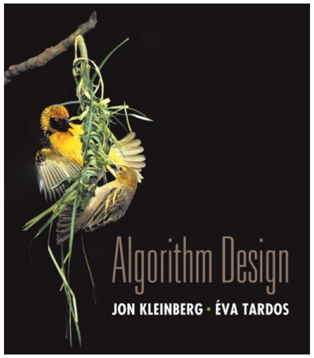
# CSE 417

## Chapter 4: Greedy Algorithms




Many Slides by Kevin Wayne.

Greed is good. Greed is right. Greed works. Greed clarifies, cuts through, and captures the essence of the evolutionary spirit.

- *Gordon Gecko (Michael Douglas)*

# Intro: Coin Changing

# Coin Changing

Goal. Given currency denominations: 1, 5, 10, 25, 100, give change to customer using *fewest* number of coins.

Ex: 34¢

Algorithm is "Greedy": One large coin better than two or more smaller ones

Cashier's algorithm. At each step, give the *largest* coin valued ≤ **the amount to be paid.**

Ex: $2.89

# Coin-Changing: Does Greedy Always Work?

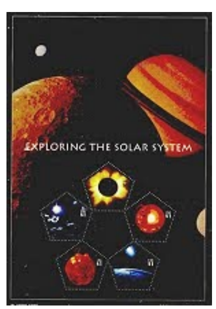
Observation. Greedy is sub-optimal for US *postal* denominations: 1, 10, 21, 34, 70, 100, 350, 1225, 1500.

Counterexample. 140¢.

- Greedy: 100, 34, 1, 1, 1, 1, 1, 1.
- Optimal: 70, 70.

Algorithm is "Greedy", but also short-sighted – attractive choice now may lead to dead ends later.

Correctness is key!

# Outline & Goals

"Greedy Algorithms"
- what they are

Pros
- intuitive
- often simple
- often fast

Cons
- often incorrect!

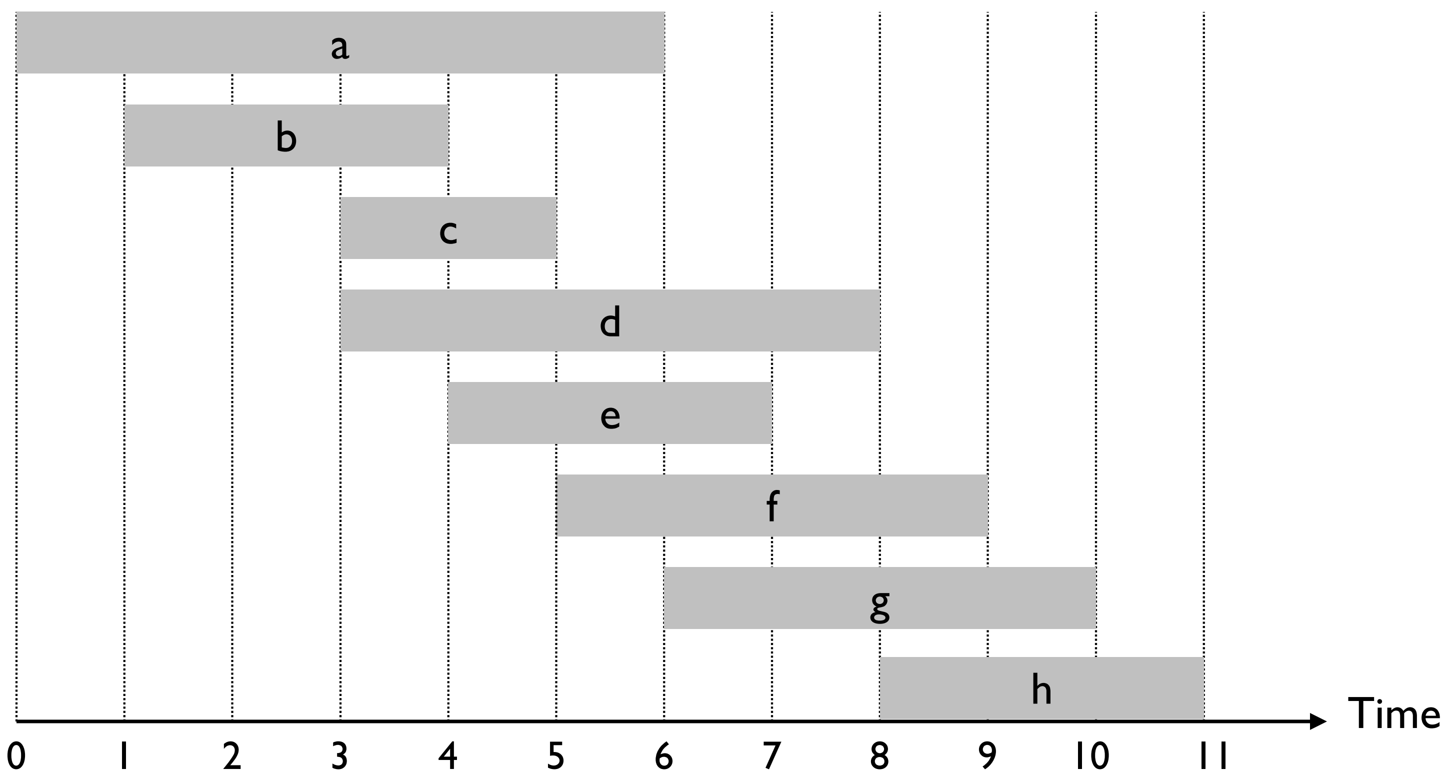
Proofs are crucial. 3 (of many) techniques:
- stay ahead
- structural
- exchange arguments

# 4.1 Interval Scheduling

## Proof Technique 1: “greedy stays ahead”

## Interval scheduling.

- Job j starts at $s_j$ and finishes at $f_j$.
- Two jobs compatible if they don't overlap.
- Goal: find max size subset of mutually compatible jobs.



Greedy template. Consider jobs in some order. Take next job provided it's compatible with the ones already taken.

- What order?
- Does that give best answer?
- Why or why not?
- Does it help to be greedy about order?

# Interval Scheduling: Greedy Algorithms

**Greedy template.** Consider jobs in some order. Take each job provided it's compatible with the ones already taken.

[Earliest start time] Order jobs by ascending start time $s_j$

[Earliest finish time] Order jobs by ascending finish time $f_j$

[Shortest interval] Order jobs by ascending interval length $f_j - s_j$

[Longest Interval] Reverse of the above

[Fewest conflicts] For each job j, let $c_j$ be the count the number of jobs in conflict with j. Order jobs by ascending $c_j$

## Can You Find Counterexamples?

E.g., Longest Interval:

Others?:

# Interval Scheduling: Greedy Algorithms

**Greedy template.** Consider jobs in some order. Take each job provided it's compatible with the ones already taken.

breaks earliest start time

breaks shortest interval

breaks fewest conflicts

Greedy algorithm. Consider jobs in *increasing order of finish time*. Take each job provided it's compatible with the ones already taken.

