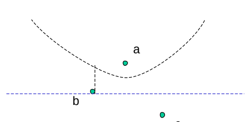


beach line
a

event queue
b, c

Example

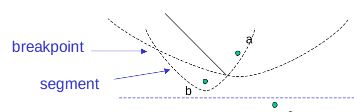
site event



beach line
a, b, a

event queue
c

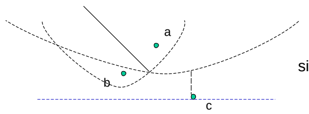
Example



beach line
a, b, a

event queue
c

Example

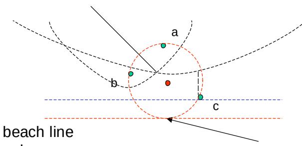


site event

beach line
a, b, a, c, a

event queue
?

Example

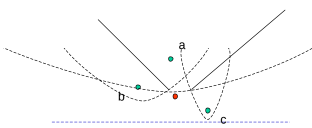


beach line
a, b, a, c, a

circle event must be added to the event queue

event queue
$c_{(b,a,c)}$

Example

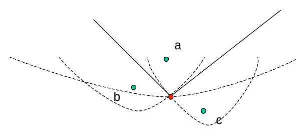


beach line
a, b, a, c, a

event queue
$c_{(b,a,c)}$

Example

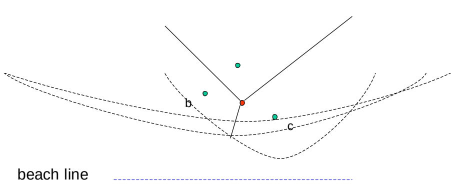
circle event



beach line
a, b, c, a

event queue

Example

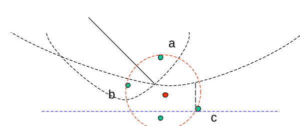


beach line
a, b, c, a

event queue

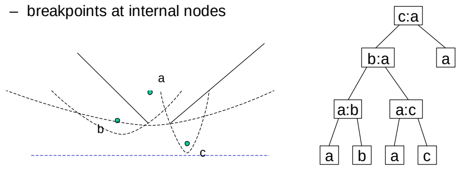
Event Queue

- Contains site events and circle events sorted by y in decreasing order, then by x in increasing order
- Circle events can be both inserted and deleted.



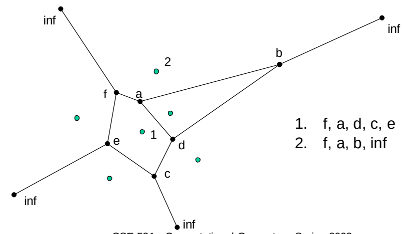
Beach Line

- Implemented as a balanced binary search tree.
  - sites at leaves
  - breakpoints at internal nodes



Output

- For each site output the vertices in clockwise order. When a circle event occurs add to the vertex list of the three (or more) sites.



1. f, a, d, c, e
2. f, a, b, inf

Complexity

- Number of segments in the beach line $\leq$ 2n
  - Each site event adds at most 2 segments.
- Number of circle event insertions $\leq$ 2n
  - Each site event creates at most 2 circle events.
- Time per event is O(log n)
  - Insert new segments into the segment tree.
  - Insert new circle events into the event queue
  - Delete circle events from the event queue
- Total time is O(n log n)

Voronoi Diagram Notes

- Voronoi diagram
  - Dirichlet (1850), Voronoi (1907)
- O(n log n) algorithm
  - Divide and conquer - Shamos and Hoey (1975)
  - Plane sweep – Fortune (1987)

Exercise

- Give an O(n log n) algorithms which given a set of n points on the plane, for each point finds its nearest neighbor.