CSE 521
Assignment 8
Due Tuesday, May 27, 2003

1. We consider the famous bin-packing problem defined as follows. We are given as input numbers $s_1, s_2, \ldots, s_n$ with $0 < s_i < 1$. The output is a partition P of $\{1, 2, \ldots, n\}$ such that for all $X \in P$, $\sum_{i \in X} s_i \leq 1$ and the cardinality of P is minimized. Equivalently, we put the numbers s_i into bins of size 1 and minimize the number of bins. Solving this problem is NP-hard. A simple heuristic is called *first-fit* which considers $s_1, s_2, \ldots, s_i, \ldots$ in order and places s_i in the first bin that it will fit. If none fit then a new bin is initiated for s_i. Let $S = \sum_{i=1}^{n} s_i$.

   (a) Show that an optimal solutions uses at least $\lceil S \rceil$ bins.

   (b) Argue that the first-fit heuristic leaves at most one bin half-full.

   (c) Show that the number of bins used by the first-fit heuristic is never more than $\lceil 2S \rceil$.

   (d) Show that the approximation ratio of the first-fit heuristic is bounded above by 2.

   (e) Show that the approximation ratio of the first-fit heuristic is bounded below by 3/2. This can be done by a simple example with four items of three different sizes.

   (f) (extra credit) Show that the approximation ratio of the first-fit heuristic is bounded below by 5/3. This is a more complicated example, but only requires three different sizes.

2. In the SET-COVER problem we are given a set U of n members and a collection $S_1, \ldots, S_m$ of subsets of U. The goal is to find a minimum-size set-cover, that is, a minimum-size collection of input subsets, whose union is U. Example: $U = \{1, 2, 3, 4, 5, 6, 7\}$, $S_1 = \{1\}, S_2 = \{7\}, S_3 = \{2, 3, 4, 5, 6\}, S_4 = \{1, 4, 5, 7\}$. Then $\{S_3, S_4\}$ is an optimal solution of size 2. The collection $\{S_1, S_2, S_3\}$ is also a cover but its size is 3 so it is not optimal.

   (a) Describe SET-COVER as an integer programming problem.

   (b) Relax the integer constraints to get a linear programming problem.

   (c) For $e \in U$, let $r(e)$ be the number of subsets that include e. In the above example $r(2) = 1$, $r(4) = 2$. Let $r = \max\{r(e) : e \in E\}$. Given a solution to the relaxed LP, take as a potential set-cover the subsets whose corresponding variables have value at least $1/r$. Show that the selected collection of subsets is a valid set-cover and that this is an r-approximation.