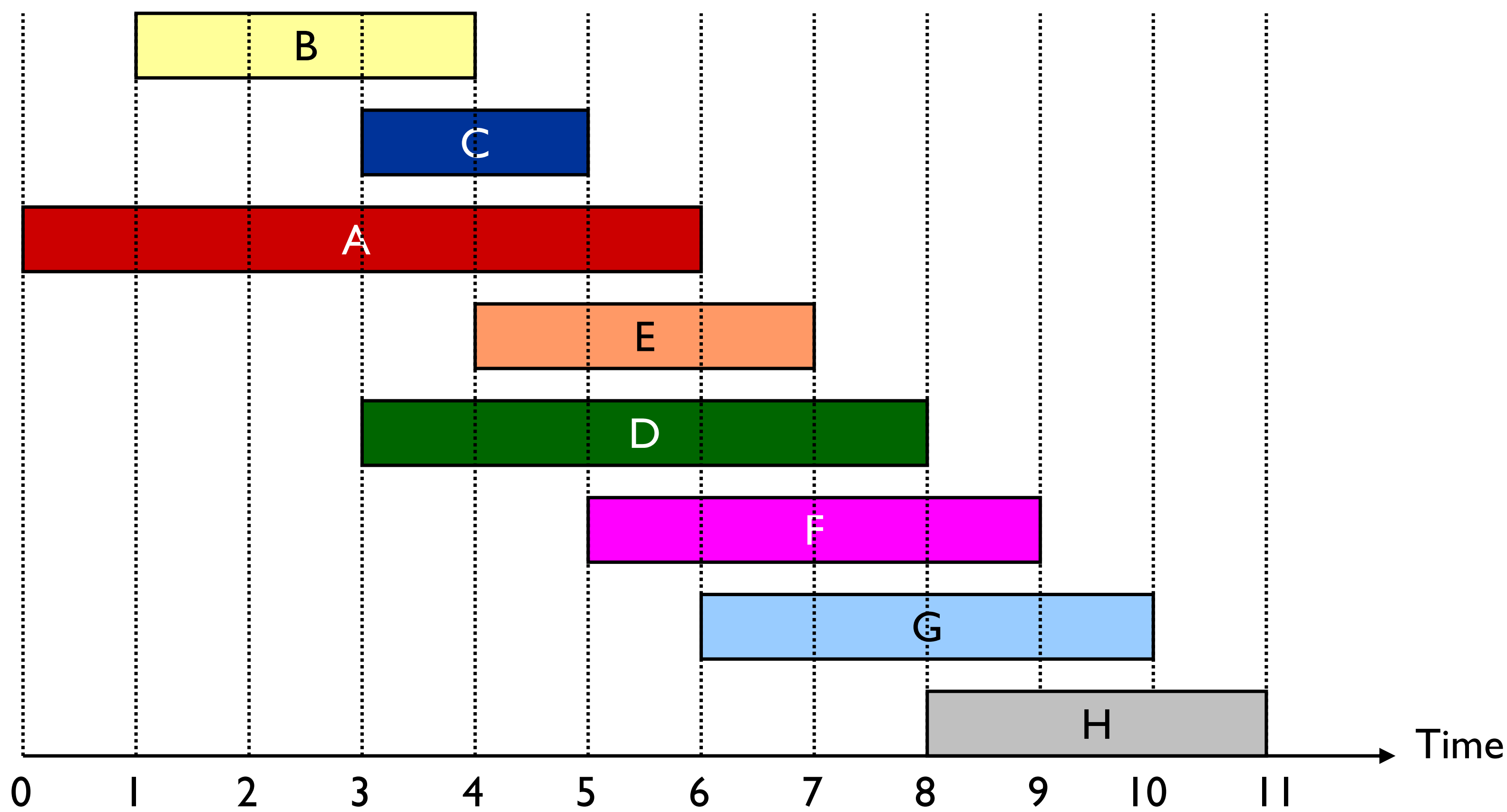
jobs selected

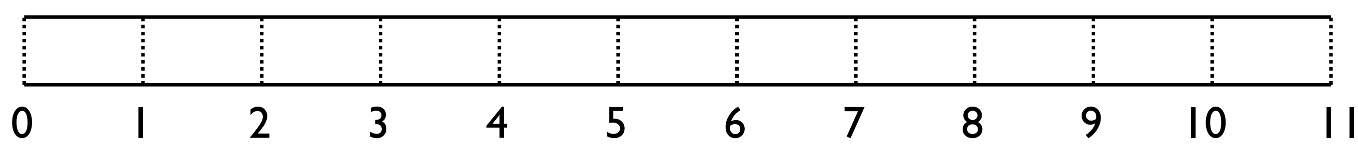
```
Sort jobs by finish times so that
f₁ ≤ f₂ ≤ ... ≤ fₙ.

A ← ϕ
for j = 1 to n {
   if (job j compatible with A)
      A ← A ∪ {j}
}
return A
```

Implementation. O(n log n).

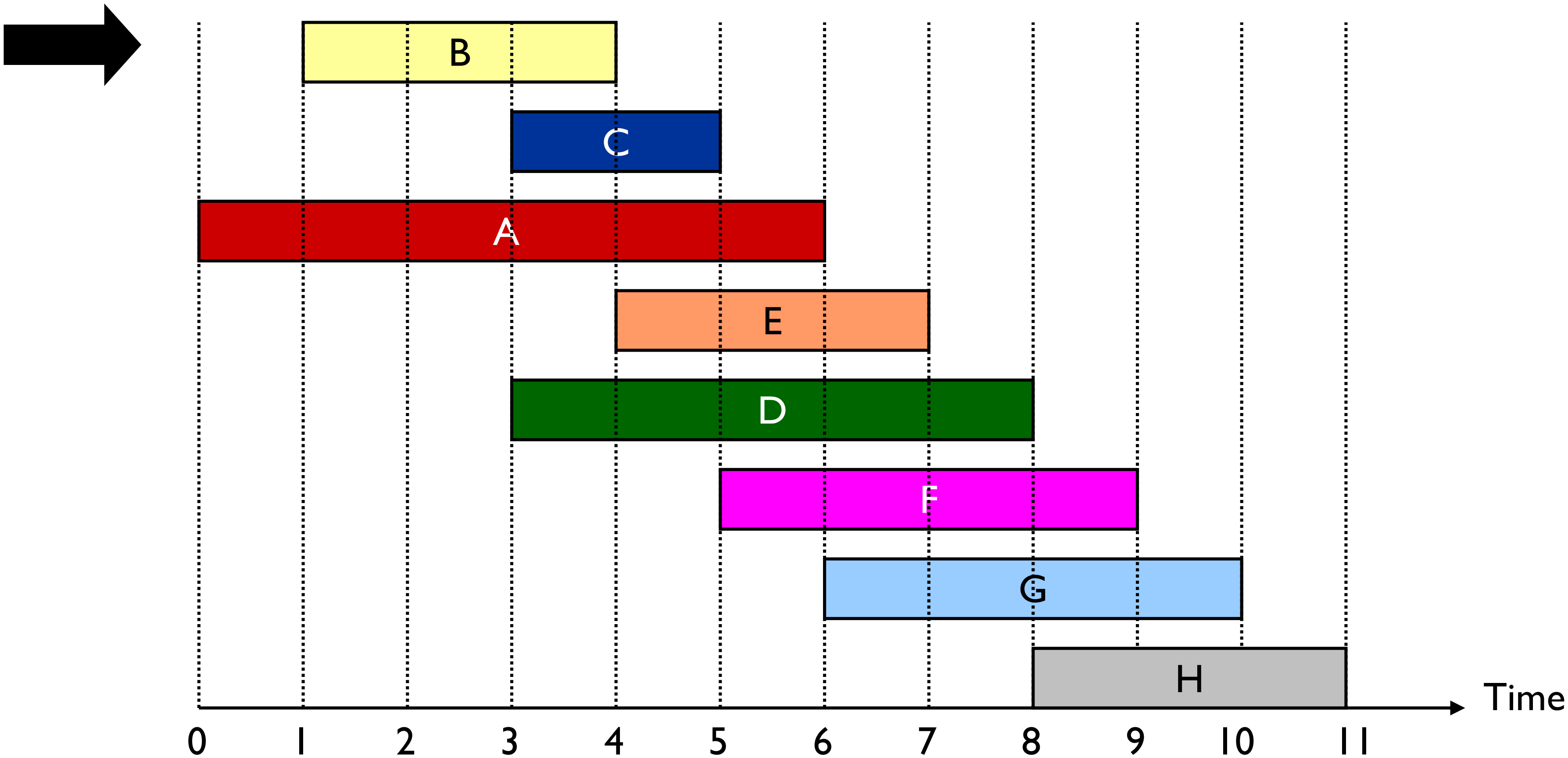
- Remember job j* that was added last to A.
- Job j is compatible with A if $s_j \geq f_{j^*}$.

