

Affine transformations

1

Reading

Required:

- ♦ Foley, et al, Chapter 5.1-5.5.

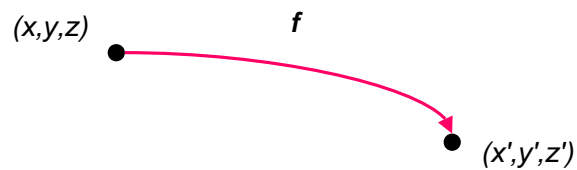
Further reading:

- ♦ David F. Rogers and J. Alan Adams, *Mathematical Elements for Computer Graphics*, 2nd Ed., McGraw-Hill, New York, 1990, Chapter 2.

2

Geometric transformations

Geometric transformations will map points in one space to points in another: $(x',y',z') = f(x,y,z)$.



These transformations can be very simple, such as scaling each coordinate, or complex, such as non-linear twists and bends.

We'll focus on transformations that can be represented easily with matrix operations.

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Representation

We can represent a **point**, $\mathbf{p} = (x,y)$ in the plane

- ♦ as a column vector $\begin{bmatrix} x \\ y \end{bmatrix}$
- ♦ as a row vector $\begin{bmatrix} x & y \end{bmatrix}$

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Representation, cont.

We can represent a **2-D transformation** M by a matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If \mathbf{p} is a column vector, M goes on the left:

$$\mathbf{p}' = M\mathbf{p}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

If \mathbf{p} is a row vector, M^T goes on the right:

$$\mathbf{p}' = \mathbf{p}M^T$$

$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

We will use **column vectors**.

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Two-dimensional transformations

Here's all you get with a 2 x 2 transformation matrix M :

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

So:

$$x' = ax + by$$

$$y' = cx + dy$$

We will develop some intimacy with the elements $a, b, c, d...$

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Identity

Suppose we choose $a=d=1, b=c=0$:

- ♦ Gives the **identity** matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- ♦ Doesn't move the points at all

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Scaling

Suppose we set $b=c=0$, but let a and d take on any *positive* value:

- ♦ Gives a **scaling** matrix:

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

- ♦ Provides **differential scaling** in x and y .

$$x' = ax$$

$$y' = dy$$

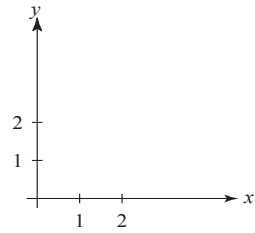
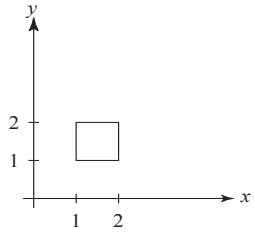
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Scaling

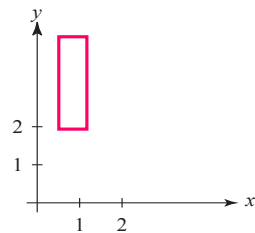
$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

$$x' = ax$$

$$y' = dy$$



$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

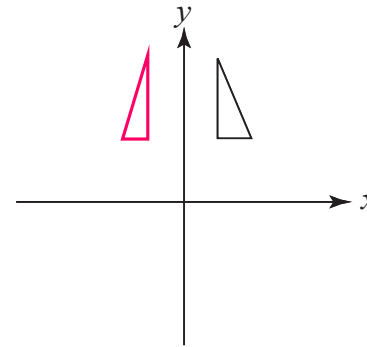


$$\begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}$$

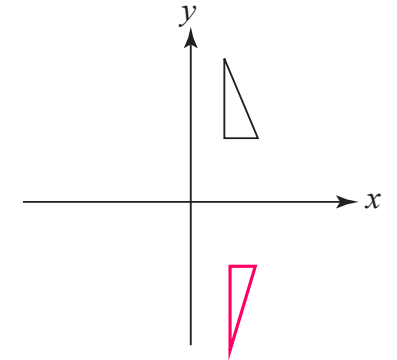
Suppose we keep $b=c=0$, but let either a or d go negative.

