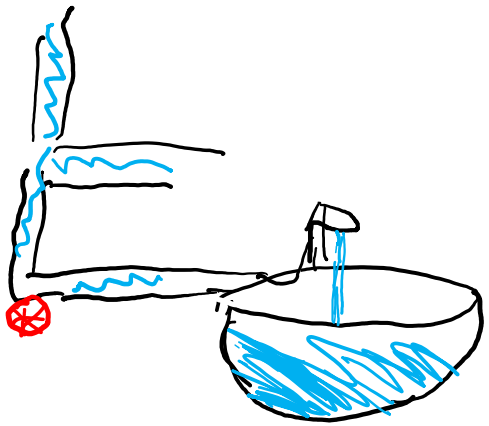
I'm teaching from home today
if my internet cuts out it'll probably
come back in ≤2 minutes :)



Network Flow

CSE 417 Winter 21
Lecture 19

Announcements

HW4 solutions on webpage

HW5 (written part) still due tonight

Remember you have the extra late days

Extra mini-project due next Friday.

HW6 comes out tonight. Due in 2 weeks.

Finish up HW5 programming

Question 2 (finding a max flow and min cut) will make sense tonight

Question 3 will make sense Monday

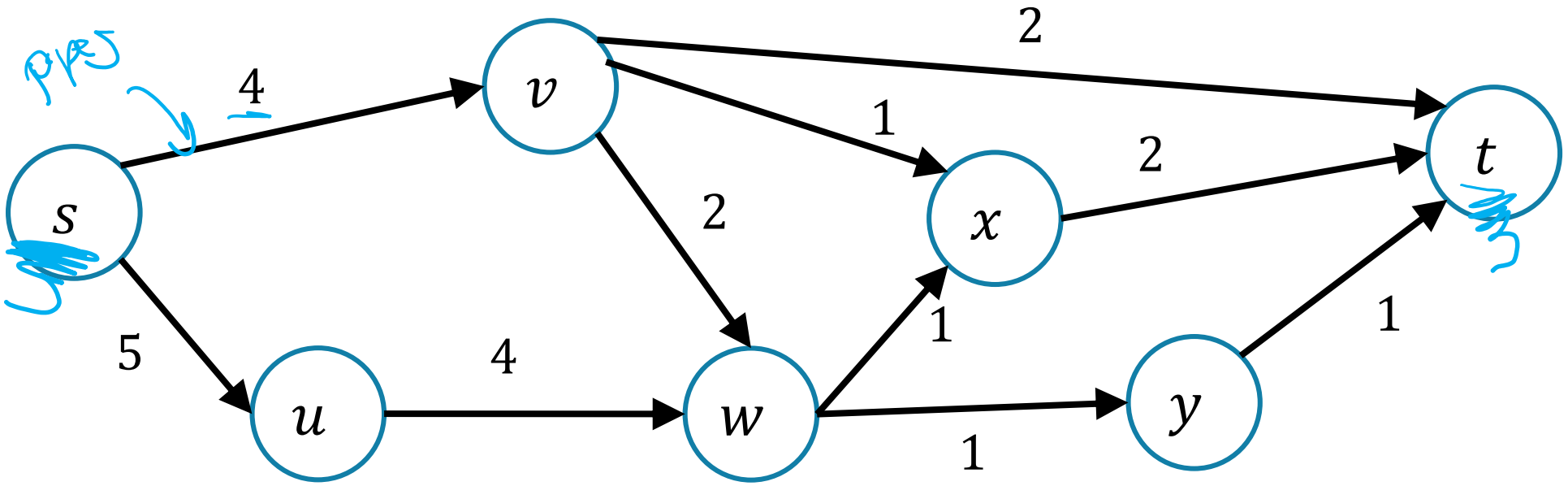
Questions 4-5 will make sense Wednesday

Max Flow

We have a directed graph G, a source vertex s and a target vertex t.

We have some commodity (water or data packets) we have to send from s to t.

Every edge has a capacity, it can only handle so much water.

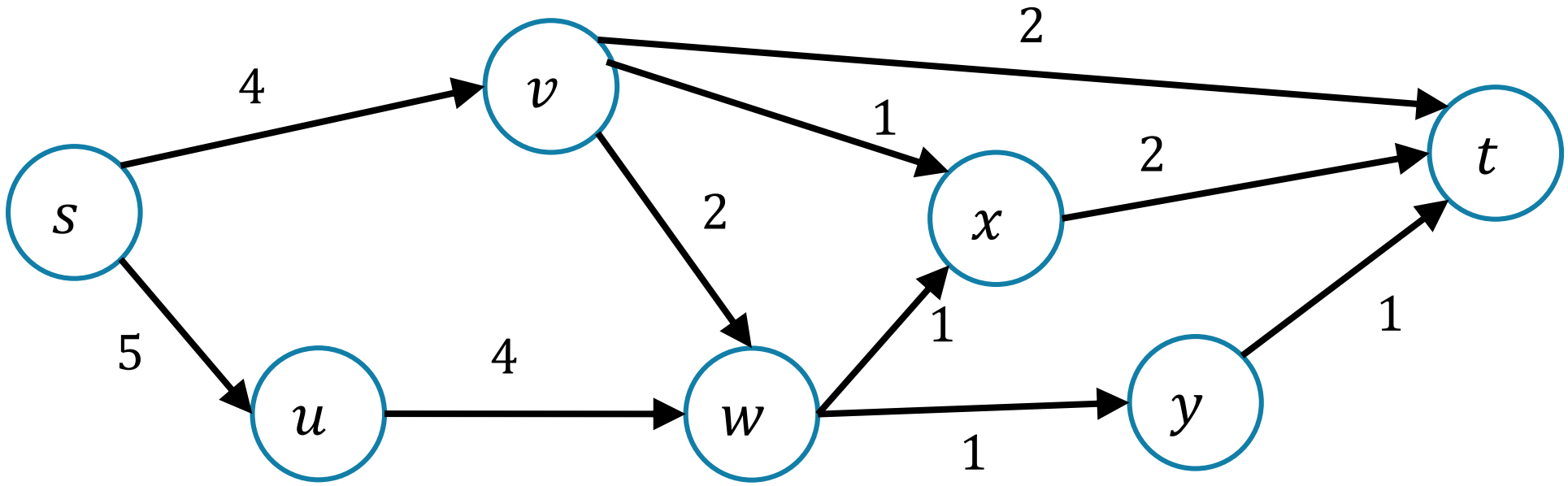


Flows

A **flow** moves units of water from s **to** t.

Water can only be created at s and only disappear at t.

And you cannot move more water than the capacity on any edge.



