Parametric curves

Reading

Required:

- Angel 10.1-10.3, 10.5.2, 10.6-10.7

Optional

- Bartels, Beatty, and Barsky. *An Introduction to Splines for use in Computer Graphics and Geometric Modeling*, 1987.
- Farin. *Curves and Surfaces for CAGD: A Practical Guide*, 4th ed., 1997.

Curves before computers

The "loftsman's spline":

- long, narrow strip of wood or metal
- shaped by lead weights called "ducks"
- gives curves with second-order continuity, usually

Used for designing cars, ships, airplanes, etc.

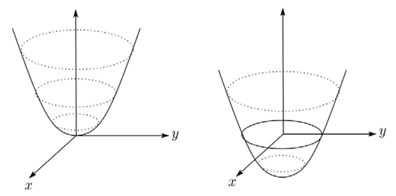
But curves based on physical artifacts can't be replicated well, since there's no exact definition of what the curve is.

Around 1960, a lot of industrial designers were working on this problem.

Today, curves are easy to manipulate on a computer and are used for CAD, art, animation, …

Mathematical curve representation

- Explicit $y=f(x)$
  - what if the curve isn't a function, e.g., a circle?
- Implicit $g(x,y) = 0$



- Parametric $(x(u),y(u))$
  - For the circle:

    $x(u) = \cos 2\pi u$

    $y(u) = \sin 2\pi u$

Parametric polynomial curves

We'll use parametric curves, *Q(u)*=(*x(u)*,*y(u)*), where the functions are all polynomials in the parameter.

$$x(u) = \sum_{k=0}^{n} a_k u^k$$

$$y(u) = \sum_{k=0}^{n} b_k u^k$$

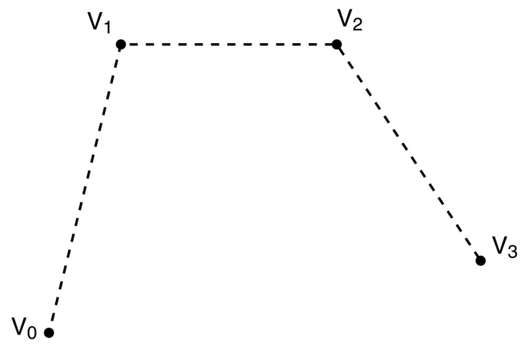
Advantages:

- easy (and efficient) to compute
- infinitely differentiable

We'll also assume that *u* varies from 0 to 1.

de Casteljau's algorithm

Recursive interpolation:

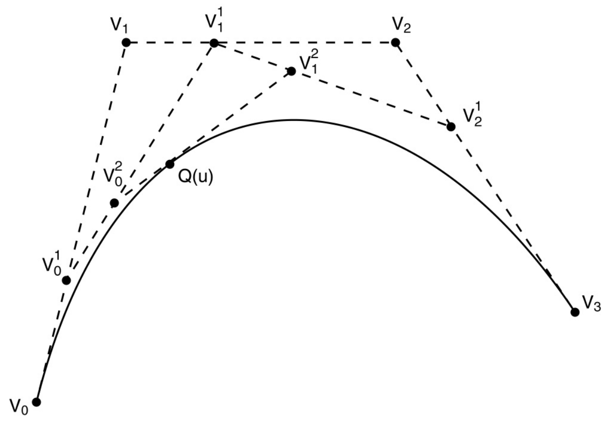


What if *u*=0?

What if *u*=1?

de Casteljau's algorithm, cont'd

Recursive notation:



What is the equation for V_0^1?

Finding Q(u)

Let's solve for **Q**(u):

$$V_0^1 = (1-u)V_0 + uV_1$$
$$V_1^1 = (1-u)V_1 + uV_2$$
$$V_2^1 = (1-u)V_2 + uV_3$$

$$V_0^2 = (1-u)V_0^1 + uV_1^1$$
$$V_1^2 = (1-u)V_1^1 + uV_2^1$$

$$\begin{aligned} Q(u) &= (1-u)V_0^2 + uV_1^2 \\ &= (1-u)[(1-u)V_0^1 + uV_1^1] + u[(1-u)V_1^1 + uV_2^1] \\ &= (1-u)[(1-u)\{(1-u)V_0 + uV_1\} + u\{(1-u)V_1 + uV_2\}] + \dots \\ &= (1-u)^3 V_0 + 3u(1-u)^2 V_1 + 3u^2(1-u)V_2 + u^3 V_3 \end{aligned}$$

Finding Q(u) (cont'd)

In general,

$$Q(u) = \sum_{i=0}^{n} \binom{n}{i} u^i (1-u)^{n-i} V_i$$

where "*n* choose *i*" is:

$$\binom{n}{i} = \frac{n!}{(n-i)!\,i!}$$

This defines a class of curves called **Bézier curves**.