Examples:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

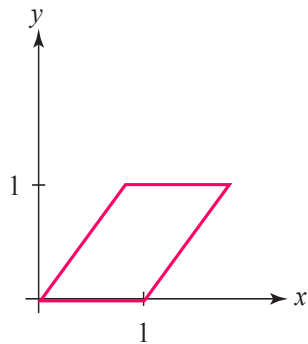
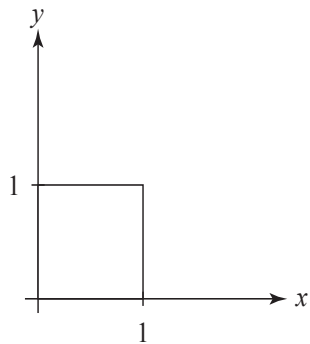


Now let's leave $a=d=1$ and experiment b . . .

The matrix $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$

$$x' = x + by$$

gives: $y' = y$



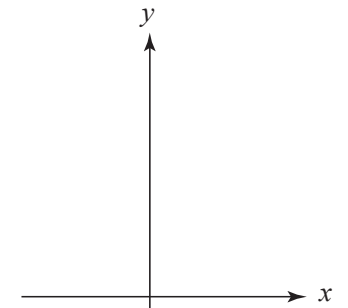
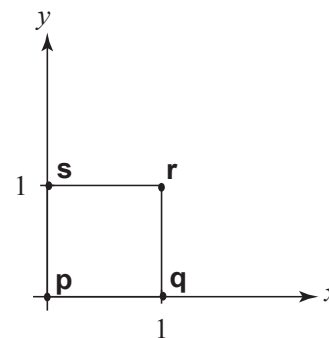
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Effect on unit square

Let's see how a general 2 x 2 transformation M affects the unit square:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q & r & s \end{bmatrix} = \begin{bmatrix} p' & q' & r' & s' \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & a & a+b & b \\ 0 & c & c+d & d \end{bmatrix}$$



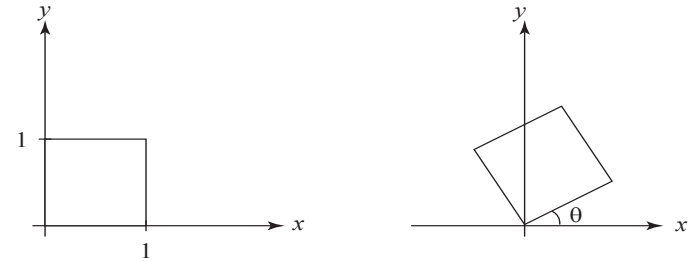
Effect on unit square, cont.

Observe:

- ♦ Origin invariant under M
- ♦ M can be determined just by knowing how the corners $(1,0)$ and $(0,1)$ are mapped
- ♦ a and d give x - and y -scaling
- ♦ b and c give x - and y -shearing

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Rotation



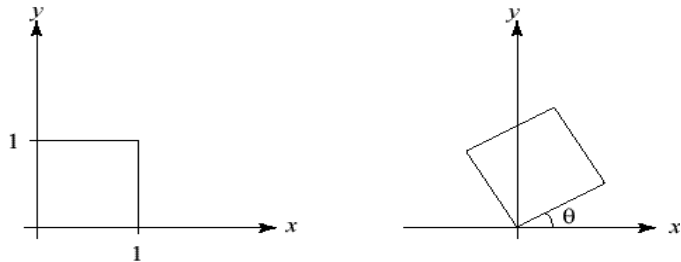
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$$

$$M = R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

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Inverse Rotation



$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$R^{-1}(\theta) = ?$$

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Limitations of the 2 x 2 matrix

A 2 x 2 matrix allows

- ♦ Scaling
- ♦ Rotation
- ♦ Reflection
- ♦ Shearing

Q: What important operation does that leave out?

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Homogeneous coordinates

Idea is to loft the problem up into 3-space, adding a third component to every point:

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

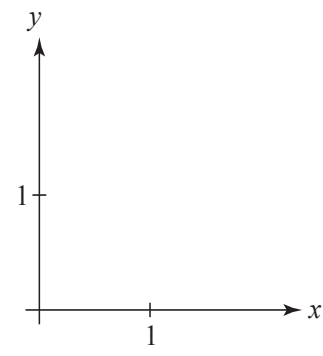
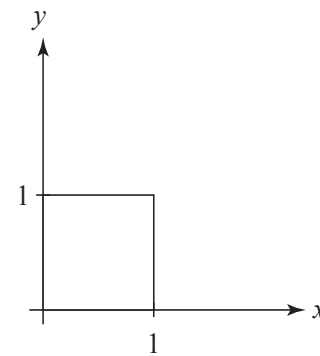
And then transform with a 3 x 3 matrix:

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = T(\mathbf{t}) \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

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Homogeneous coordinates

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = T(\mathbf{t}) \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

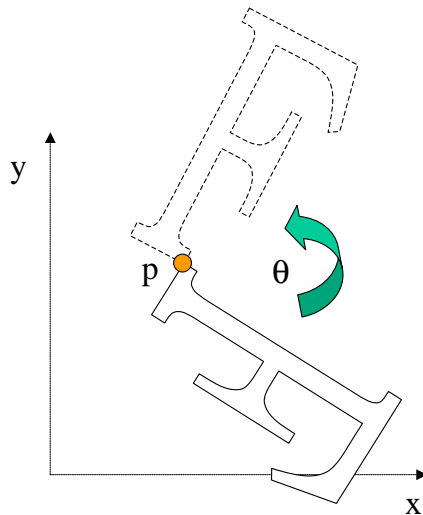


$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

... gives **translation!**

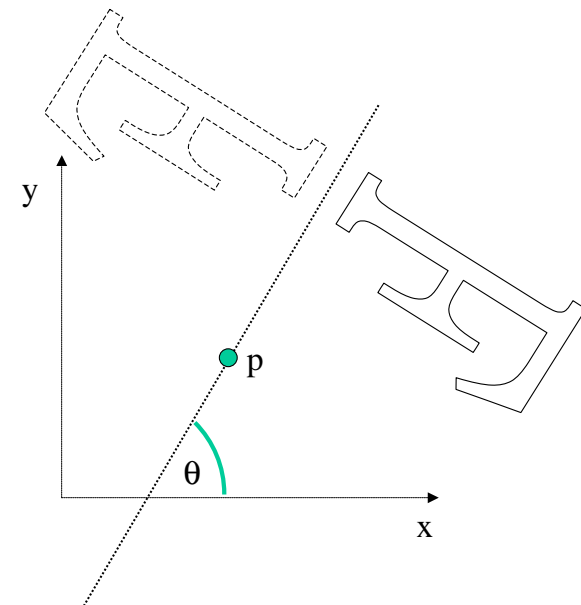
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Rotation around arbitrary point



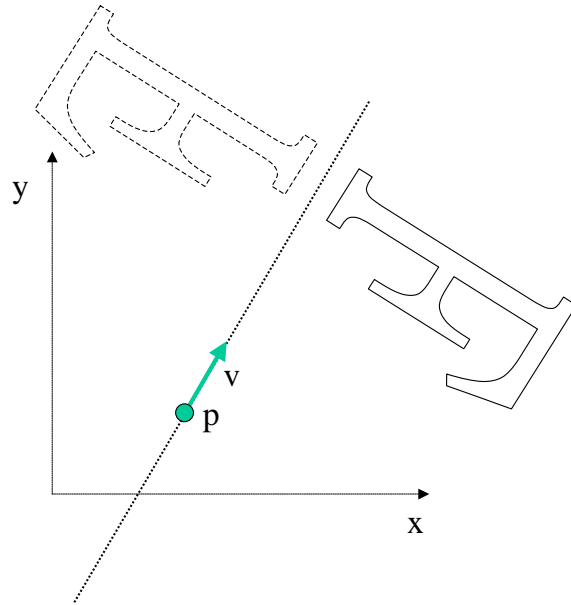
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Reflection around arbitrary axis



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Reflection around arbitrary axis



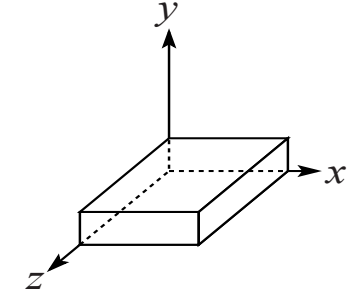
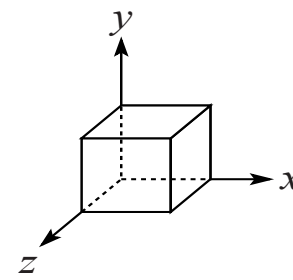
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Basic 3-D transformations: scaling

Some of the 3-D transformations are just like the 2-D ones.