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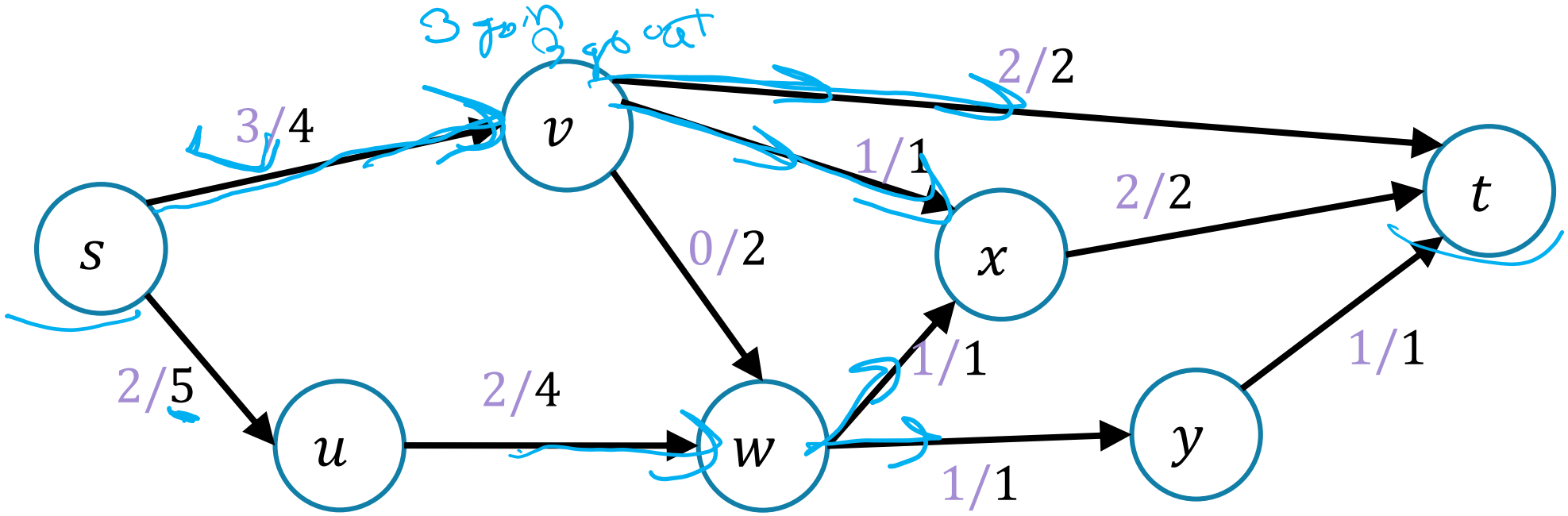
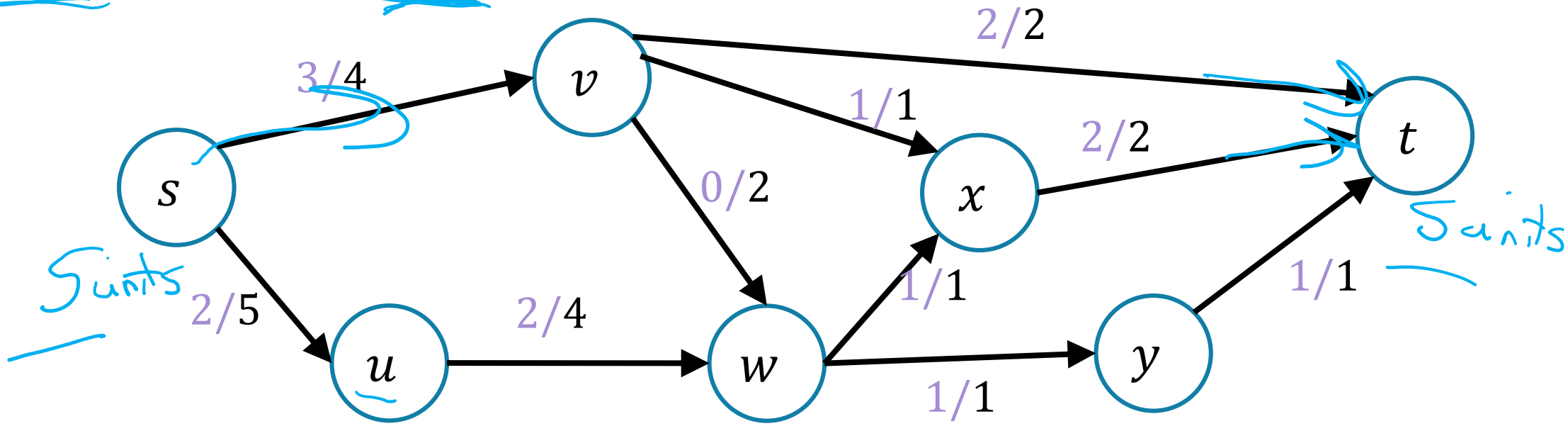


Flows

The **value** or **size** of a flow is the net flow leaving s

Or, equivalently the net flow entering t.

Value of this flow is 5.



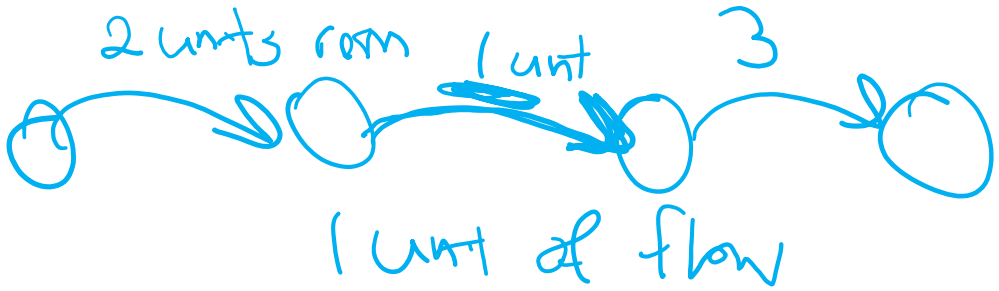
Finding the Max Flow

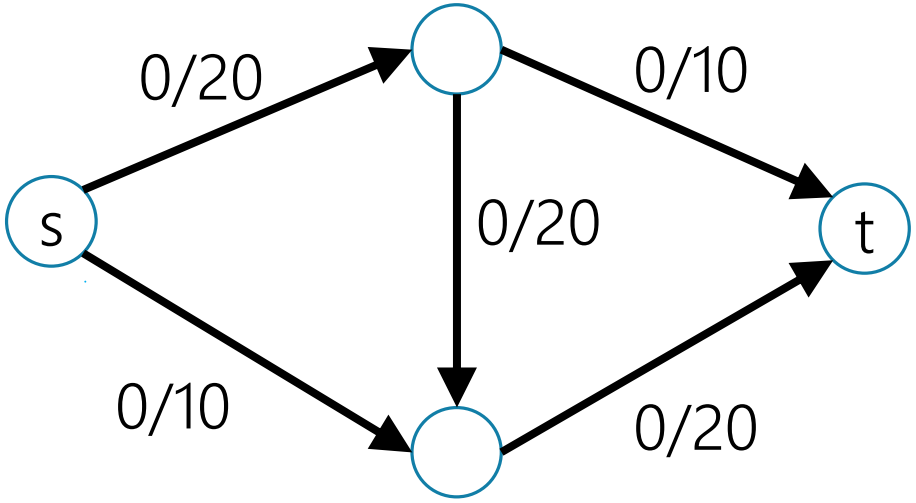
Idea: find a path that we can push flow along.

Start from s, follow an edge with remaining capacity until you get to t

How much flow can you add? The minimum **remaining** capacity on any of the edges we used.

Does this work?





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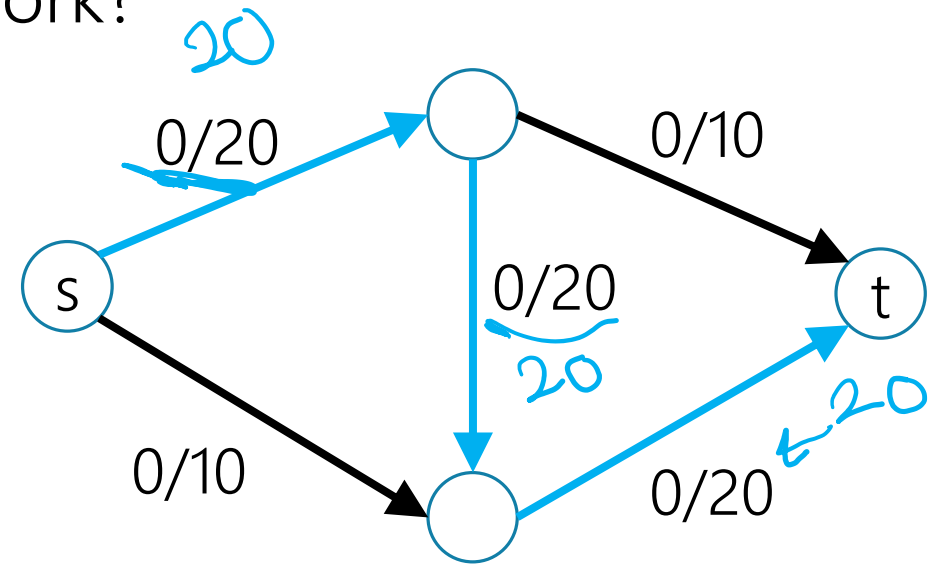

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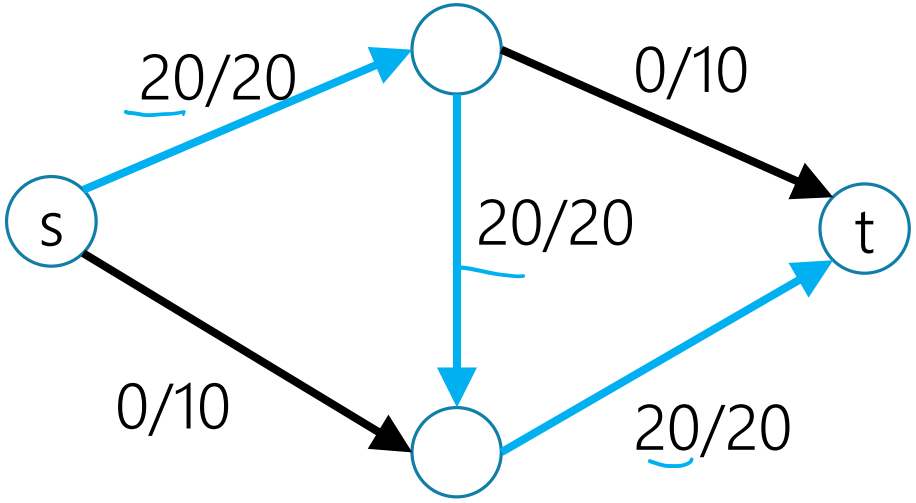