What's the relationship between the number of control points and the degree of the polynomials?

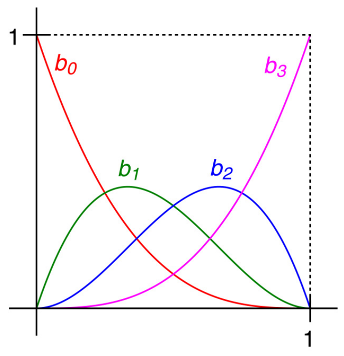
Bernstein polynomials

The coefficients of the control points are a set of functions called the **Bernstein polynomials:**

$$Q(u) = \sum_{i=0}^{n} b_i(u) V_i$$

For degree 3, we have:

$$b_0(u) = (1-u)^3$$
$$b_1(u) = 3u(1-u)^2$$
$$b_2(u) = 3u^2(1-u)$$
$$b_3(u) = u^3$$



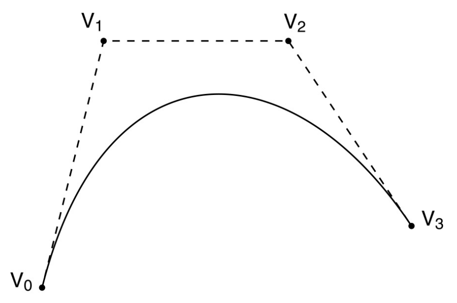
Useful properties on the interval [0,1]:

- each is between 0 and 1
- sum of all four is exactly 1 (a.k.a., a "partition of unity")

These together imply that the curve lies within the **convex hull** of its control points.

Displaying Bézier curves

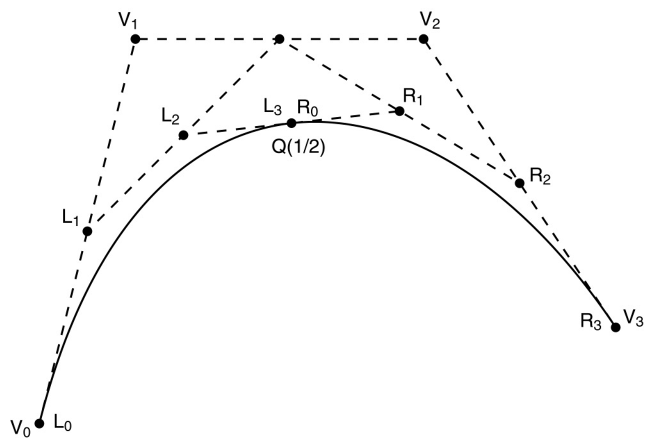
How could we draw one of these things?



It would be nice if we had an *adaptive* algorithm, that would take into account flatness.

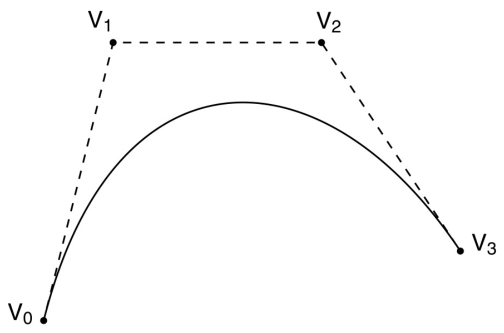
```
DisplayBezier( V0, V1, V2, V3 )
begin
    if ( FlatEnough( V0, V1, V2, V3 ) )
        Line( V0, V3 );
    else
        something;
end;
```

Subdivide and conquer



```
DisplayBezier( V0, V1, V2, V3 )
begin
    if ( FlatEnough( V0, V1, V2, V3 ) )
        Line( V0, V3 );
    else
        Subdivide(V[]) $\Rightarrow$ L[], R[]
        DisplayBezier( L0, L1, L2, L3 );
        DisplayBezier( R0, R1, R2, R3 );
end;
```

Testing for flatness



Compare total length of control polygon to length of line connecting endpoints:

$$\frac{|V_0 - V_1| + |V_1 - V_2| + |V_2 - V_3|}{|V_0 - V_3|} < 1 + \varepsilon$$

