CSE 521
Assignment 4
Due Tuesday, April 29, 2003

1. Solve the following problem using the two phase Simplex Algorithm. The first phase finds a feasible slack form, and the second phase finds the optimum. Maximize $3x_1 + x_2$ subject to

$$\begin{aligned} x_1 - x_2 &\leq -1 \\ -x_1 - x_2 &\leq -3 \\ 2x_1 + x_2 &\leq 4 \end{aligned}$$

   and $x_1, x_2 \geq 0$.

2. Consider the min cost multicommodity flow problem defined as follows: $G = (V, E)$ is a graph with designated vertices $s_1, \ldots, s_k$, and $t_1, \ldots, t_k$, capacities $c(u, v) \geq 0$ for $(u, v) \in E$, costs $p_i(u, v) \geq 0$ for $(u, v) \in E$, and $d_1, \ldots, d_k$ for each of the commodities. The capacity $c(u, v)$ is an upper bound on the total flow for all commodities that flow on the edge (u, v). The cost $p_i(u, v)$ is the cost per flow unit of commodity i on the edge (u, v). The demand d_i is the demand by vertex t_i for commodity i. We assume source s_i has an unbounded supply of commodity i.

   (a) Express the problem of finding a minimum cost multicommodity flow as a linear program with $O(|E|)$ constraints. The cost of a flow is the sum over all $(u, v) \in E$ and i of $p_i(u, v)f_i(u, v)$ where $f_i(u, v)$ is the flow of commodity i on the edge (u, v). (Some new constrains will have to replace the skew symmetry constraints because skew symmetry requires $O(|V|^2)$ constraints.)

   (b) Describe how the Simplex Algorithm can be used to solve the problem of minimum cost flow, including the possibility that there is no flow with the desired demand.

3. In this problem we will explore the *currency arbitrage problem*, which is the problem of determining if there is some way to exchange money between countries so as to make a profit. Suppose we have n countries indexed 1,2,...,n. Each country i has its own currency which can be exchanged for currency of country j. When exchanging x units of i dollars you get $a_{ij}x$ dollars of country j currency. Currency exchange normally has the property that $a_{ji}a_{ij} < 1$, that is, exchanging currency from country i to j, then back to i again, results in a loss. However, there may be a sequence of currency exchanges that results in a profit.

   (a) Use linear programming to determine if there is a potential profit in country 1 dollars that one can achieve by a sequence of currency exchanges starting with 1 dollar of country 1 currency. Think of this as a kind of flow problem where the flow starts in country 1, ends in country 1, and no more than 1 dollar in country 1 currency flows out of country 1.

(b) Distiguish the situations, using linear programming concepts, where there is no possibility of profit, bounded profit, and unbounded profit starting with 1 dollar of country 1 currency.

4. In this problem you will prove the *complementary slackness condition*. Let $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ be a feasible solution to the standard linear program

$$\begin{array}{rl} \text{maximize} & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \text{ for } 1 \leq i \leq m \\ & x_j \geq 0 \text{ for } 1 \leq j \leq n \end{array}$$

and let $\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_m$ be a feasible solution to its dual. Prove that $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ and $\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_m$ are optimal for their respective problems if and only if both

$$\begin{array}{l} \sum_{j=1}^{n} a_{ij} \bar{x}_j = b_i \text{ or } \bar{y}_i = 0,\ 1 \leq i \leq m \text{ and} \\ \sum_{i=1}^{m} a_{ij} \bar{y}_i = c_j \text{ or } \bar{x}_j = 0,\ 1 \leq j \leq n. \end{array}$$

You may assume weak duality, $\sum_{j=1}^{n} c_j \bar{x}_j \leq \sum_{i=1}^{m} b_i \bar{y}_i$ for any feasible $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ and $\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_m$.