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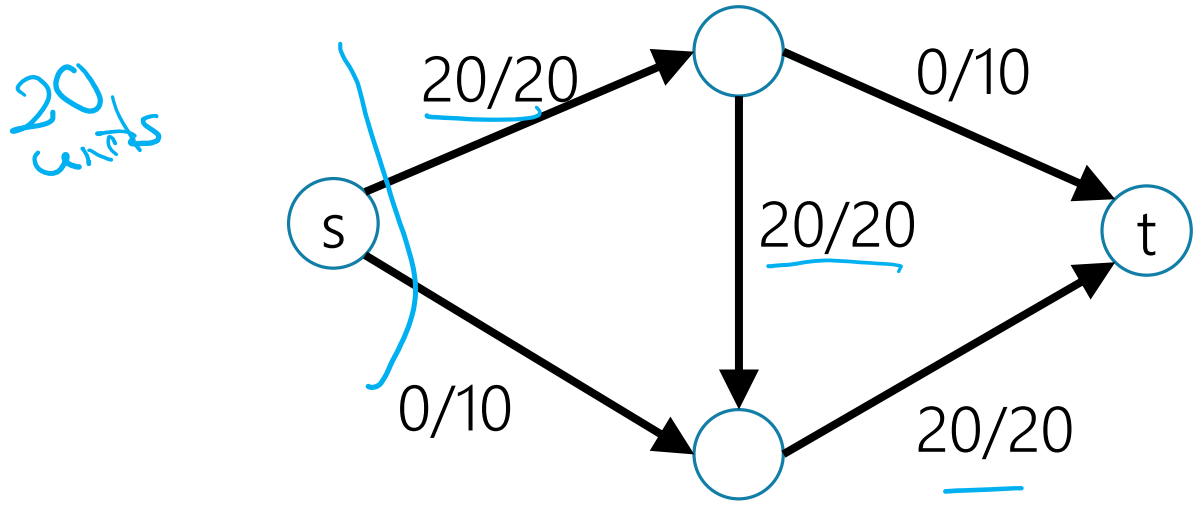
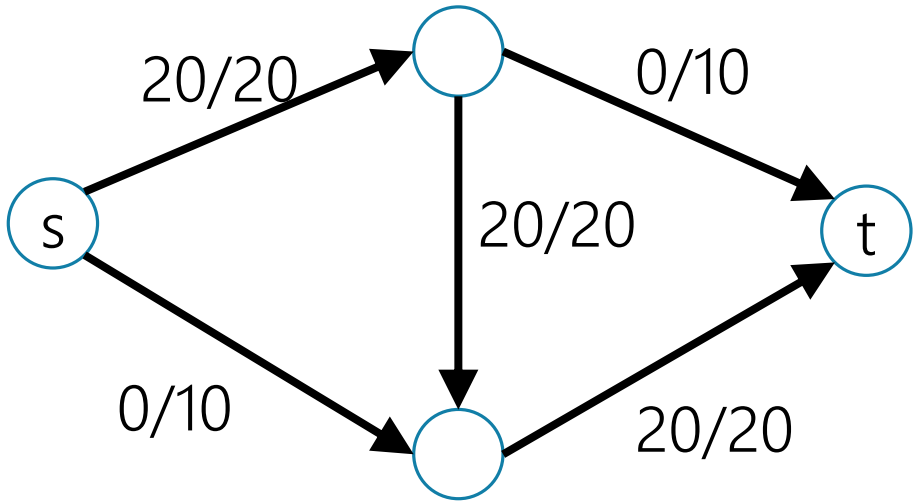


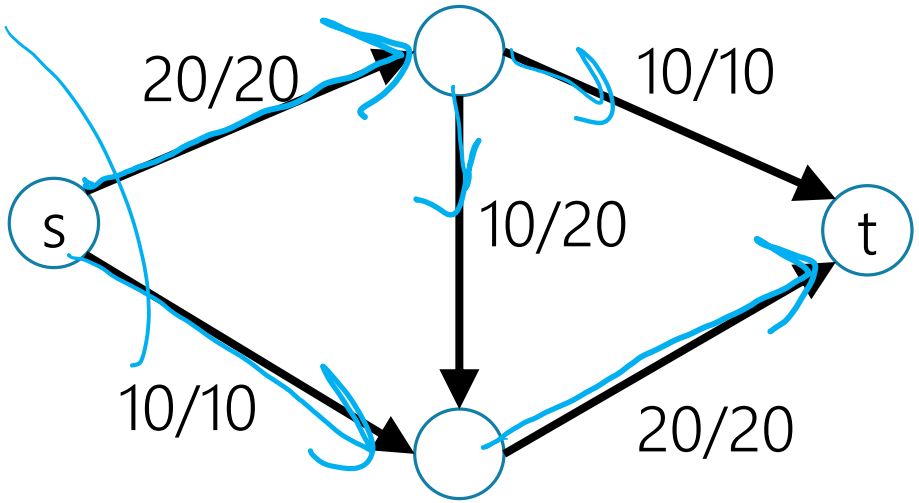
Finding the Max Flow

We find a valid flow…but it might not be the maximum one.

What our first idea found



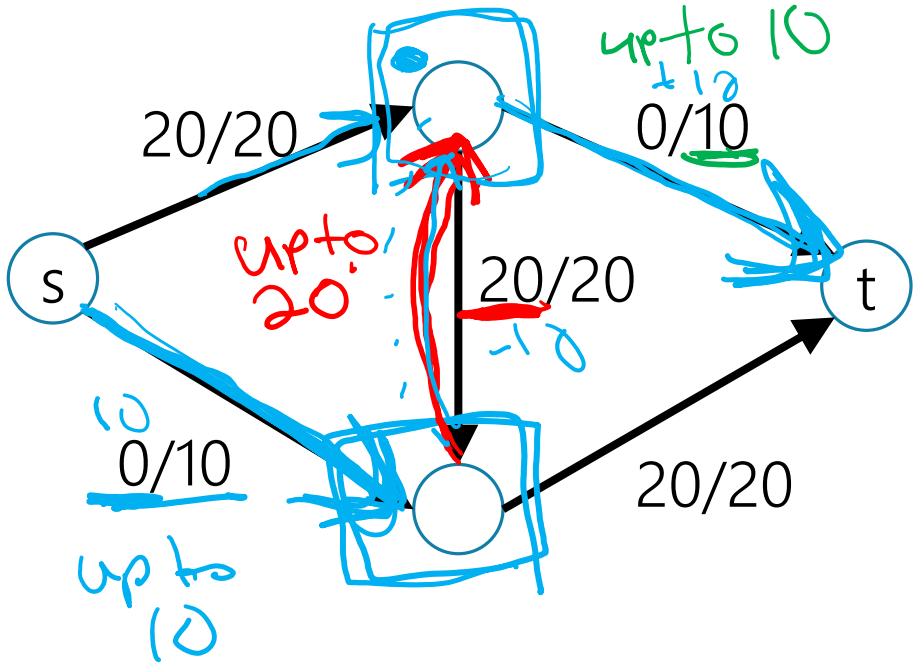
The true optimum



Finding the Max Flow

How would we fix what our first idea found?

What our first idea found



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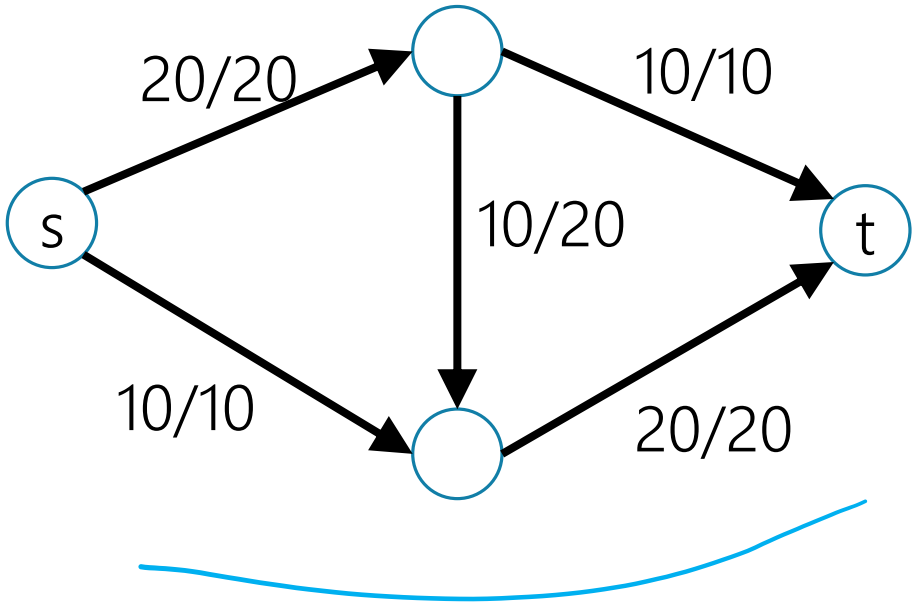