Curve desiderata

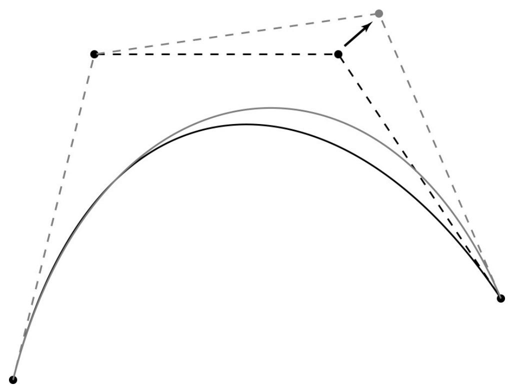
Bézier curves offer a fairly simple way to model parametric curves.

But, let's consider some general properties we would like curves to have…

Local control

One problem with Béziers is that every control point affects every point on the curve (except the endpoints).

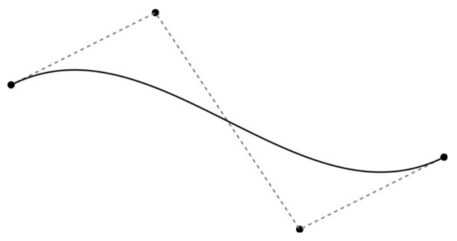
Moving a single control point affects the whole curve!



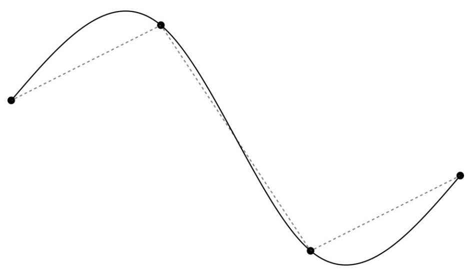
We'd like to have **local control**, that is, have each control point affect some well-defined neighborhood around that point.

Interpolation

Bézier curves are **approximating**. The curve does not (necessarily) pass through all the control points. Each point pulls the curve toward it, but other points are pulling as well.



We'd like to have a curve that is **interpolating**, that is, that always passes through every control point.

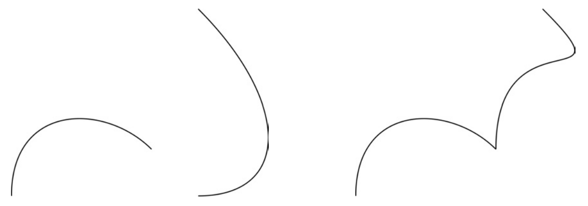


Continuity

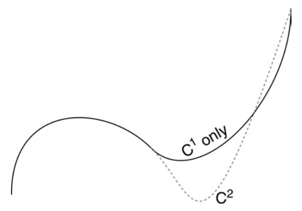
We want our curve to have **continuity**: there shouldn't be any abrupt changes as we move along the curve.

There are *nested* degrees of continuity:

C^{-1}: C^{0}:



C^{1}, C^{2}: C^{3}, C^{4}, …:



Bézier curves → splines

Bézier curves have C-infinity continuity on their interiors, but we saw that they do not exhibit local control or interpolate their control points.

It is possible to define points that we want to interpolate, and then solve for the Bézier control points that will do the job.

But, you will need as many control points as interpolated points -> high order polynomials -> wiggly curves. (And you still won't have local control.)

Instead, we'll splice together a curve from individual Béziers segments, in particular, cubic Béziers.

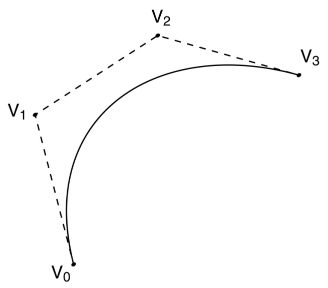
We call these curves **splines**. When splicing Béziers together, the first thing we need to worry about is continuity where one curve segment meets the next…

Ensuring C^0 continuity

Suppose we have a cubic Bézier defined by (V_0,V_1,V_2,V_3), and we want to attach another curve (W_0,W_1,W_2,W_3) to it, so that there is C^0 continuity at the joint.