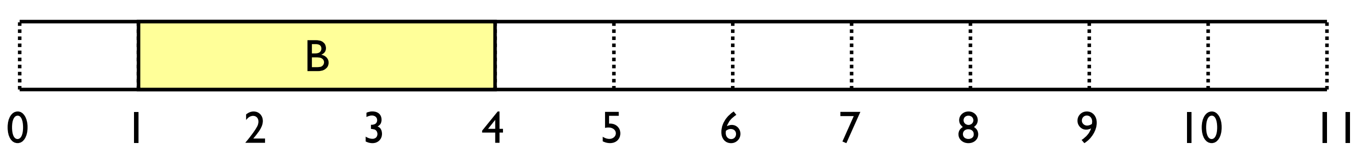
# Interval Scheduling



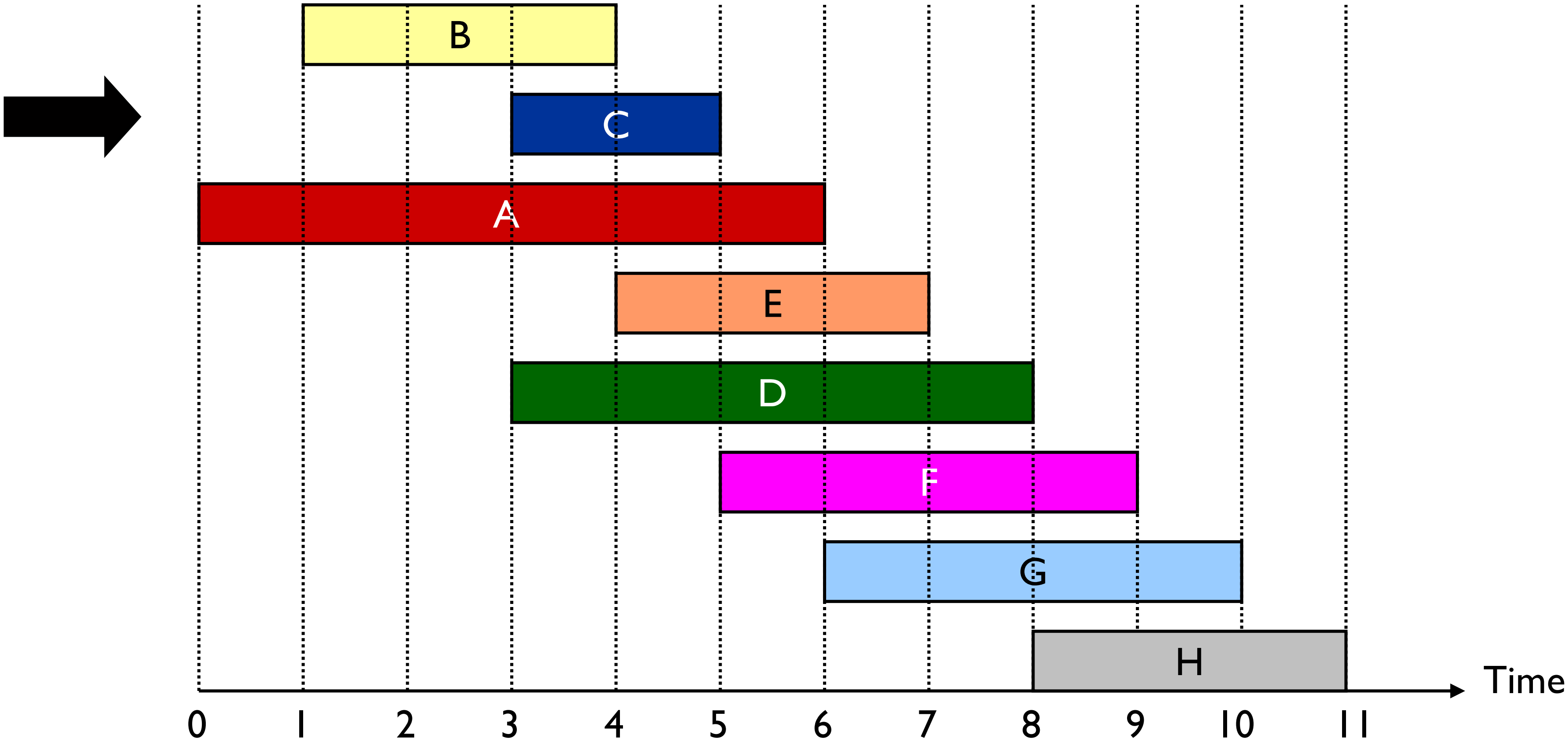



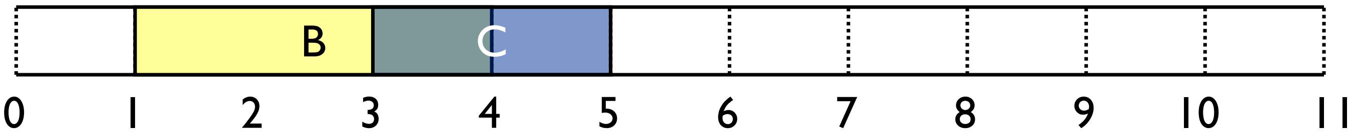
# Interval Scheduling



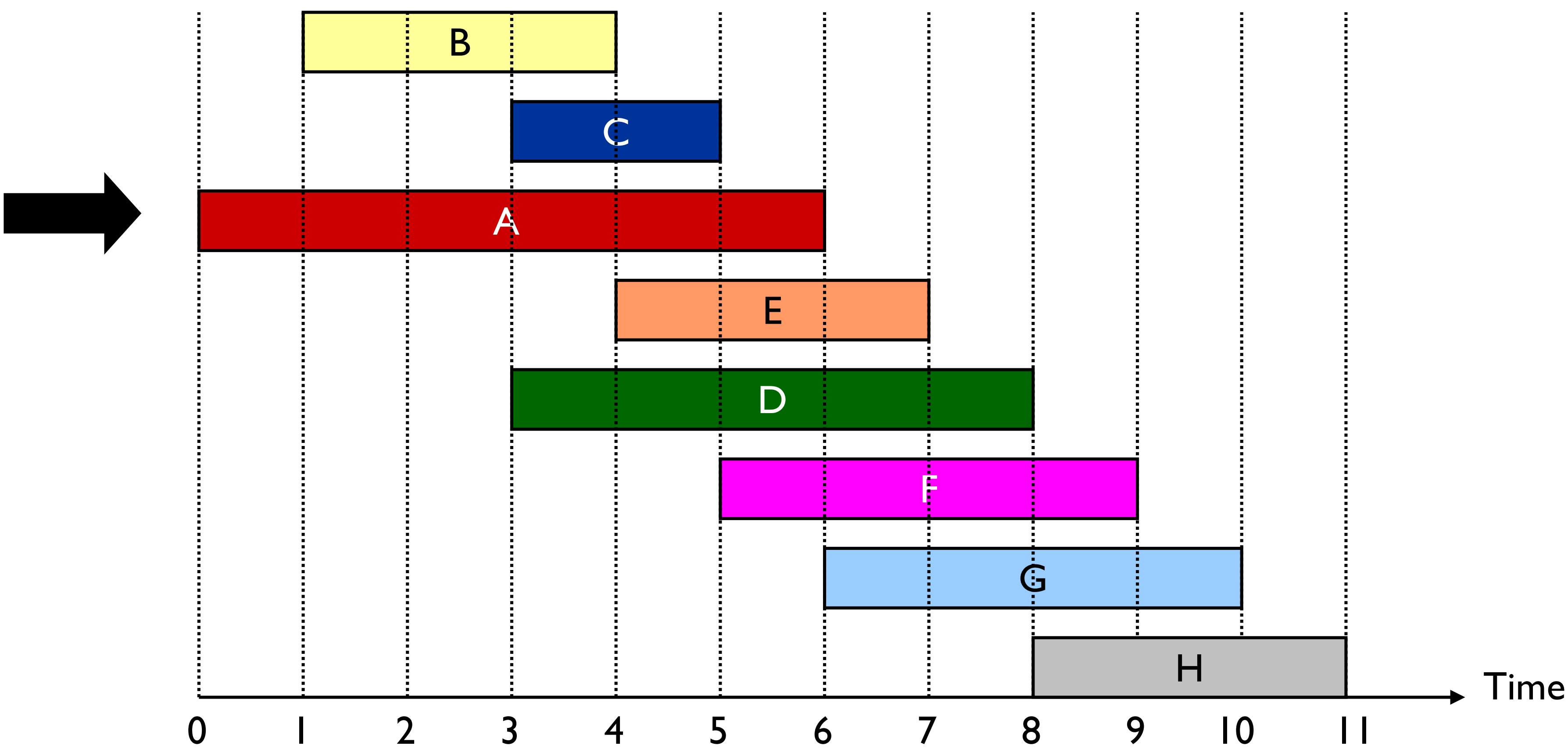



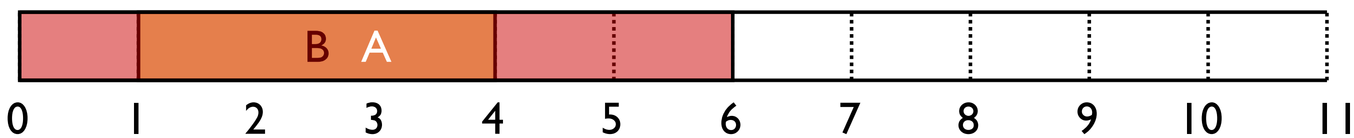
# Interval Scheduling



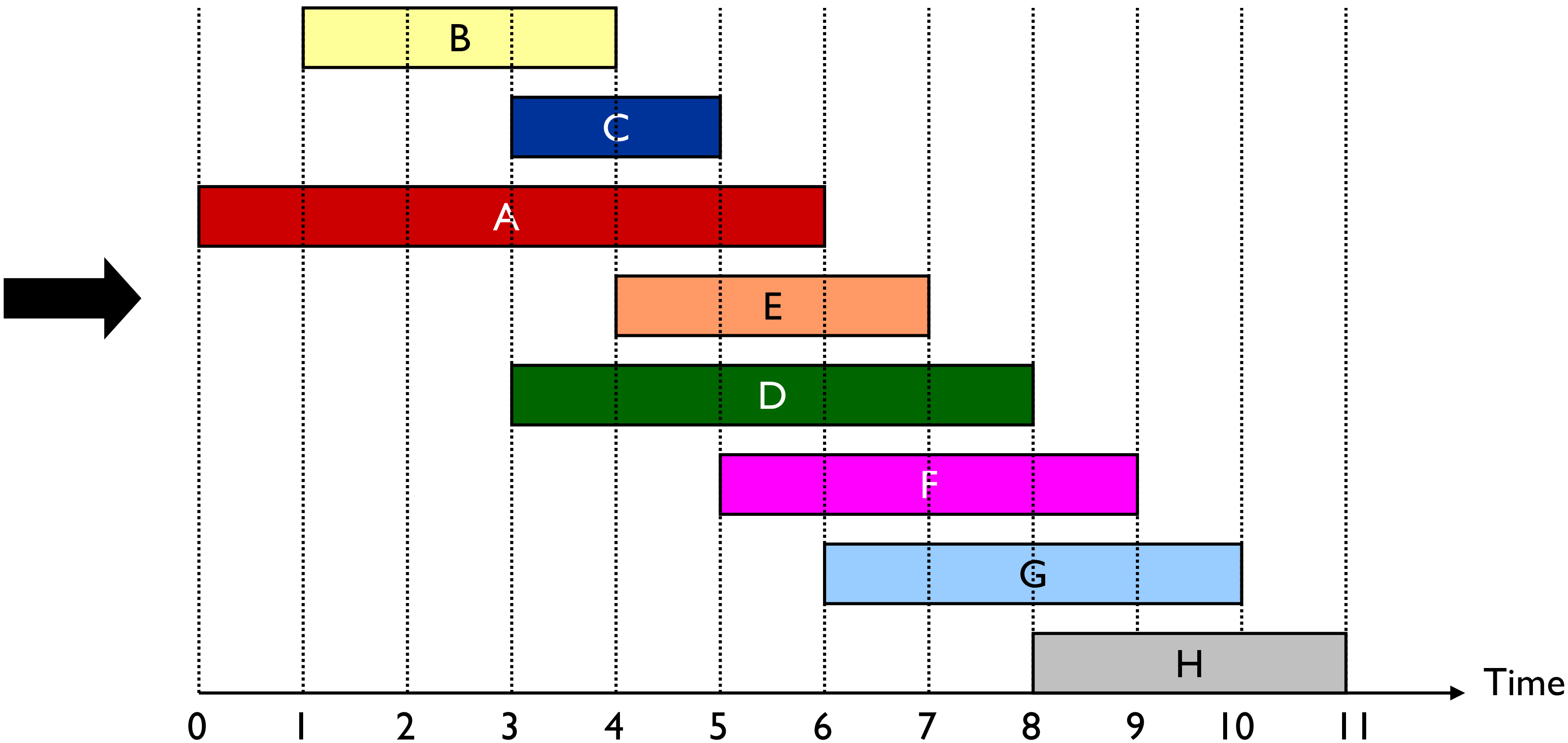



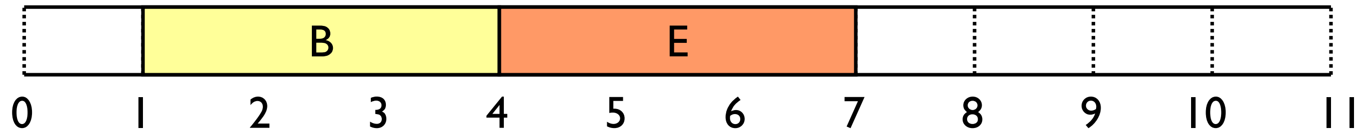
# Interval Scheduling



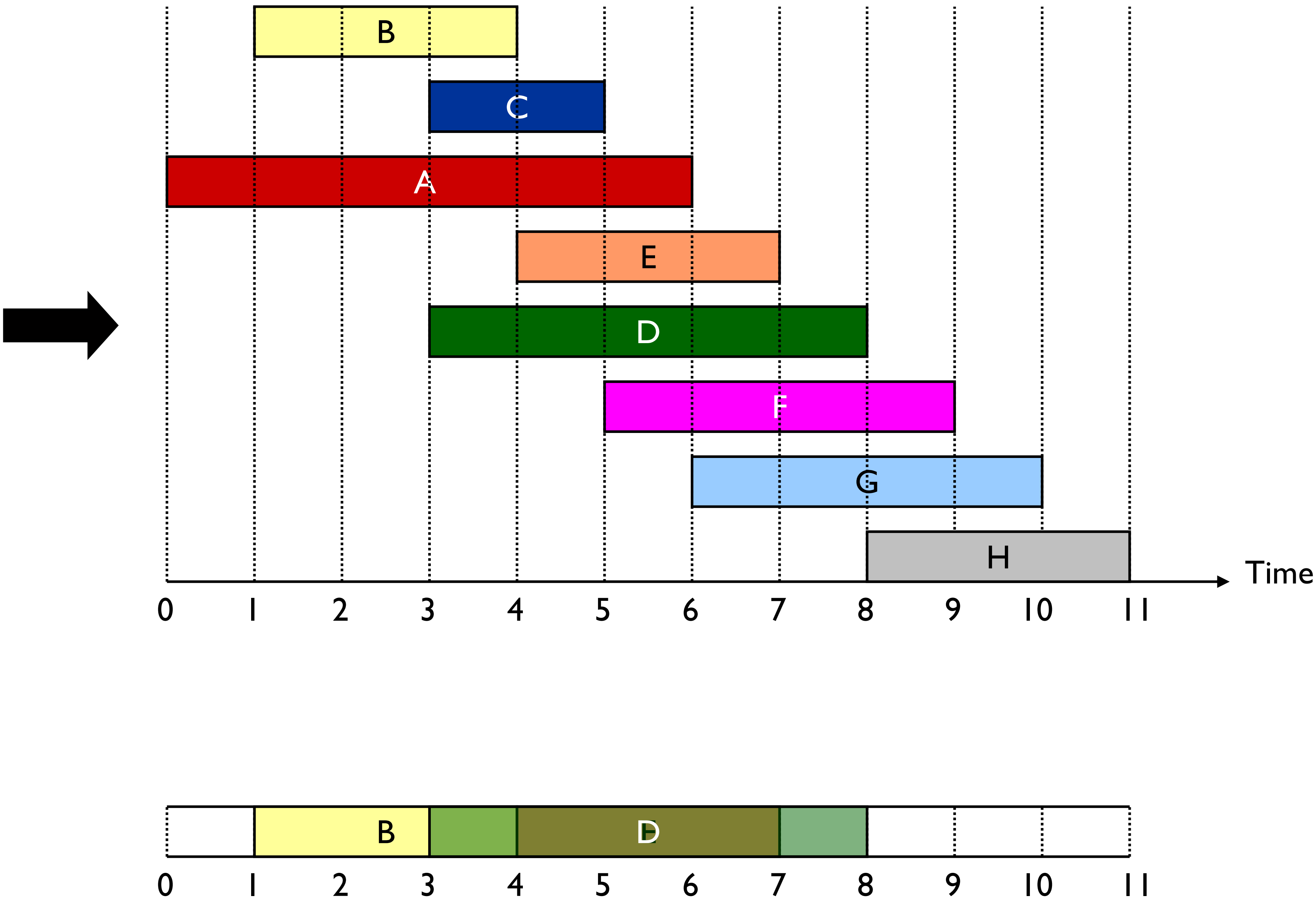



# Interval Scheduling

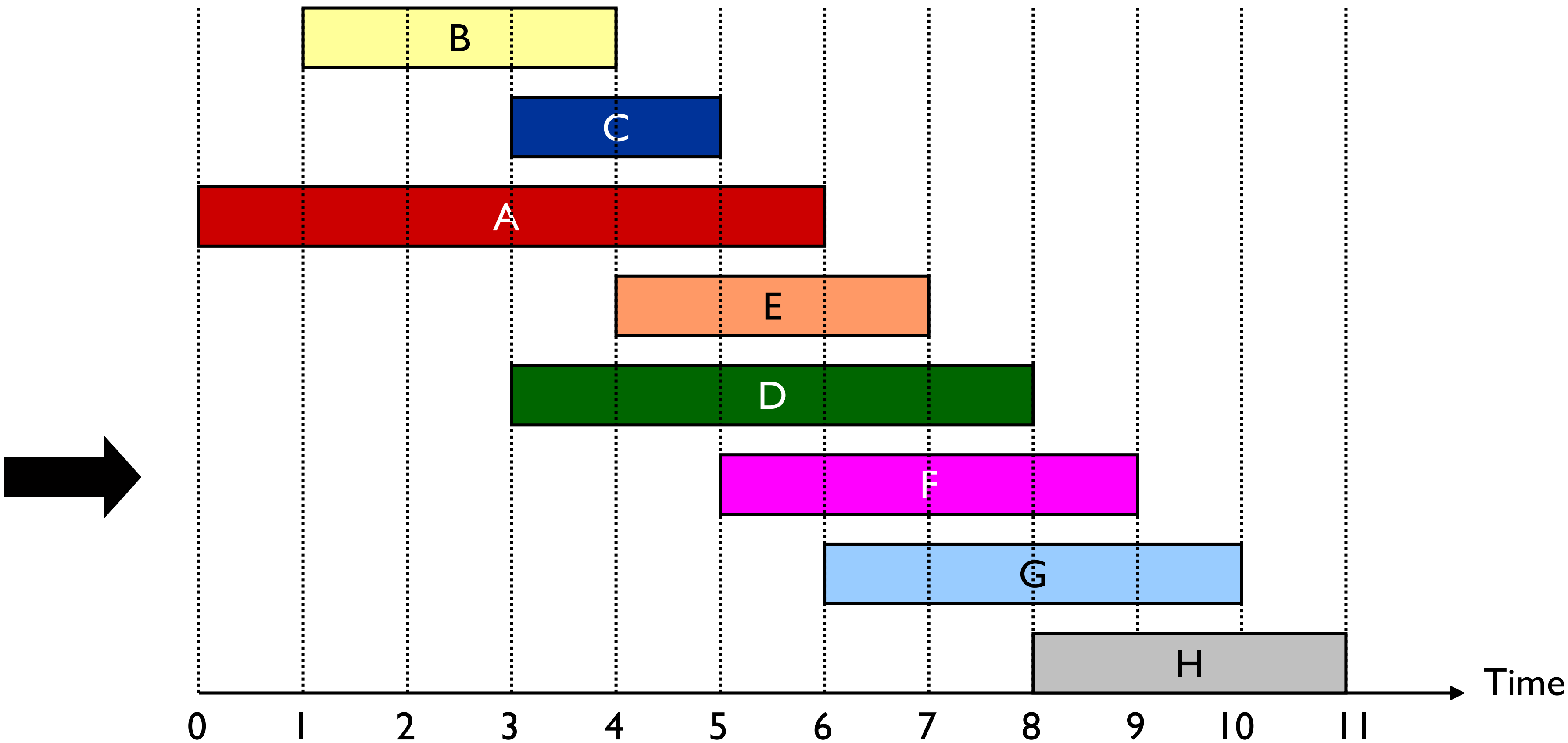