Finding the Max Flow

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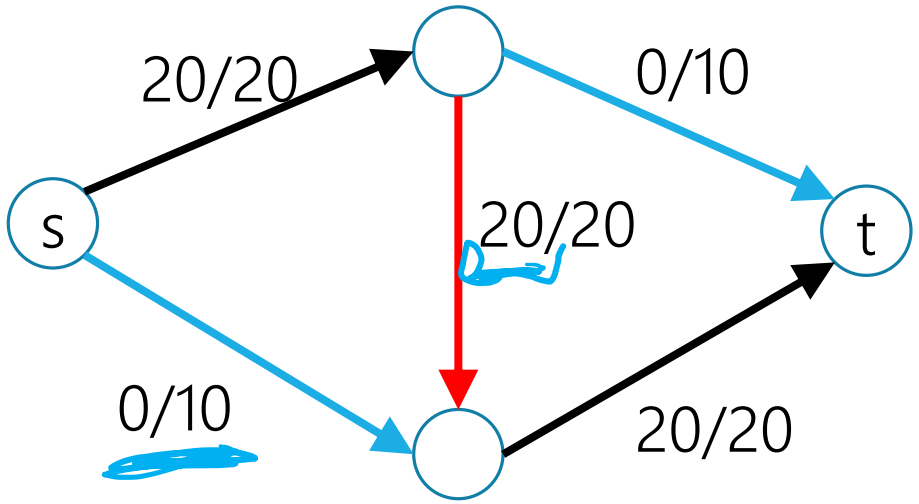

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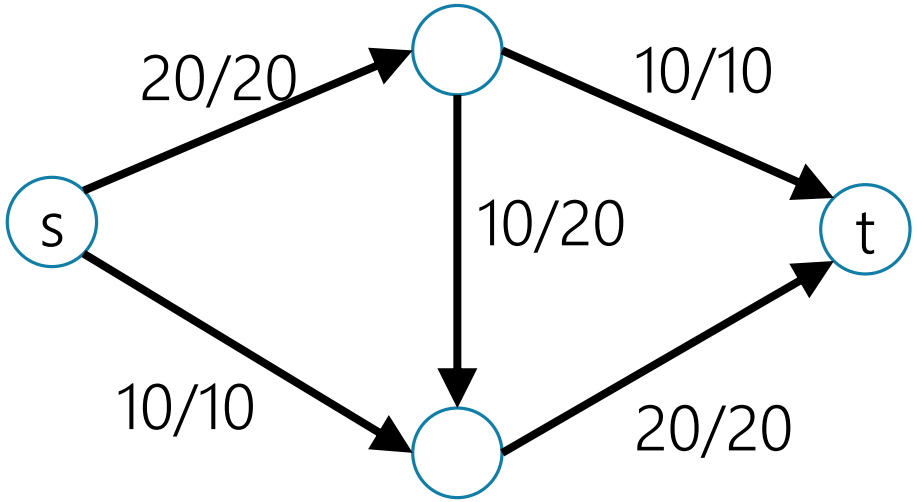


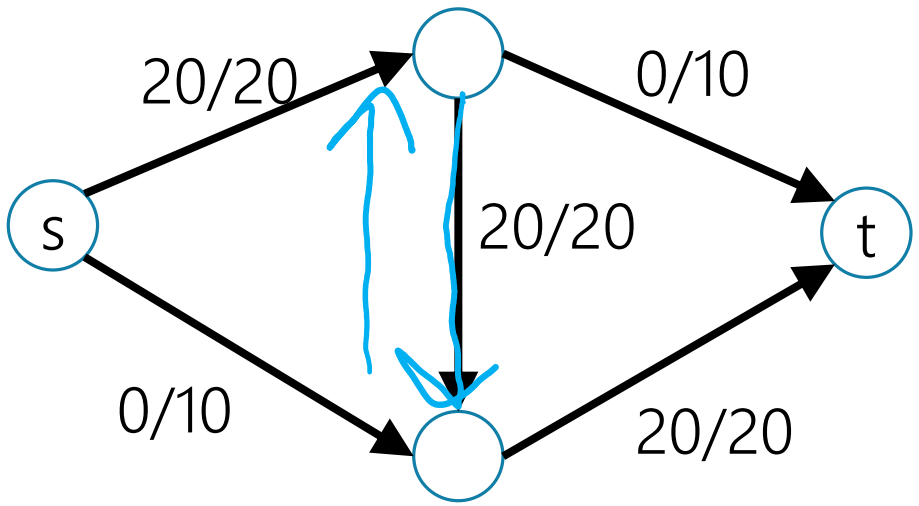
We can send extra flow from s and decrease the flow along another edge with existing flow

And we'll still get a valid flow!

Finding the Max Flow

How would we fix what our first idea found?

What our first idea found



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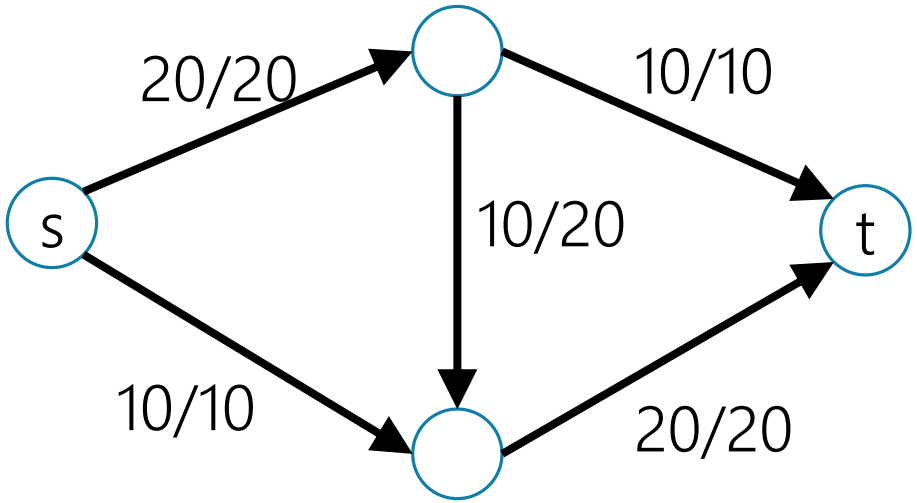


Fixing Our First Idea

When can we send flow in a direction?

When there is unused capacity OR

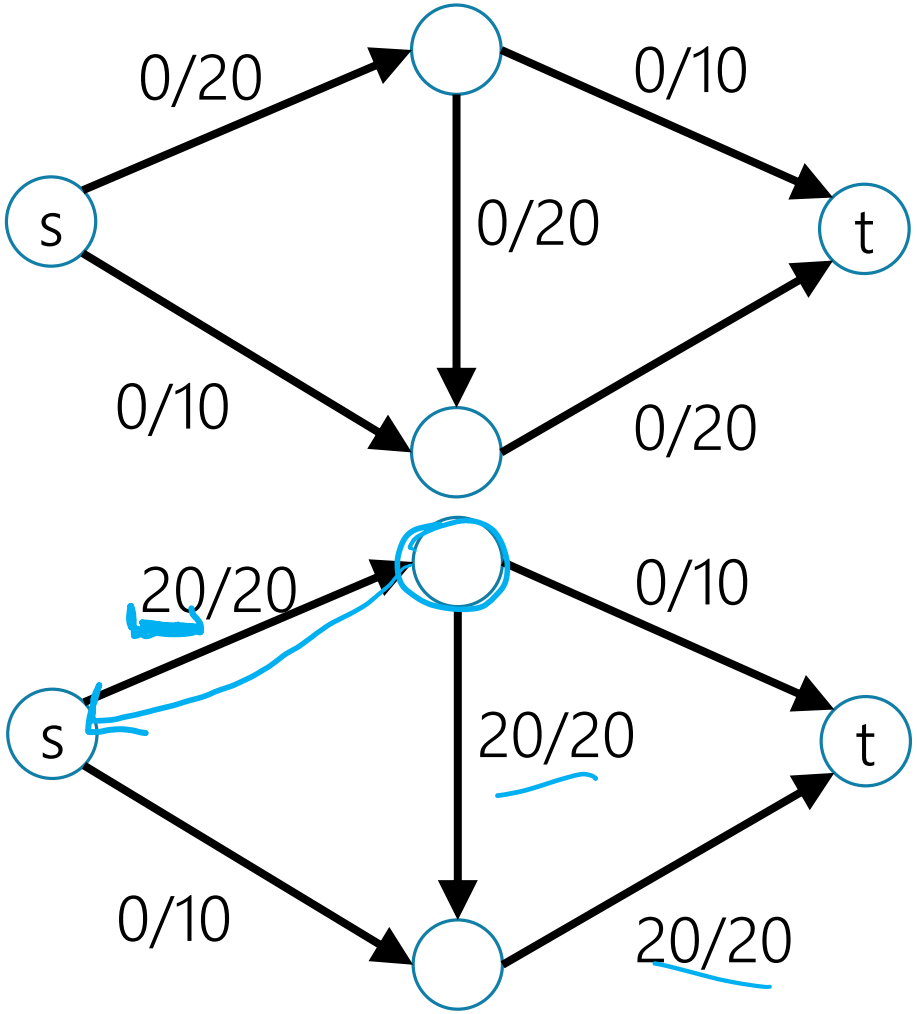
When there is flow going the other direction we can delete it.

