Affine transformations

Reading

Required:

- Angel 4.6, 4.7.1-4.7.4, 4.8-4.8.3, 4.9

Further reading:

- Angel, the rest of Chapter 4
- Foley, et al, Chapter 5.1-5.5.
- David F. Rogers and J. Alan Adams, *Mathematical Elements for Computer Graphics*, 2nd Ed., McGraw-Hill, New York, 1990, Chapter 2.

Geometric transformations

Geometric transformations will map points in one space to points in another: $(x',y',z') = \boldsymbol{f}(x,y,z)$.

These transformations can be very simple, such as scaling each coordinate, or complex, such as non-linear twists and bends.

We'll focus on transformations that can be represented easily with matrix operations.

We'll start in 2D...

Representation

We can represent a **point**, **p** = (x,y), in the plane

- as a column vector $\begin{bmatrix} x \\ y \end{bmatrix}$
- as a row vector $\begin{bmatrix} x & y \end{bmatrix}$

Representation, cont.

We can represent a **2-D transformation** *M* by a matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If **p** is a column vector, *M* goes on the left:

$$\mathbf{p}' = M\mathbf{p}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

If **p** is a row vector, M^T goes on the right:

$$\mathbf{p}' = \mathbf{p}M^T$$

$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

We will use **column vectors**.

Two-dimensional transformations

Here's all you get with a 2 x 2 transformation matrix:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

So:

$$x' = ax + by$$
$$y' = cx + dy$$

We will develop some intimacy with the elements *a, b, c, d…*

Identity

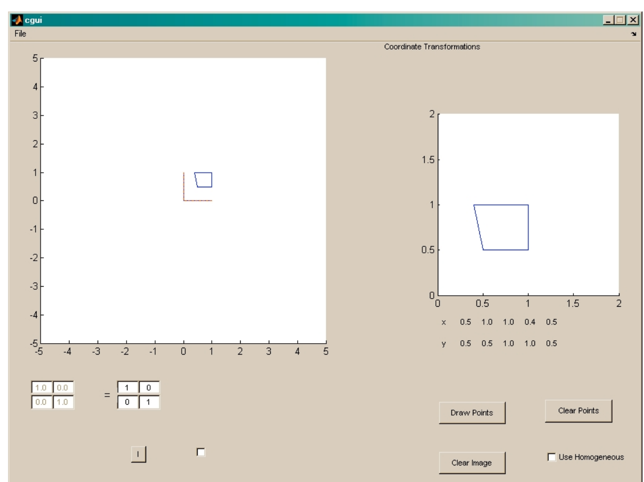
Suppose we choose *a=d=1, b=c=0:*

- Gives the **identity** matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Doesn't move the points at all

Coordinate Transformation



Scaling

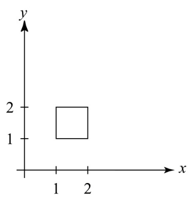
Suppose we set $b=c=0$, but let a and d take on any *positive* value:

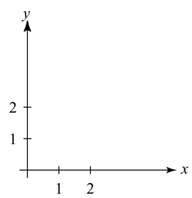
- Gives a **scaling** matrix:

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

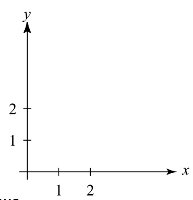
- Provides **uniform scaling** or **differential (non-uniform) scaling** in x and y:

$$x' = ax$$
$$y' = dy$$





$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

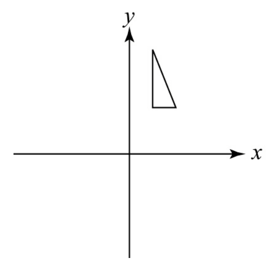


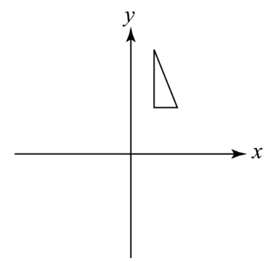
$$\begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}$$

Suppose we keep $b=c=0$, but let either a or d go negative.

Examples:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$





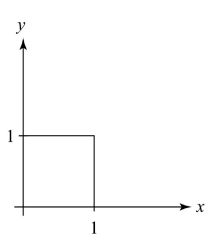
Now let's leave $a=d=1$ and experiment with b. . . .