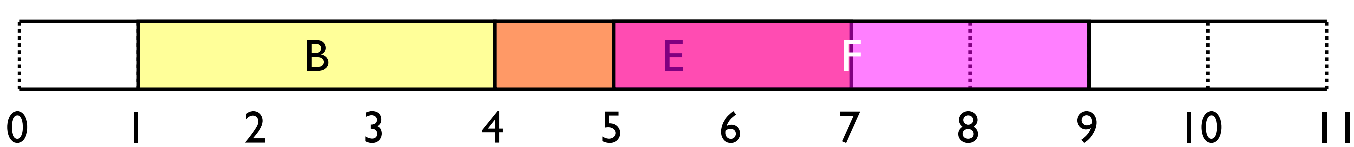



# Interval Scheduling

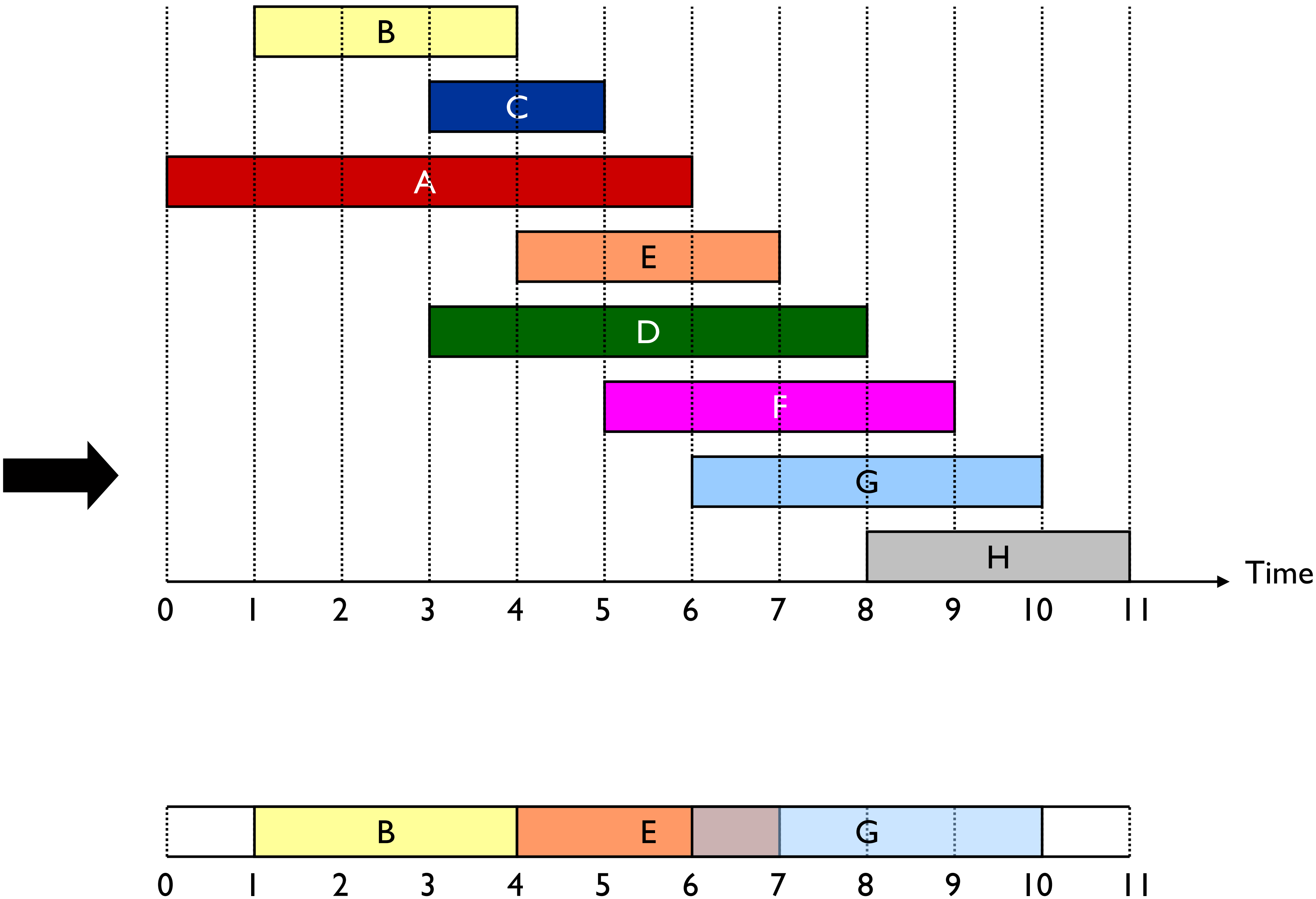


# Interval Scheduling

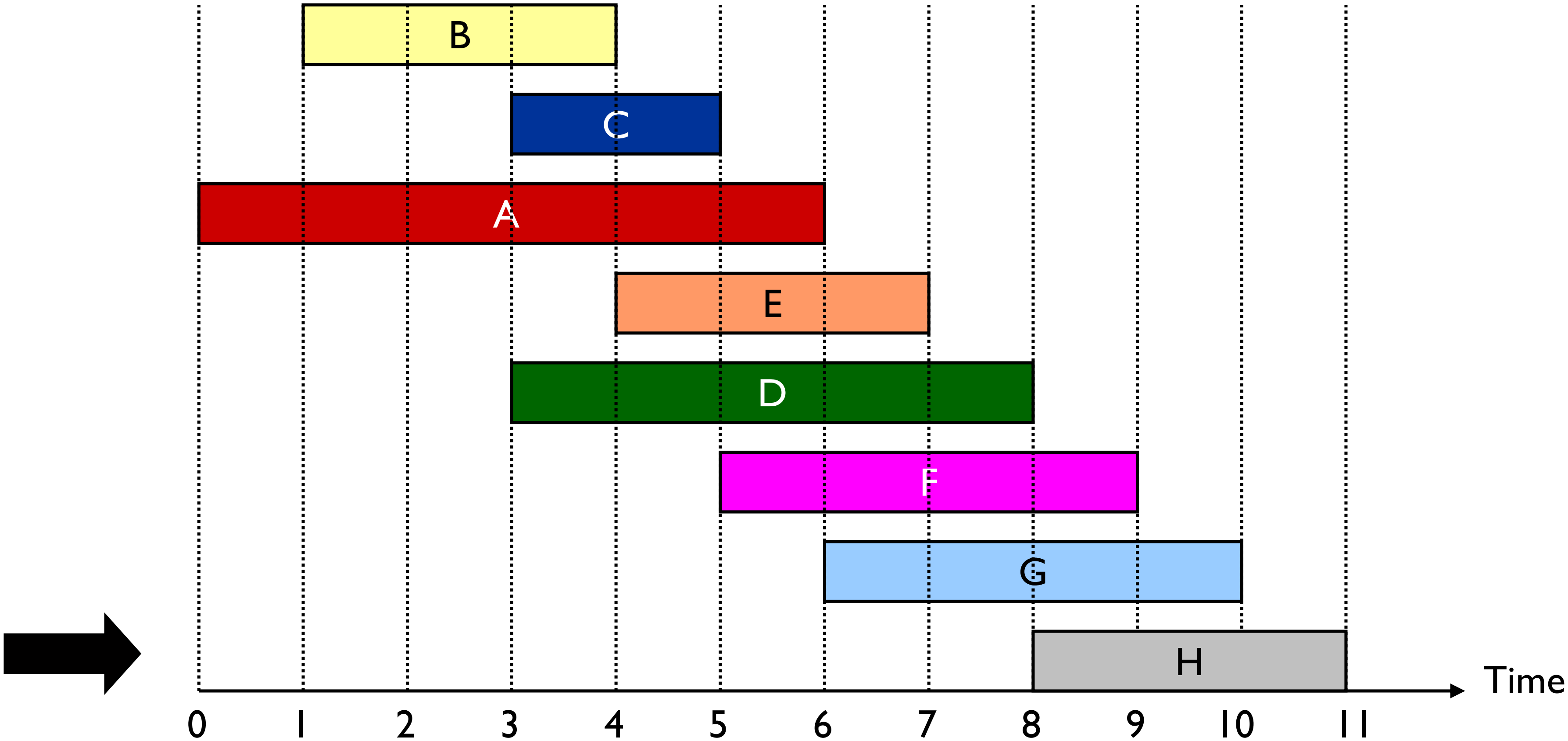


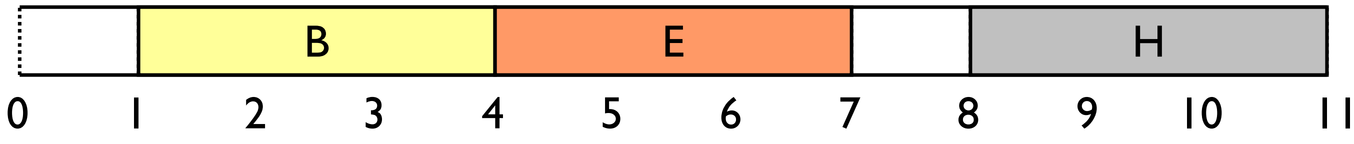



# Interval Scheduling



# Interval Scheduling



## Interval Scheduling: Correctness

**Theorem.** *Earliest Finish First* Greedy algorithm is optimal.

**Pf.** ("greedy stays ahead")

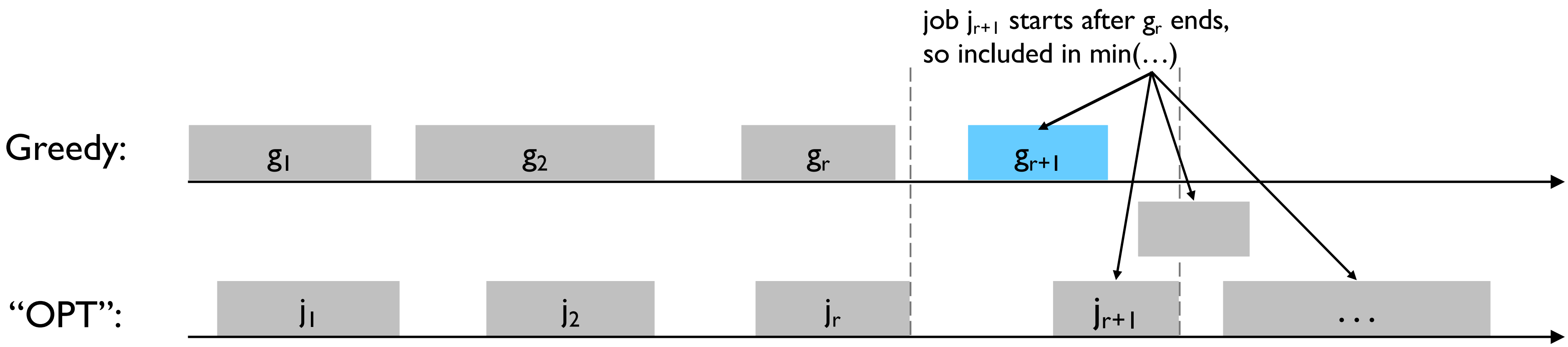
Let $g_1, \ldots g_k$ be greedy's job picks, $j_1, \ldots j_m$ those in some optimal solution