Let's update the graph to take that into account.

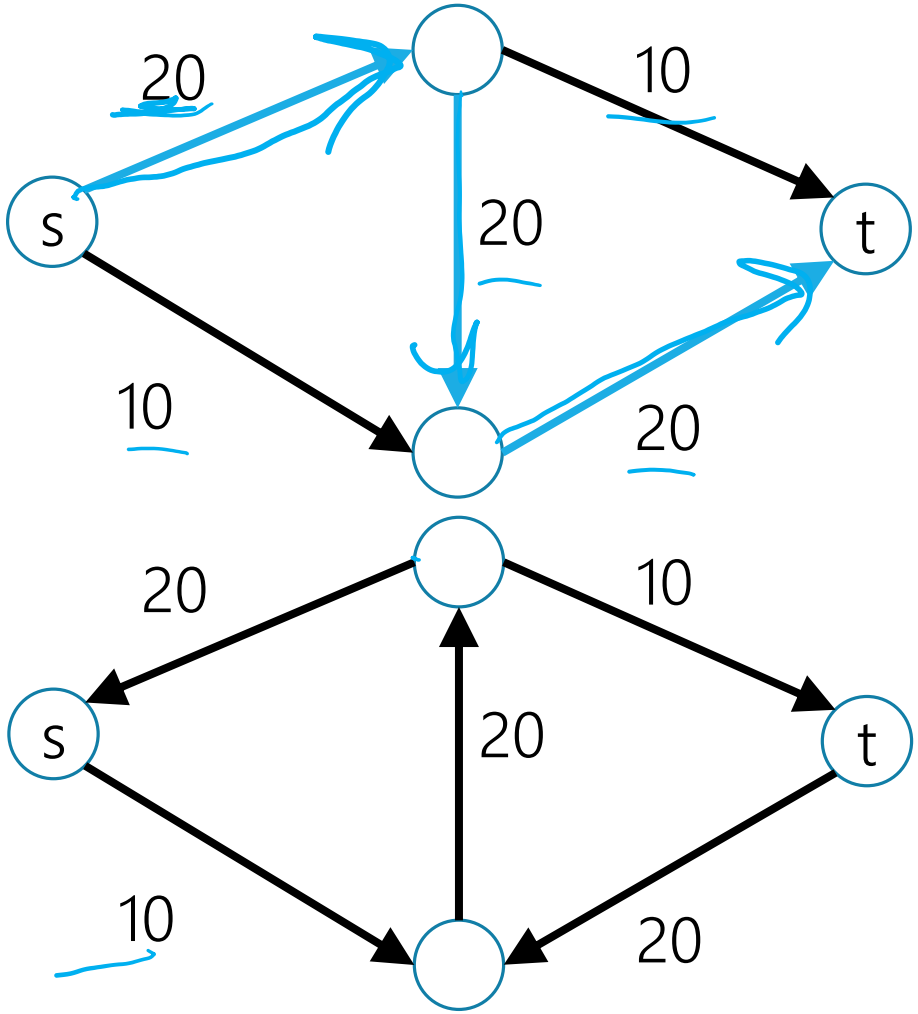
Have an edge weight demonstrate the changeable capacity (the "residual")

An Example

Graph, with flow



"Residual Graph"



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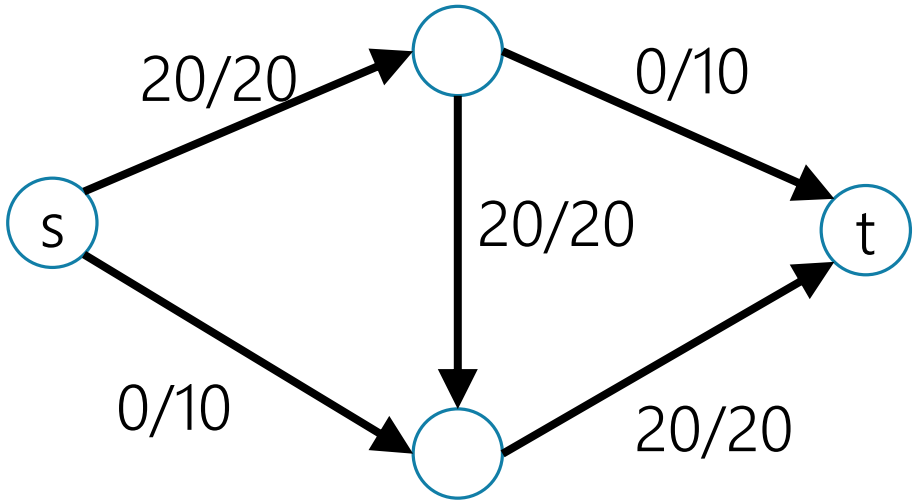

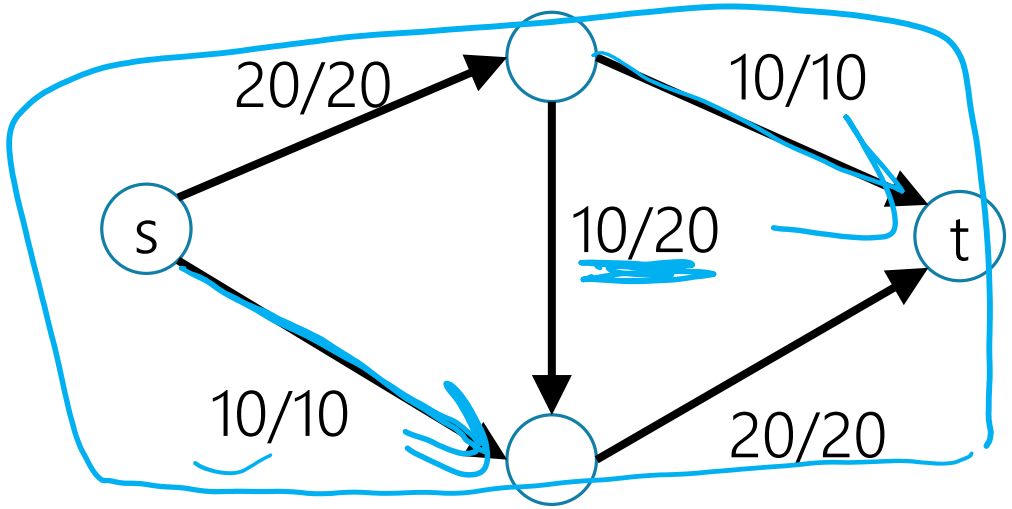

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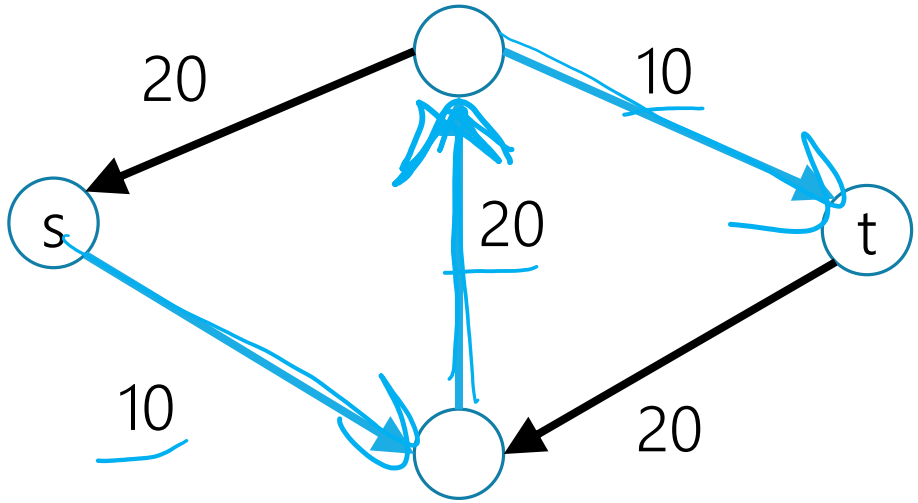
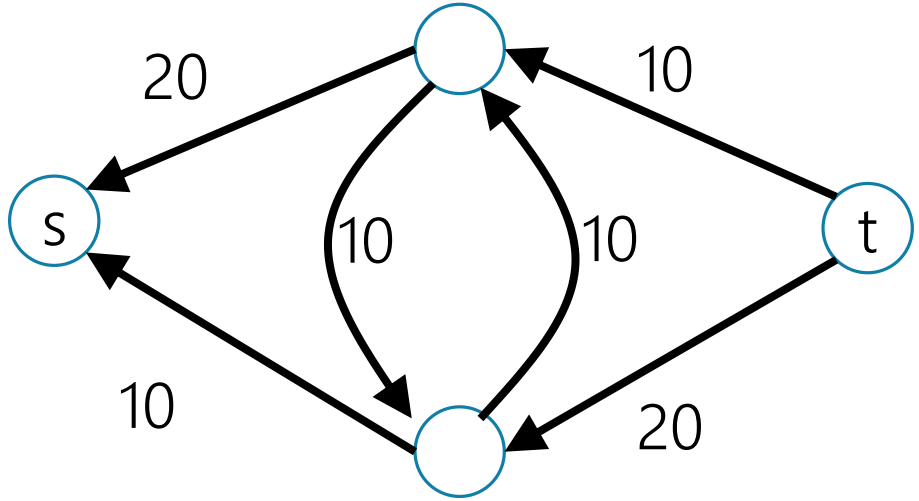