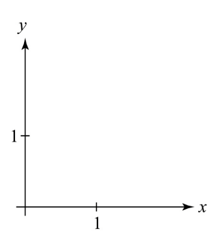
The matrix

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

gives:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{aligned} x' &= x + by \\ y' &= y \end{aligned}$$





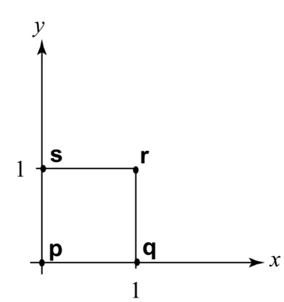
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

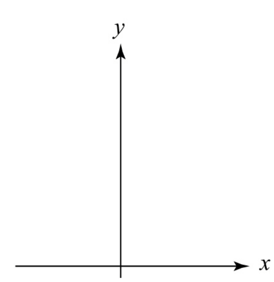
Effect on unit square

Let's see how a general 2 x 2 transformation M affects the unit square:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} [\mathbf{p} \quad \mathbf{q} \quad \mathbf{r} \quad \mathbf{s}] = [\mathbf{p'} \quad \mathbf{q'} \quad \mathbf{r'} \quad \mathbf{s'}]$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & a & a+b & b \\ 0 & c & c+d & d \end{bmatrix}$$





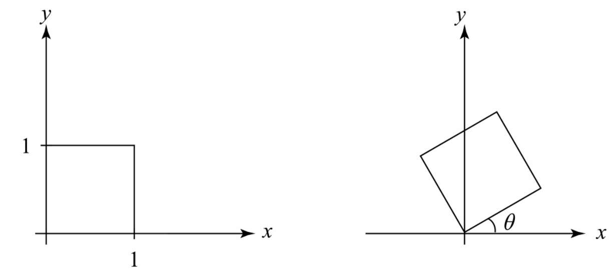
Effect on unit square, cont.

Observe:

- Origin invariant under *M*
- *M* can be determined just by knowing how the corners (1,0) and (0,1) are mapped
- *a* and *d* give *x*- and *y*-scaling
- *b* and *c* give *x*- and *y*-shearing

Rotation

From our observations of the effect on the unit square, it should be easy to write down a matrix for “rotation about the origin”:



- $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow$

- $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow$

Thus,

$$M = R(\theta) = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

Limitations of the 2 x 2 matrix

A 2 x 2 **linear transformation** matrix allows

- Scaling
- Rotation
- Reflection
- Shearing

Q: What important operation does that leave out?

Homogeneous coordinates

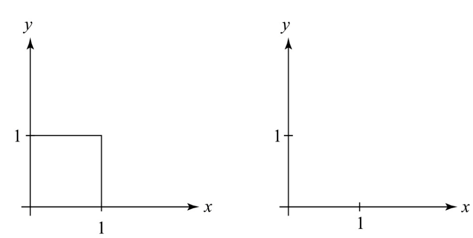
We can loft the problem up into 3-space, adding a third component to every point:

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Adding the third “*w*” component puts us in **homogenous coordinates**.

Then, transform with a 3 x 3 matrix:

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = T(\mathbf{t}) \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

. . . gives **translation**!

Affine transformations

The addition of translation to linear transformations gives us **affine transformations**.

In matrix form, 2D affine transformations always look like this:

$$M = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{c|c} A & \mathbf{t} \\ \hline 0 \quad 0 & 1 \end{array}\right]$$

2D affine transformations always have a bottom row of [0 0 1].

An "affine point" is a "linear point" with an added *w*-coordinate which is always 1:

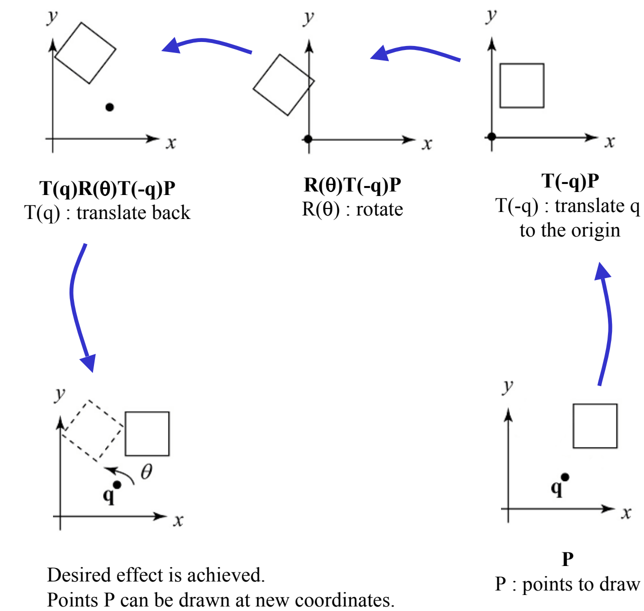
$$\mathbf{p}_{\text{aff}} = \begin{bmatrix} \mathbf{p}_{\text{lin}} \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Applying an affine transformation gives another affine point:

$$M\mathbf{p}_{\text{aff}} = \begin{bmatrix} A\mathbf{p}_{\text{lin}} + \mathbf{t} \\ 1 \end{bmatrix}$$

Rotation about arbitrary points

Until now, we have only considered rotation about the origin. With homogeneous coordinates, we can specify a rotation, θ, about any point $\mathbf{q} = [q_x \; q_y \; 1]^T$ with a matrix developed as follows.