Show $f(g_r) \leq f(j_r)$ by induction on r.

Basis: $g_1$ chosen to have min finish time, so $f(g_1) \leq f(j_1)$

Ind: $f(g_r) \leq f(j_r) \leq s(j_{r+1})$, so $j_{r+1}$ is among the candidates considered by greedy when it picked $g_{r+1}$, & it picks min finish, so $f(g_{r+1}) \leq f(j_{r+1})$

Similarly, $k \geq m$, else $j_{k+1}$ is among (nonempty) set of candidates for $g_{k+1}$



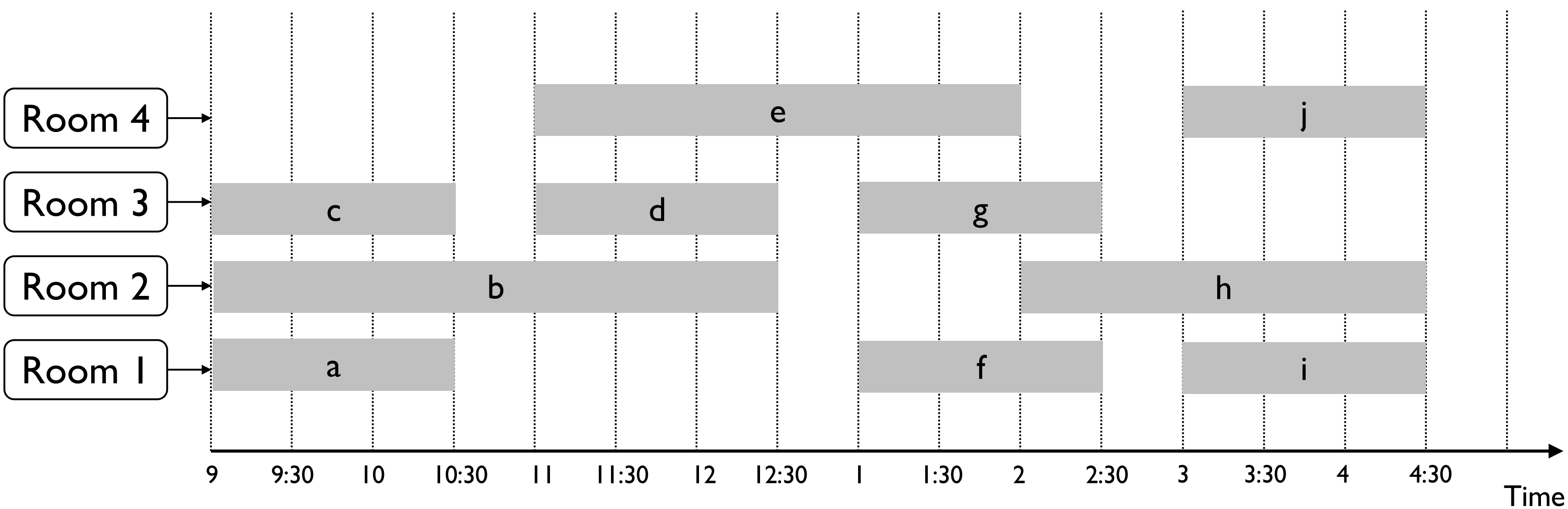
# 4.1 Interval Partitioning

Proof Technique 2: "Structural"