For example, scaling:

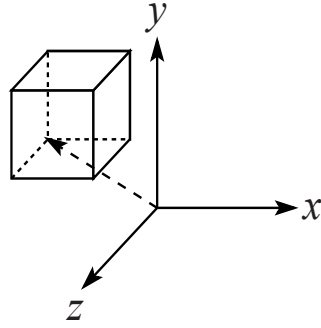
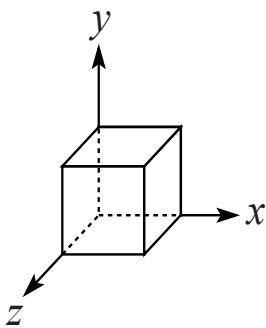
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



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Translation in 3D

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



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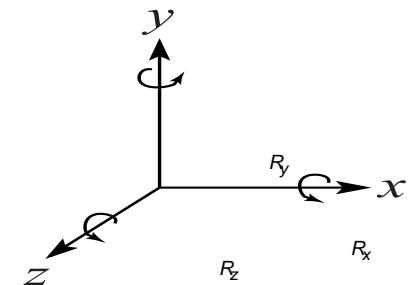
Rotation in 3D

Rotation now has more possibilities in 3D:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos \theta & 0 & \sin \theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \theta & 0 & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Use right hand rule

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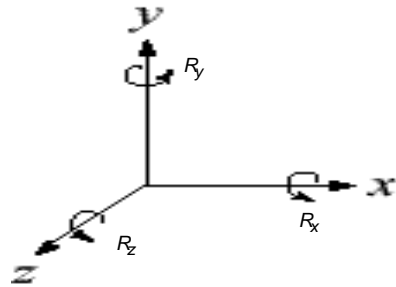
Rotation in 3D

What about the inverses of 3D rotations?

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos \theta & 0 & \sin \theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \theta & 0 & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

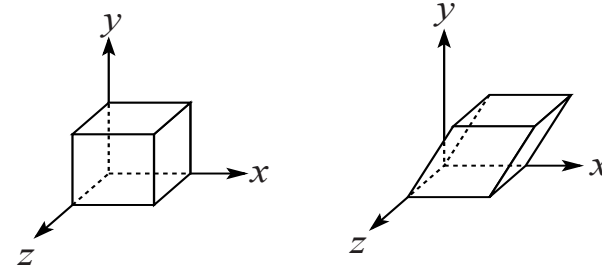


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Shearing in 3D

Shearing is also more complicated. Here is one example:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



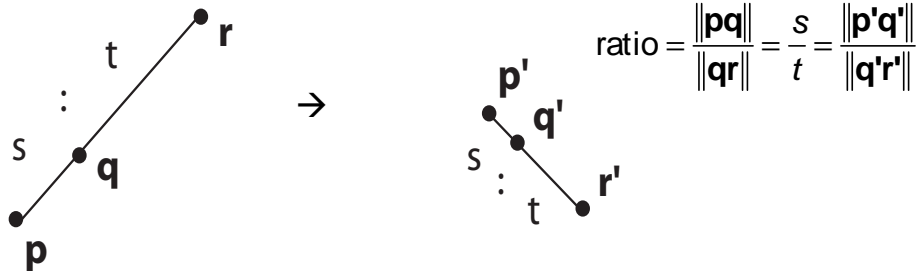
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Properties of affine transformations

All of the transformations we've looked at so far are examples of "affine transformations."

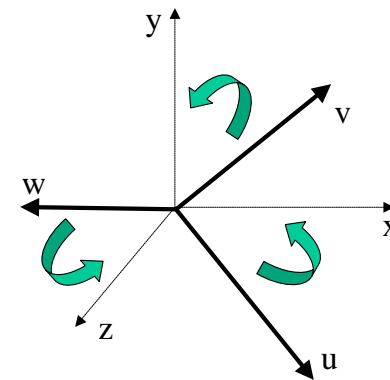
Here are some useful properties of affine transformations:

- Lines map to lines
- Parallel lines remain parallel
- Midpoints map to midpoints (in fact, ratios are always preserved)



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Rotation that aligns 3 orthonormal vectors with the principal axes



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Summary

What to take away from this lecture:

- ♦ All the names in boldface.
- ♦ How points and transformations are represented.
- ♦ What all the elements of a 2×2 transformation matrix do and how these generalize to 3×3 transformations.
- ♦ What homogeneous coordinates are and how they work for affine transformations.
- ♦ How to concatenate transformations.
- ♦ The mathematical properties of affine transformations.