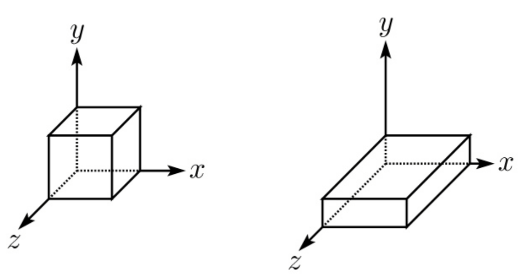
Basic 3-D transformations: scaling

Some of the 3-D affine transformations are just like the 2-D ones.

In this case, the bottom row is always [0 0 0 1].

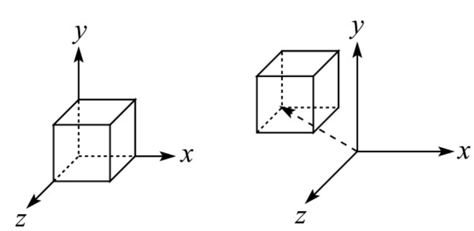
For example, scaling:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Translation in 3D

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



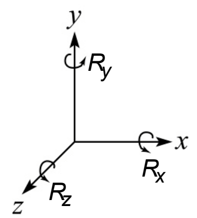
Rotation in 3D

Rotation now has more possibilities in 3D:

$$R_x(\theta)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta & -\sin\theta & 0\\ 0 & \sin\theta & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$R_y(\theta)=\begin{bmatrix}\cos\theta & 0 & \sin\theta & 0\\ 0 & 1 & 0 & 0\\ -\sin\theta & 0 & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$R_z(\theta)=\begin{bmatrix}\cos\theta & -\sin\theta & 0 & 0\\ \sin\theta & \cos\theta & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$



Use right hand rule

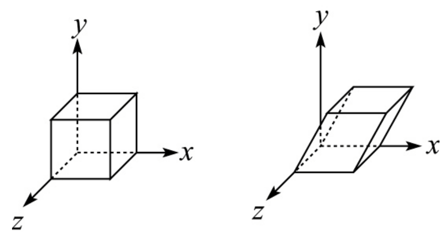
How many degrees of freedom are there in an arbitrary rotation?

How else might you specify a rotation?

Shearing in 3D

Shearing is also more complicated. Here is one example:

$$\begin{bmatrix}x'\\ y'\\ z'\\ 1\end{bmatrix}=\begin{bmatrix}1 & b & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\\ 1\end{bmatrix}$$

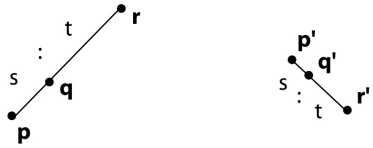


We call this a shear with respect to the x-z plane or a shear along the y-direction.

Properties of affine transformations

Here are some useful properties of affine transformations:

- Lines map to lines
- Parallel lines remain parallel
- Midpoints map to midpoints (in fact, ratios are always preserved)



$$\text{ratio}=\frac{\|\mathbf{pq}\|}{\|\mathbf{qr}\|}=\frac{s}{t}=\frac{\|\mathbf{p'q'}\|}{\|\mathbf{q'r'}\|}$$

Affine transformations in OpenGL

OpenGL maintains a “modelview” matrix that holds the current transformation **M.**

The modelview matrix is applied to points (usually vertices of polygons) before drawing.

It is modified by commands including:

- `glLoadIdentity()` $\mathbf{M} \leftarrow \mathbf{I}$
  – set **M** to identity
- `glTranslatef(t_x, t_y, t_z)` $\mathbf{M} \leftarrow \mathbf{MT}$
  – translate by (t_x, t_y, t_z)
- `glRotatef(θ, x, y, z)` $\mathbf{M} \leftarrow \mathbf{MR}$
  – rotate by angle θ about axis (x, y, z)
- `glScalef(s_x, s_y, s_z)` $\mathbf{M} \leftarrow \mathbf{MS}$
  – scale by (s_x, s_y, s_z)

Note that OpenGL adds transformations by *postmultiplication* of the modelview matrix.

Summary

What to take away from this lecture:

- All the names in boldface.
- How points and transformations are represented.
- What all the elements of a 2 x 2 transformation matrix do and how these generalize to 3 x 3 transformations.
- What homogeneous coordinates are and how they work for affine transformations.
- How to concatenate transformations.
- The mathematical properties of affine transformations.