## Interval partitioning.

- Lecture j starts at $s_j$ and finishes at $f_j$.
- Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

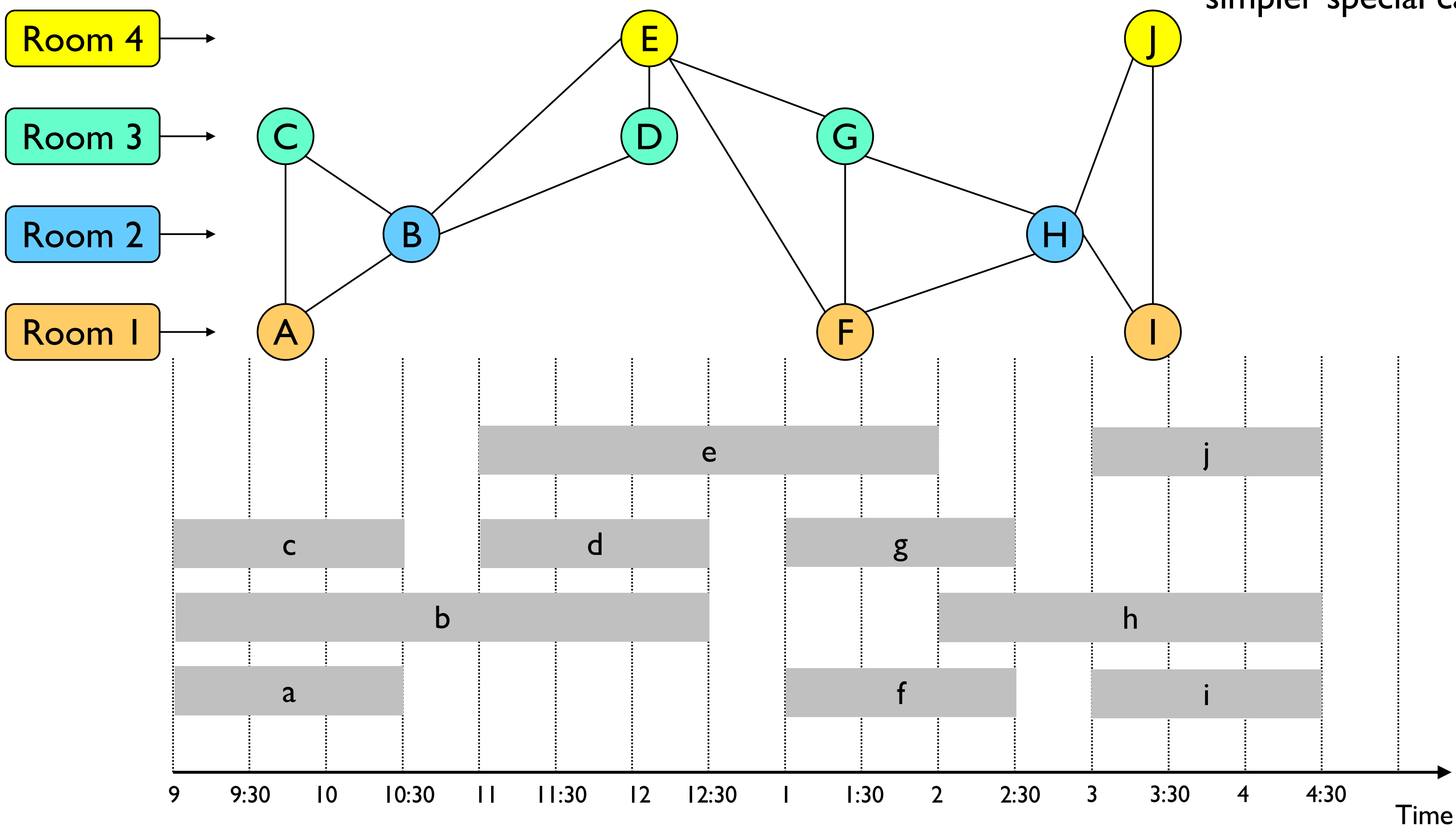
Ex: This schedule uses 4 classrooms to schedule 10 lectures.



# Interval Partitioning as Interval Graph Coloring

Vertices = classes;
Edges = conflicting class pairs;
Different colors = different assigned rooms

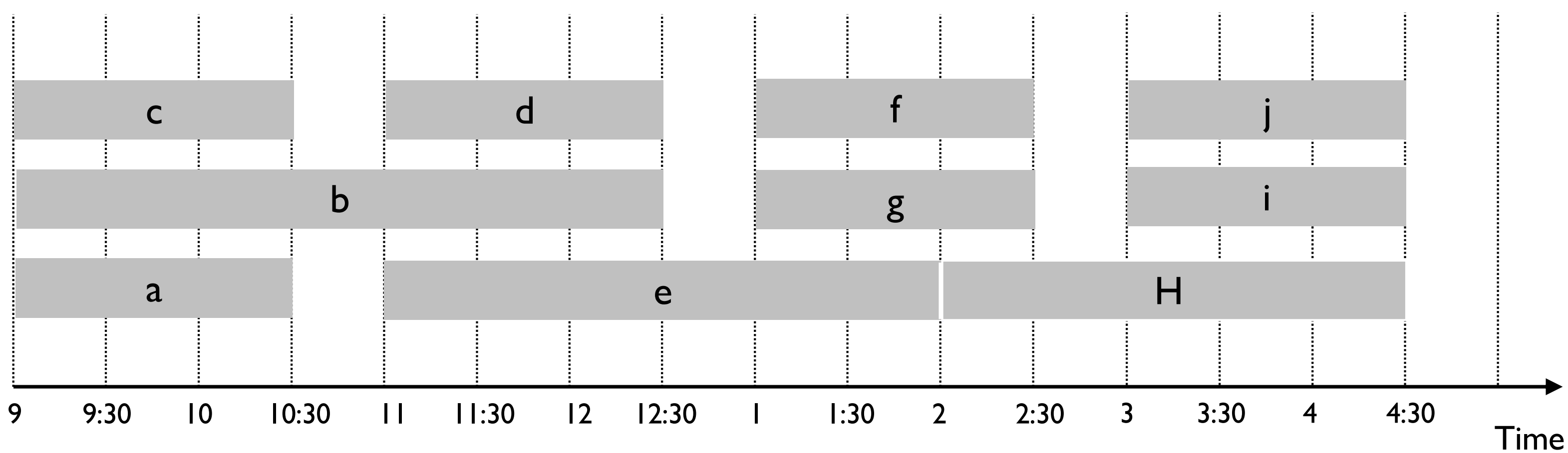
Note: graph coloring is very hard in general, but graphs corresponding to interval intersections are a much simpler special case.



## Interval partitioning.

- Lecture j starts at $s_j$ and finishes at $f_j$.
- Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

Ex: Same classes, but this schedule uses only 3 rooms.



## Interval Partitioning: A "Structural" Lower Bound on Optimal Solution

Def. The depth of a set of open intervals is the maximum number that contain any given time.

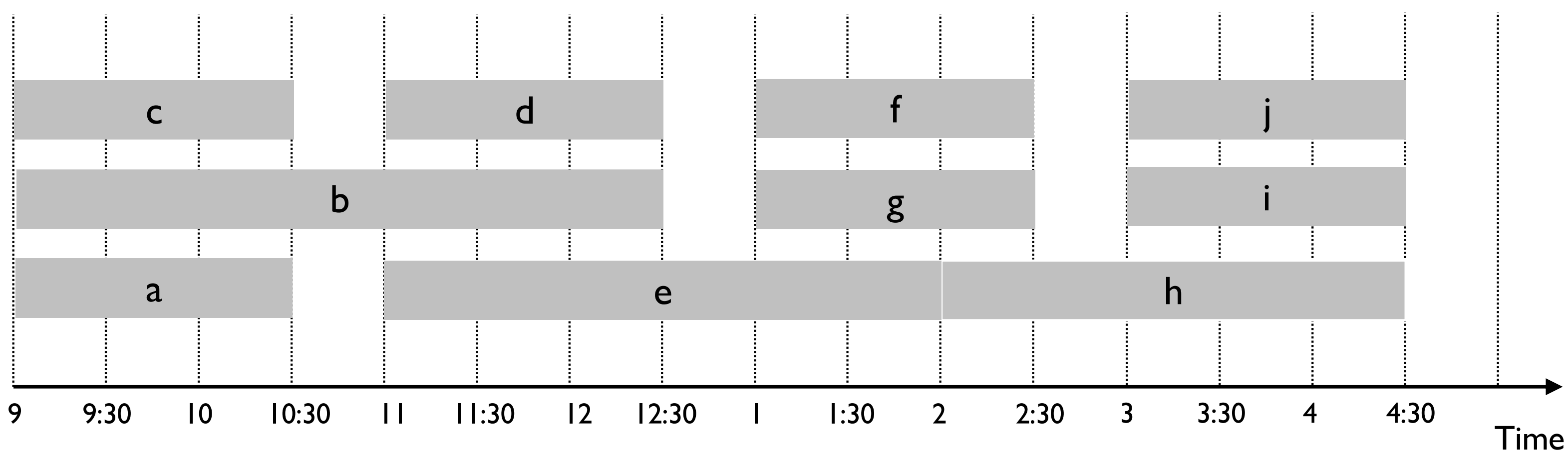
no collisions at ends

Key observation. Number of classrooms needed $\geq$ depth.