Residual Graph

In general:

If the original graph has an edge (u, v) of capacity c, and the flow sends $f_{u,v}$ along (u, v):

Include (u, v) in the residual with capacity $c - f_{u,v}$ as long as $c - f_{u,v} > 0$ (if equal to zero, don't include the edge)

Include (v, u) [the edge going in the reverse direction] with capacity $f_{u,v}$ as long as $f_{u,v} > 0$

Ford-Fulkerson Algorithm

While(flow is not maximum)

- Run BFS in residual graph starting from s.
- Record predecessors to find an s, t-path
- Iterate through path, finding c minimum residual capacity on path.
- Add c to every edge on path in flow
- Update residual graph

Ford-Fulkerson Algorithm

While(true)

- Run BFS in residual graph starting from s.
- Record predecessors to find an s, t-path
- If you don't reach t, break. //otherwise you can still augment
- Iterate through path, finding c minimum residual capacity on path.
- Add c to every edge on path in flow
- Update residual graph

Assuming $E \geq V$ (isolated vertices won't affect the flow, so this is reasonable).

Running Time:? $O(E)$ per iteration. Number of iterations?

Ford-Fulkerson Algorithm

If we have all integer capacities at the start…

The residual graph will always have integer capacities.

Why? The minimum capacity on the first path is an integer.

So we subtract or add integers to the residual graph.

And the result is more integers!

So in every iteration we add at least 1 unit of flow!

Ford-Fulkerson Algorithm

While(true)

Run BFS in residual graph starting from s.

Record predecessors to find an s, t-path

If you don't reach t, break. //otherwise you can still augment

Iterate through path, finding c minimum residual capacity on path.

Add c to every edge on path in flow

Update residual graph

Running Time:? $O(E)$ per iteration. Number of iterations? $O(f)$, where f is the value of the maximum flow. Total $O(Ef)$

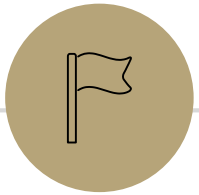
Wait… f?

We haven't seen a running time before that depends on the **answer**

Normally, we want the running time directly in terms of the input.

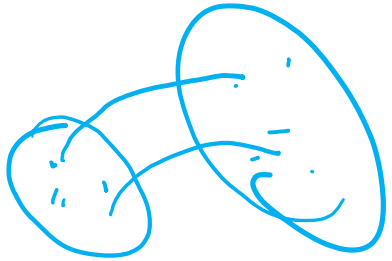
There are tricks to speed up Ford-Fulkerson so you don't take too long if f is really big.

Some optional content about this at the end of this deck.



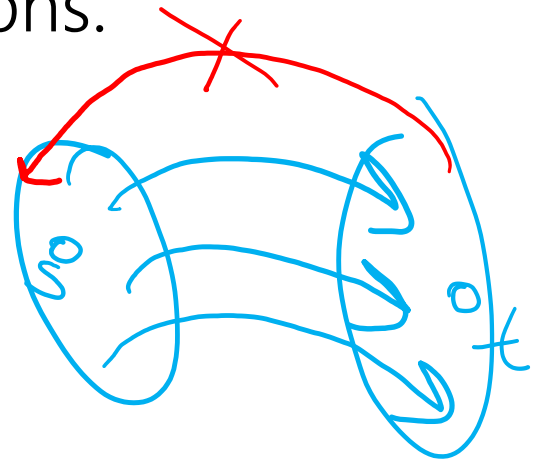
Minimum Cut

Remember cuts?



We talked about them a long time ago, when we were defining "safe edges" for MST algorithms. That was for undirected graphs.

For this problem:



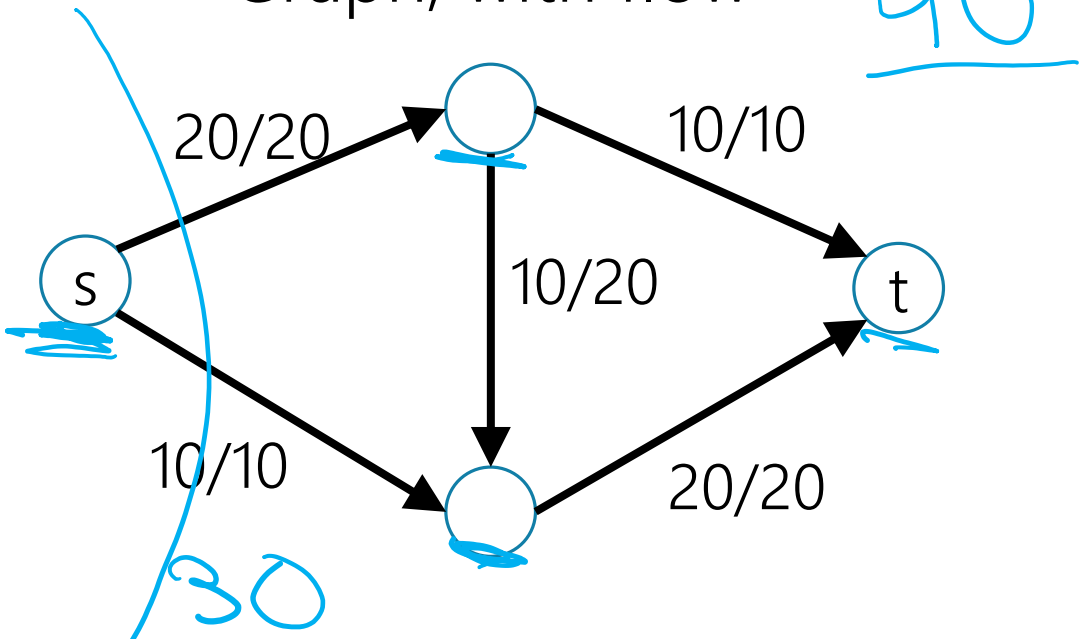
An (s, t)-cut, is a split of the vertices into two sets (S, T)

So that s is in S, t is in T,
and every other vertex is in exactly one of S and T.

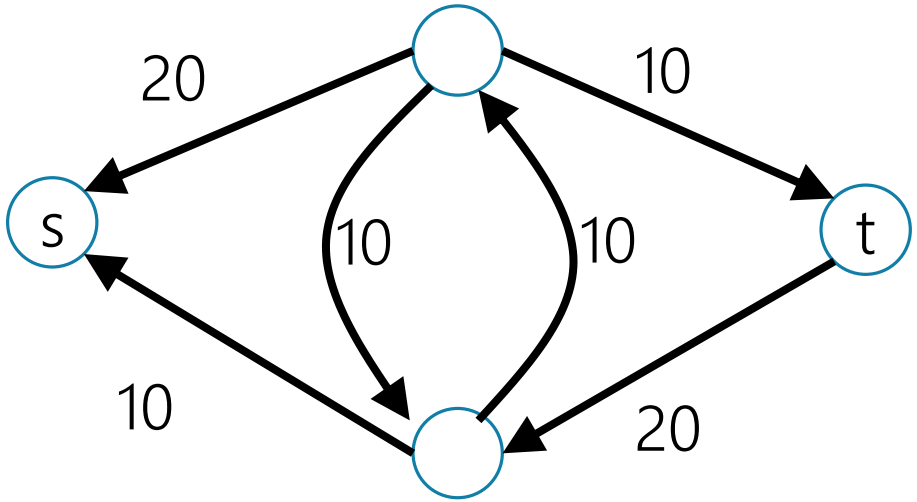
The capacity of a cut (or size of a cut) is the capacity of the edges going from S to T (don't count capacity from T to S).