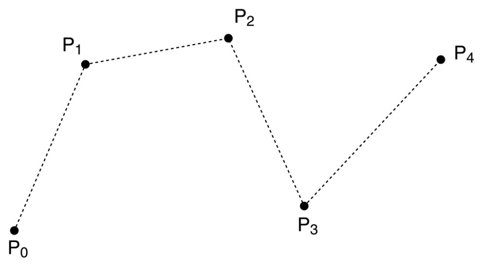
$$C^0 : Q_V(1) = Q_W(0)$$

What constraint(s) does this place on (W_0,W_1,W_2,W_3)?



The C^0 Bezier spline

How then could we construct a curve passing through a set of points $P_1 \ldots P_n$?



We call this curve a **spline**. The endpoints of the Bezier segments are called **joints**.

In the animator project, you will construct such a curve by specifying all the Bezier control points directly.

1st derivatives at the endpoints

For degree 3 (cubic) curves, we have already shown that we get:

$$Q(u) = (1-u)^3 V_0 + 3u(1-u)^2 V_1 + 3u^2(1-u)V_2 + u^3 V_3$$

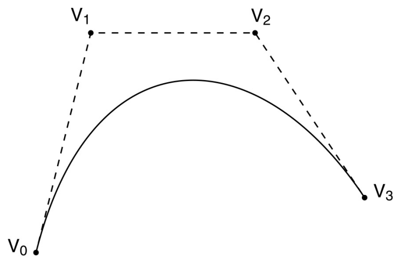
We can expand the terms in *u* and rearrange to get:

$$\begin{aligned} Q(u) = \quad & (-V_0 + 3V_1 - 3V_2 + V_3)u^3 \;+ \\ & (3V_0 - 6V_1 + 3V_2)u^2 \;+\; (-3V_0 + 3V_1)u \;+\; V_0 \end{aligned}$$

What then is the first derivative when evaluated at each endpoint, *u*=0 and *u*=1?

$Q'(0) =$

$Q'(1) =$

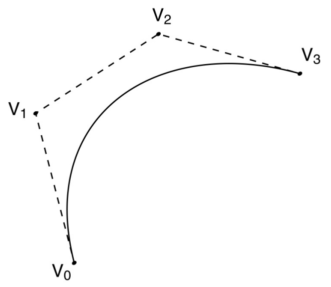


Ensuring C^1 continuity

Suppose we have a cubic Bézier defined by (V_0,V_1,V_2,V_3), and we want to attach another curve (W_0,W_1,W_2,W_3) to it, so that there is C^1 continuity at the joint.

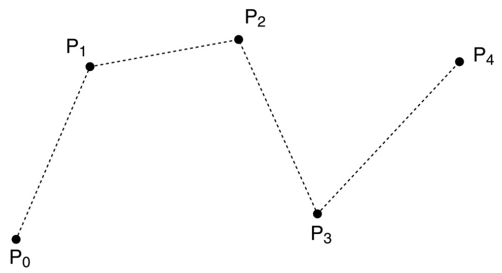
$$\begin{aligned} C^0 &: Q_V(1) = Q_W(0) \\ C^1 &: Q_V'(1) = Q_W'(0) \end{aligned}$$

What constraint(s) does this place on (W_0,W_1,W_2,W_3)?



The C^1 Bezier spline

How then could we construct a curve passing through a set of points $P_1 \ldots P_n$?



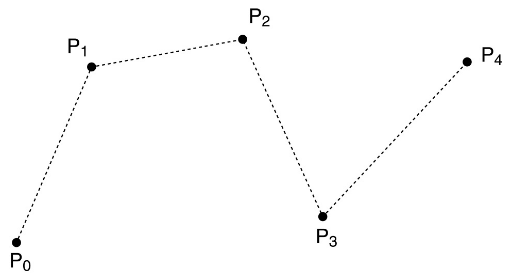
We can specify the Bezier control points directly, or we can devise a scheme for placing them automatically…

Catmull-Rom splines

If we set each derivative to be one half of the vector between the previous and next controls, we get a **Catmull-Rom spline**.