Ex: Depth of schedule below = 3 $\Rightarrow$ schedule is optimal.

e.g., a, b, c all contain 9:30

Q. Does a schedule equal to depth of intervals always exist?



# Interval Partitioning: Earliest Start First Greedy Algorithm

Greedy algorithm. Consider lectures *in increasing order of start time*: assign lecture to any compatible classroom.

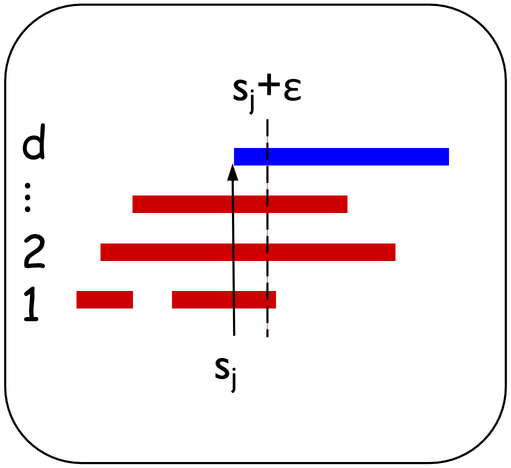
```
Sort intervals by start time so s₁ ≤ s₂ ≤ ... ≤ sₙ.
d ← 0   ← number of allocated classrooms

for j = 1 to n {
   if (lect j is compatible with some room k, 1≤k≤d)
      schedule lecture j in classroom k
   else
      allocate a new classroom d + 1
      schedule lecture j in classroom d + 1
      d ← d + 1
}
```

Implementation? Run-time?
Exercises

Observation. Earliest Start First Greedy algorithm never schedules two incompatible lectures in the same classroom.

Theorem. Earliest Start First Greedy algorithm is optimal.
Pf (exploit structural property).

- Let d = number of rooms the greedy algorithm allocates.
- Classroom d opened when we needed to schedule a job, say j, incompatible with all d-1 previously used classrooms.
- We sorted by start time, so all incompatibilities are with lectures starting no later than $s_j$.
- So, d lectures overlap at time $s_j + \varepsilon$, i.e. depth $\geq$ d
- "Key observation" on earlier slide $\Rightarrow$ all schedules use $\geq$ depth rooms, so d = depth and greedy is optimal



# 4.2 Scheduling to Minimize Lateness

Proof Technique 3: “Exchange” Arguments

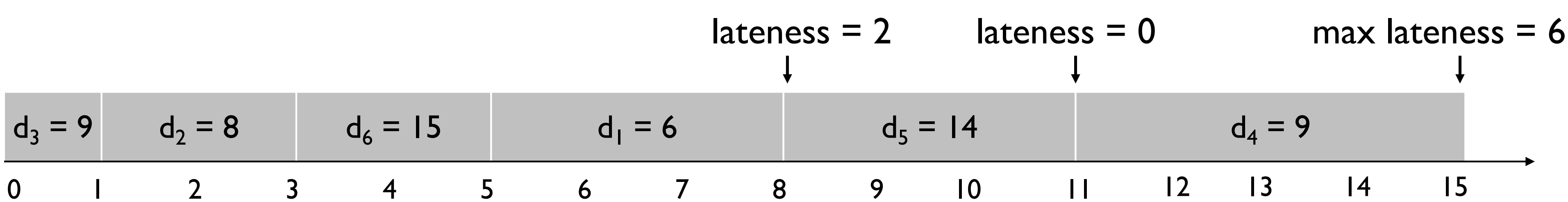
# Scheduling to Minimize Lateness

## Minimizing lateness problem.

- Single resource processes one job at a time.
- Job j requires $t_j$ units of processing time & is due at time $d_j$.
- If j starts at time $s_j$, it finishes at time $f_j = s_j + t_j$.
- Lateness: $\ell_j = \max \{ 0, f_j - d_j \}$.
- Goal: schedule all to minimize max lateness $L = \max \ell_j$.

## Ex:

| j | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $t_j$ | 3 | 2 | 1 | 4 | 3 | 2 |
| $d_j$ | 6 | 8 | 9 | 9 | 14 | 15 |



## Minimizing Lateness: Greedy Algorithms

Greedy template. Consider jobs in some order.

[Shortest job first]
Consider jobs in ascending order of processing time $t_j$.

[Earliest deadline first]
Consider jobs in ascending order of deadline $d_j$.

[Smallest slack first]
Consider jobs in ascending order of *slack* $d_j - t_j$.

Greedy template. Consider jobs in some order.

[Shortest job first] Consider in ascending order of processing time $t_j$.

| | 1 | 2 |
|---|---|---|
| $t_j$ | 1 | 10 |
| $d_j$ | 100 | 10 |

counterexample

[Smallest slack] Consider in ascending order of slack $d_j - t_j$.

| | 1 | 2 |
|---|---|---|
| $t_j$ | 1 | 10 |
| $d_j$ | 2 | 10 |

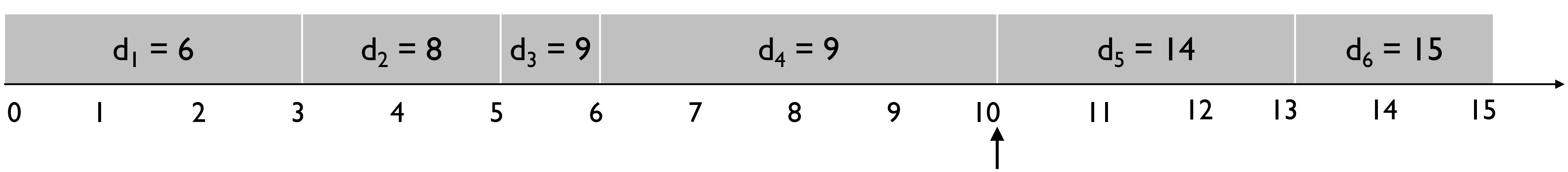
counterexample

# Minimizing Lateness: Greedy Algorithm

**Greedy algorithm.** Earliest deadline first.

```
Sort n jobs by deadline so that d_1 ≤ d_2 ≤ … ≤ d_n

t ← 0
for j = 1 to n
   // Assign job j to interval [t, t + t_j]:
   s_j ← t, f_j ← t + t_j
   t ← t + t_j
output intervals [s_j, f_j]
```

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $t_j$ | 3 | 2 | 1 | 4 | 3 | 2 |
| $d_j$ | 6 | 8 | 9 | 9 | 14 | 15 |




max lateness = 1 (also true if jobs 3 & 4 flipped)

# Proof Strategy

A *schedule* is an ordered list of jobs

Suppose $S_1$ is any schedule & let G be the/a the greedy algorithm's schedule

To show: Lateness($S_1$) ≥ Lateness(G)

Idea: find simple changes that successively transform $S_1$ into other schedules increasingly like G, each better (or at least no worse) than the last, until we reach G. I.e.

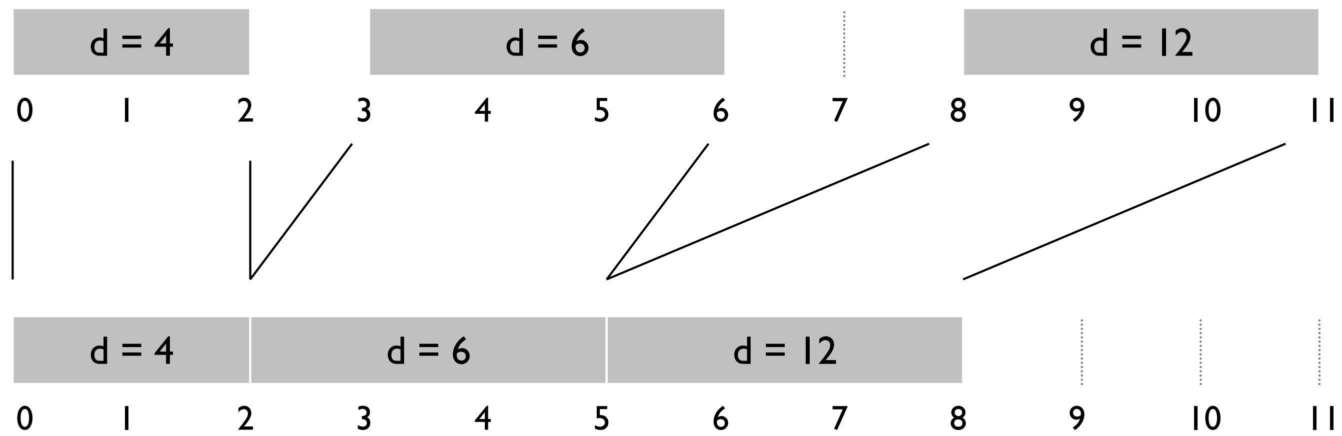
Lateness($S_1$) ≥ Lateness($S_2$) ≥ Lateness($S_3$) ≥ … ≥ Lateness(G)

If it works for *any* $S_1$, it will work for an *optimal* $S_1$, so G is optimal