An Example

Graph, with flow



"Residual Graph"

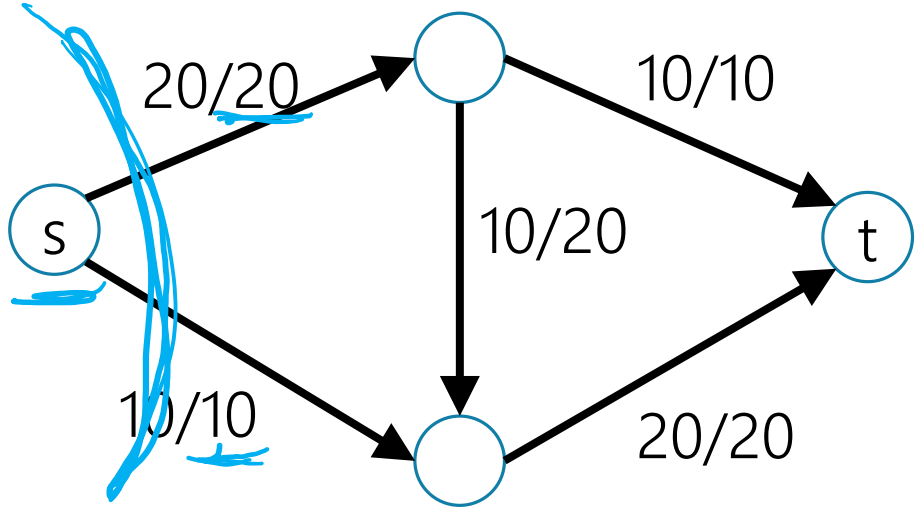


We can't get from s to t anymore in the residual graph.
We're done! That's a maximum flow.

But…why?

An Example

Graph, with flow



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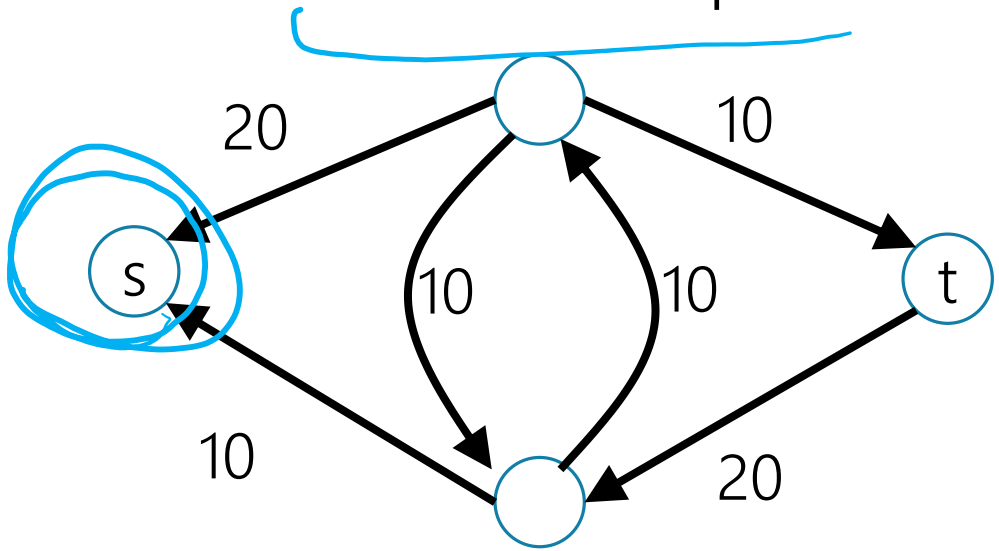


Cut is $(s, V - s)$ (i.e. s on one side, everything else on the other)

Edges from s to everything else? Capacity $\mathbf{30}$
We can't get more than $\mathbf{30}$ units of flow from s to t. Because it all (simultaneously) must cross from one side to the other.

How Do We Know?

How do we know the algorithm is done?

When we can't we get from s to t?

We've **cut** the (residual) graph.
s and all the vertices you can reach from it on one side
and t and all the vertices s can't reach on the other.

Take a look at the edges spanning the cut in the original graph.

In our first graph, that capacity was equal to the value of the max flow.

Finding the min-cut

Maintain the residual graph.

When you search from s and can't get to t:

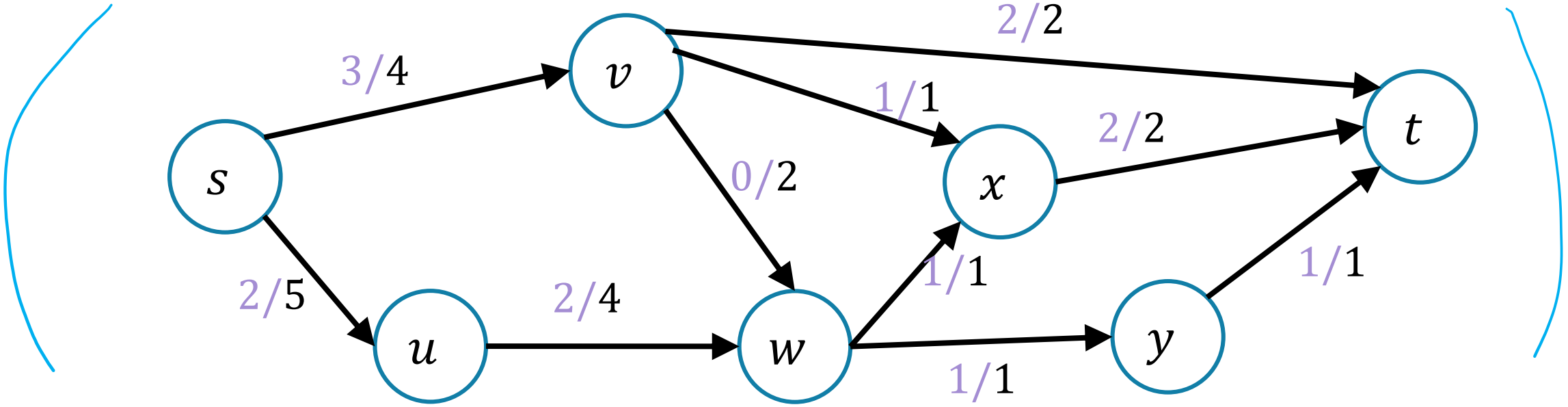
s and everything you can reach from s is on one side of the cut

t and everything you can't reach from s is on the other side.

Another Example

We started lecture with this flow. What's the residual graph?

What's the residual graph? What's the cut?



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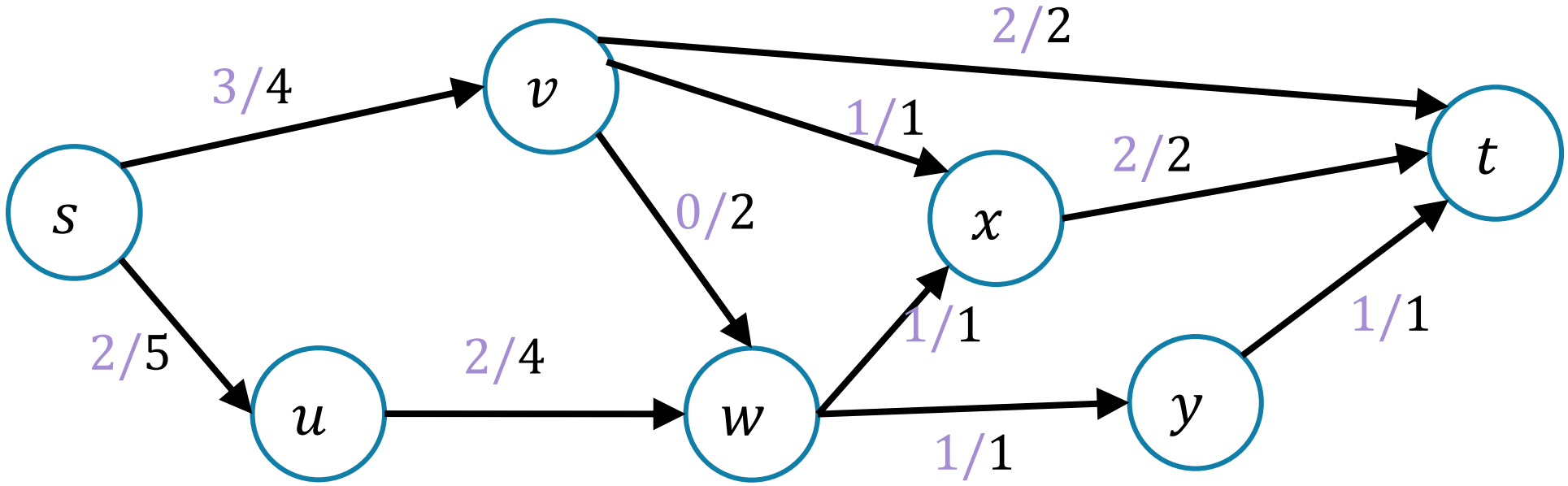