This leads to:

$$\begin{aligned} V_0 &= P_1 \\ V_1 &= P_1 + \tfrac{1}{6}(P_2 - P_0) \\ V_2 &= P_2 - \tfrac{1}{6}(P_3 - P_1) \\ V_3 &= P_2 \end{aligned}$$

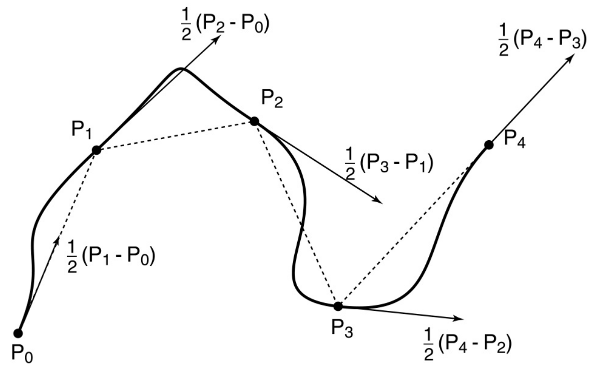


Tension control

We can give more control by exposing the derivative scale factor as a parameter:

$$
\begin{aligned}
V_0 &= P_1 \\
V_1 &= P_1 + \tfrac{\tau}{3}(P_2 - P_0) \\
V_2 &= P_2 - \tfrac{\tau}{3}(P_3 - P_1) \\
V_3 &= P_2
\end{aligned}
$$

The parameter τ controls the tension. Catmull-Rom uses $\tau = 1/2$. Here's an example with $\tau = 3/2$.

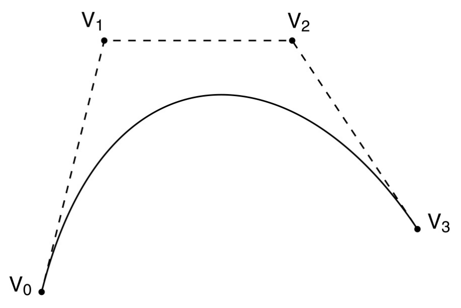


2nd derivatives at the endpoints

Finally, we'll want to develop C^2 splines. To do this, we'll need second derivatives of Bezier curves.

Taking the second derivative of $Q(u)$ yields:

$$
\begin{aligned}
Q''(0) &= 6(V_0 - 2V_1 + V_2) \\
&= -6[(V_1 - V_0) + (V_1 - V_2)] \\
Q''(1) &= 6(V_1 - 2V_2 + V_3) \\
&= -6[(V_2 - V_3) + (V_2 - V_1)]
\end{aligned}
$$

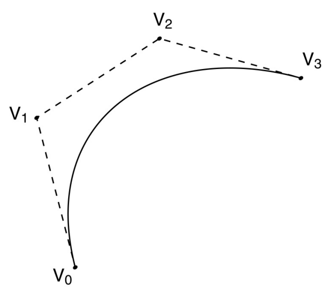


Ensuring C^2 continuity

Suppose we have a cubic Bézier defined by (V_0,V_1,V_2,V_3), and we want to attach another curve (W_0,W_1,W_2,W_3) to it, so that there is C^2 continuity at the joint.

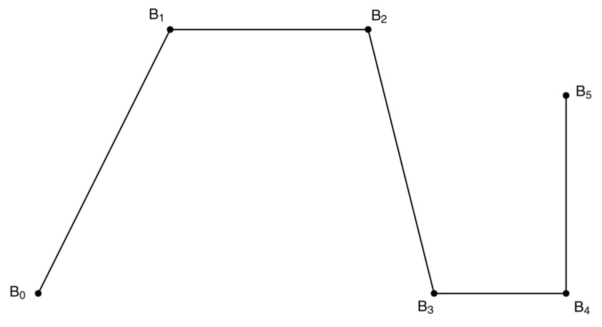
$$
\begin{aligned}
C^0 &: Q_V(1) = Q_W(0) \\
C^1 &: Q_V'(1) = Q_W'(0) \\
C^2 &: Q_V''(1) = Q_W''(0)
\end{aligned}
$$

What constraint(s) does this place on (W_0,W_1,W_2,W_3)?



Building a complex spline

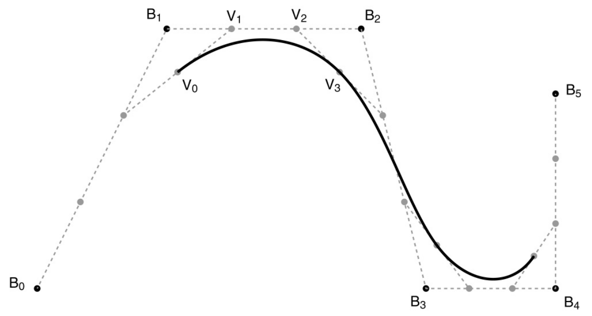
Instead of specifying the Bézier control points themselves, let's specify the corners of the A-frames in order to build a C^2 continuous spline.