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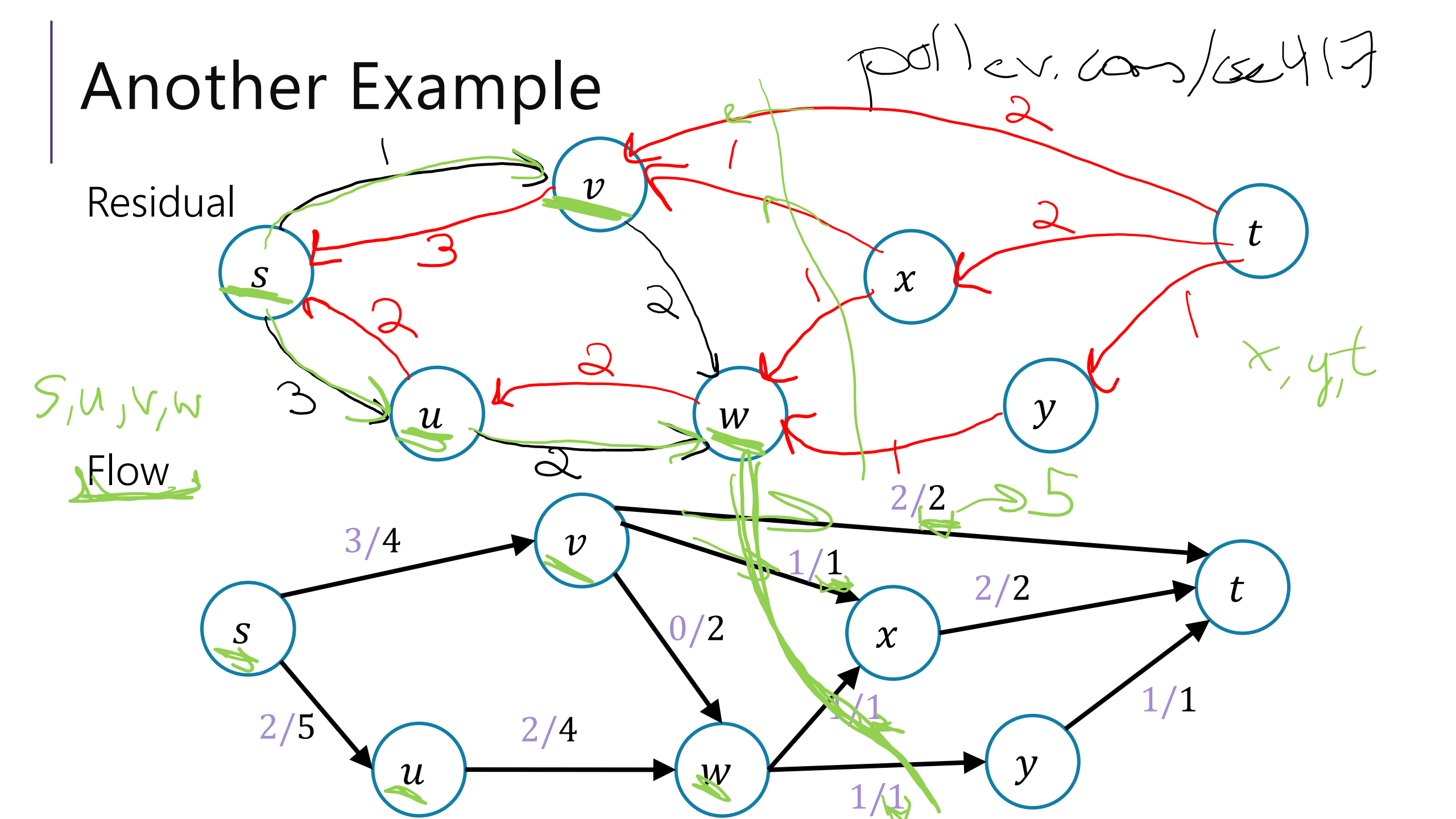

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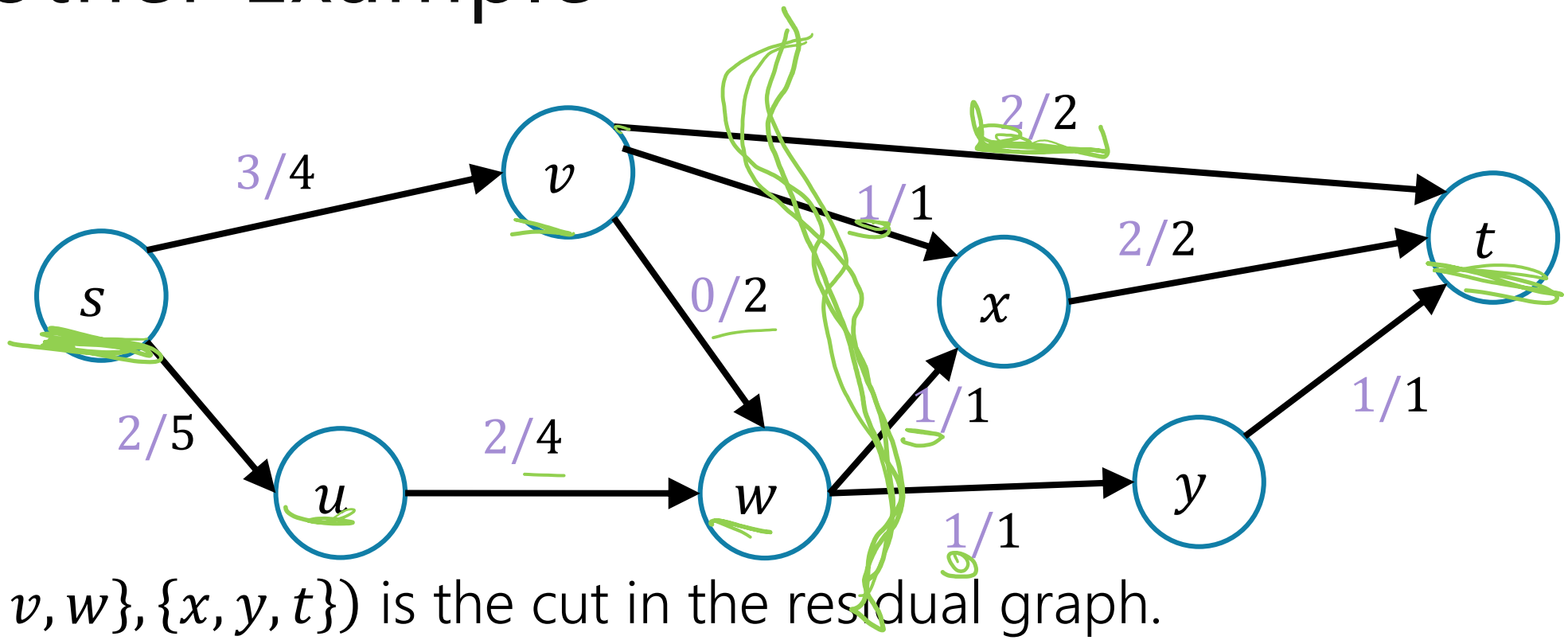


$(\{s, u, v, w\}, \{x, y, t\})$ is the cut in the residual graph.

What edges span?

$(v, x), (v, t), (w, x), (w, y)$

Total capacity? 5. What's the value of the flow? 5

Max Flow-Min Cut Theorem

Max-Flow-Min-Cut Theorem

The value of the maximum flow from s to t is equal to the value of the minimum cut separating s and t.

No proof, but here's some intuition:

Every cut is an upper-bound on the value of the flow (you can't have the total flow value higher than the cut).

Cut will be saturated (i.e. full flow going through all the edges) otherwise residual graph will have another edge. So flow and cut are equal values.

So What?

Max-flow and min-cut are each interesting algorithmic problems.

They were first studied in the 1950s

The U.S. military wanted to know how much the Soviets could ship on their rail network.

And also which rail lines they would target to *disrupt* the network.

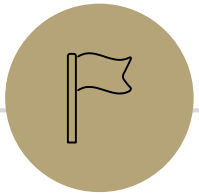
So What?

Great quick check for if you've found the maximum flow (or min-cut).
Check the other and see if the value is the same!

We'll see examples next time of max-flow used for modeling. In those cases the min-cut can be interpreted as a "barrier" to a good assignment.

It's also a nice example of **duality**

If you know what a "dual linear program" is – flows and cuts are dual LPs.



Some Practice

Write an LP for finding the biggest flow

Let $c(e)$ be the capacity on edge e

Write an LP for finding the biggest flow

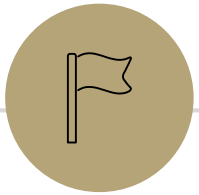
Let $c(e)$ be the capacity on edge e

Max $\sum_{e:e \text{ enters } t} x_e$

Subject to:

$\sum_{e:e \text{ enters } u} x_e = \sum_{e:e \text{ leaves } u} x_e$ for every u other than s and t

$0 \leq x_e \leq c(e)$ for every edge e



Speeding Up Ford-Fulkerson

Speeding Up Ford-Fulkerson

Ford-Fulkerson is only slow if f is big and we keep doing very small updates.

If instead of finding **any** s, t path you find a particular one (the one with the fewest edges, or the one that will lead to the biggest increase) you can guarantee you'll do fewer iterations.

- Each iteration will take a little longer to find the good path, but the tradeoff is often worth it.
- In particular some algorithms end up with running times independent of f.

In practice, Ford-Fulkerson is usually fine!

The fastest-known (theoretically) is Orlin's algorithm: running time $O(VE)$.