These are called **B-splines**. The starting set of points are called **de Boor points**.

B-splines

Here is the completed B-spline.



What are the Bézier control points, in terms of the de Boor points?

$$
\begin{aligned}
V_0 &= \_\_\_\_[\_\_\_\_ B_0 + \_\_\_\_ B_1] \\
&\quad + \_\_\_\_[\_\_\_\_ B_1 + \_\_\_\_ B_2] \\
&= \_\_\_\_ B_0 + \_\_\_\_ B_1 + \_\_\_\_ B_2 \\
V_1 &= \_\_\_\_ B_1 + \_\_\_\_ B_2 \\
V_2 &= \_\_\_\_ B_1 + \_\_\_\_ B_2 \\
V_3 &= \_\_\_\_ B_1 + \_\_\_\_ B_2 + \_\_\_\_ B_3
\end{aligned}
$$

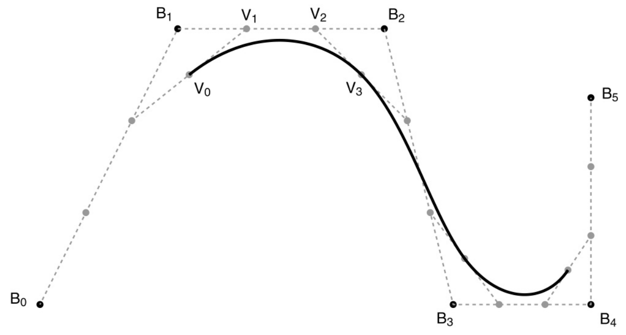
Endpoints of B-splines

We can see that B-splines don't interpolate the de Boor points.

It would be nice if we could at least control the *endpoints* of the splines explicitly.

There's a trick to make the spline begin and end at control points by repeating them.

In the example below, let's force interpolation of the last endpoint:

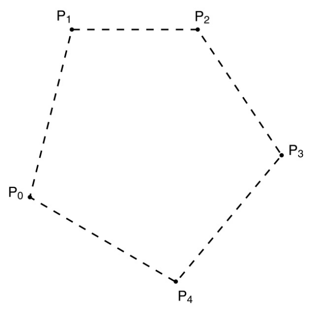


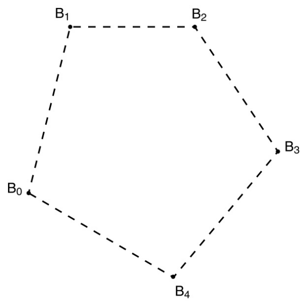
$B_0 \quad B_1 \quad B_2 \quad B_3 \quad B_4 \quad B_5$

Closing the loop

What if we want a closed curve, i.e., a loop?

With Catmull-Rom and B-spline curves, this is easy:





Curves in the animator project

In the animator project, you will draw a curve on the screen:

$$\mathbf{Q}(u) = \left(x(u), y(u)\right)$$

You will actually treat this curve as:

$$\theta(u) = y(u)$$
$$t(u) = x(u)$$

Where θ is a variable you want to animate. We can think of the result as a function:

$$\theta(t)$$

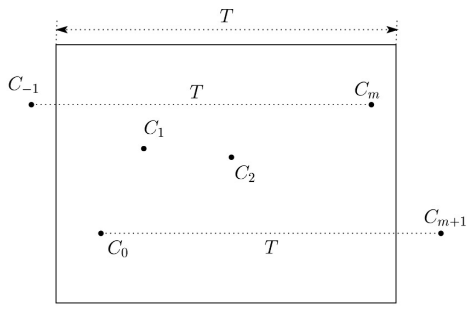
In general, you have to apply some constraints to make sure that $\theta(t)$ actually is a *function*.

"Wrapping"

One of the extra credit options in the animator project is to implement "wrapping" so that the animation restarts smoothly when looping back to the beginning.

This is a lot like making a closed curve: the calculations for the θ-coordinate are exactly the same.

The *t*-coordinate is a little trickier: you need to create "phantom" *t*-coordinates before and after the first and last coordinates.



Summary

What to take home from this lecture:

- Geometric and algebraic definitions of Bézier curves.
- Basic properties of Bézier curves.
- How to display Bézier curves with line segments.
- Meanings of C^k continuities.
- Geometric conditions for continuity of cubic splines.
- Properties of B-splines and Catmull-Rom splines.
- Geometric construction of B-splines and Catmull-Rom splines.
- How to construct closed loop splines.