HOW?: *exchange* pairs of jobs

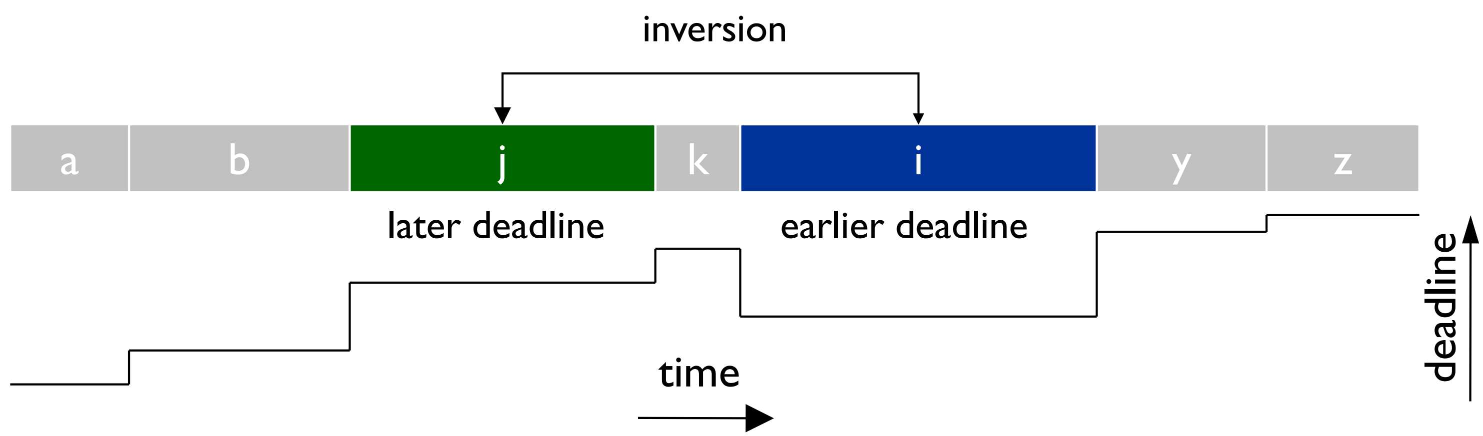
## Notes:

1. There is an optimal schedule with no idle time.




2. The greedy schedule has no idle time.

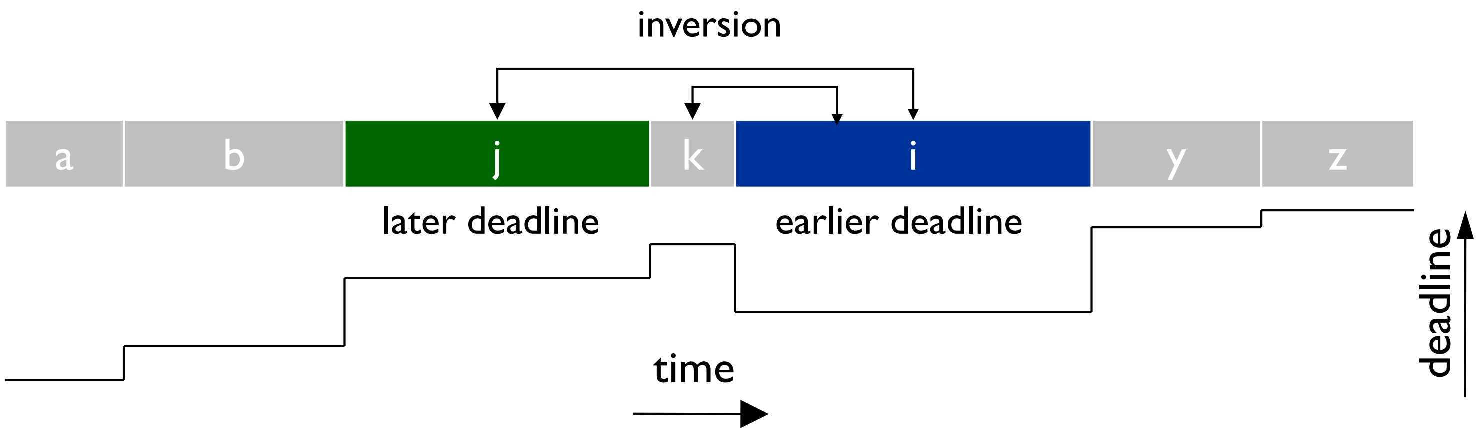
Def. An *inversion* in schedule S is a pair of jobs i and j s.t.: deadline i < deadline j but j scheduled before i.




- Greedy schedule has no inversions.
- Claim: If a schedule has an inversion, it has an adjacent inversion, i.e., a pair of inverted jobs scheduled consecutively.

  (Pf: If j & i aren't consecutive, then look at the job k scheduled right after j. If $d_k < d_j$, then (j,k) is a consecutive inversion; if not, then (k,i) is an inversion, & nearer to each other - repeat.)

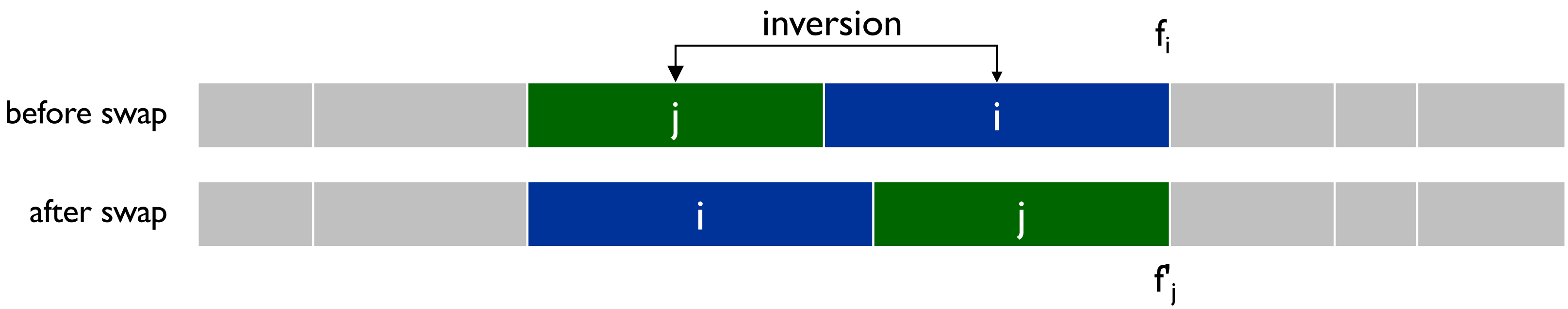
Def. An *inversion* in schedule S is a pair of jobs i and j s.t.: deadline i < deadline j but j scheduled before i.




- Claim: Swapping an *adjacent* inversion reduces tot # invs by 1 (exactly)

  Pf: Let i,j be an adjacent inversion. For any pair (p,q), inversion status of (p,q) is unchanged by i↔j swap unless {p, q} = {i, j}, and the i,j inversion is removed by that swap.

**Def.** An *inversion* in schedule S is a pair of jobs i and j s.t.: deadline i < j but j scheduled before i.




(j had later deadline, so is less tardy than i was)

**Claim.** *Swapping* two adjacent, inverted jobs does not increase the max lateness.

**Pf.** Let $\ell$ / $\ell'$ be the lateness before / after swap, resp.

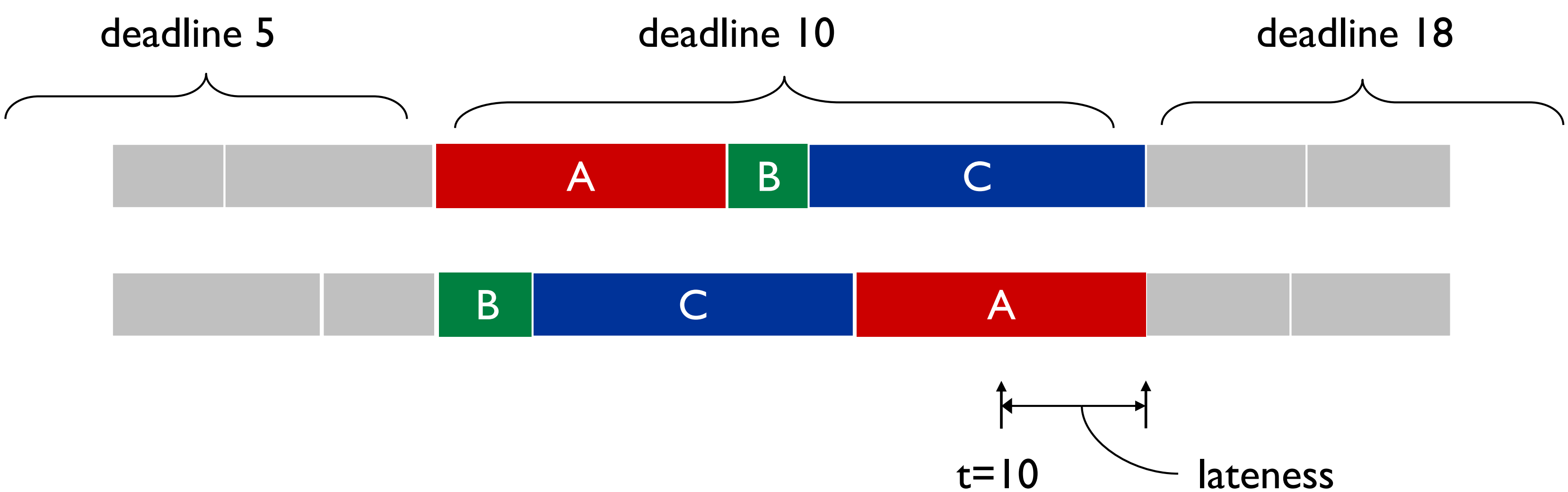
- $\ell'_k = \ell_k$ for all $k \neq i, j$
- $\ell'_i \leq \ell_i$
- If job j is now late:

$$\begin{aligned}
\ell'_j &= f'_j - d_j && \text{(definition)}\\
&= f_i - d_j && (j \text{ finishes at time } f_i)\\
&\leq f_i - d_i && (d_i \leq d_j)\\
&= \ell_i && \text{(definition)}
\end{aligned}$$

only j moves later, but it's no later than i was, so max not increased

Claim. All idle-free, inversion-free schedules S have the same max lateness.

Pf. If S has no inversions, then deadlines of scheduled jobs are monotonically nondecreasing (i.e., increase or stay the same) as we walk through the schedule from left to right. Two such schedules can differ only in the order of jobs with the same deadlines. Within a group of jobs with the same deadline, the max lateness is the lateness of the last job in the group - order within the group doesn't matter.



Theorem. Greedy schedule G is optimal

Pf. Let S* be an optimal schedule with the fewest number of inversions among all optimal schedules

- Can assume S* has no idle time.
- If S* has an inversion, let i-j be an adjacent inversion
- Swapping i and j does not increase the maximum lateness and strictly decreases the number of inversions
  - This contradicts definition of S*
- So, S* has no inversions. Hence Lateness(G) = Lateness(S*)

# Minimizing Lateness

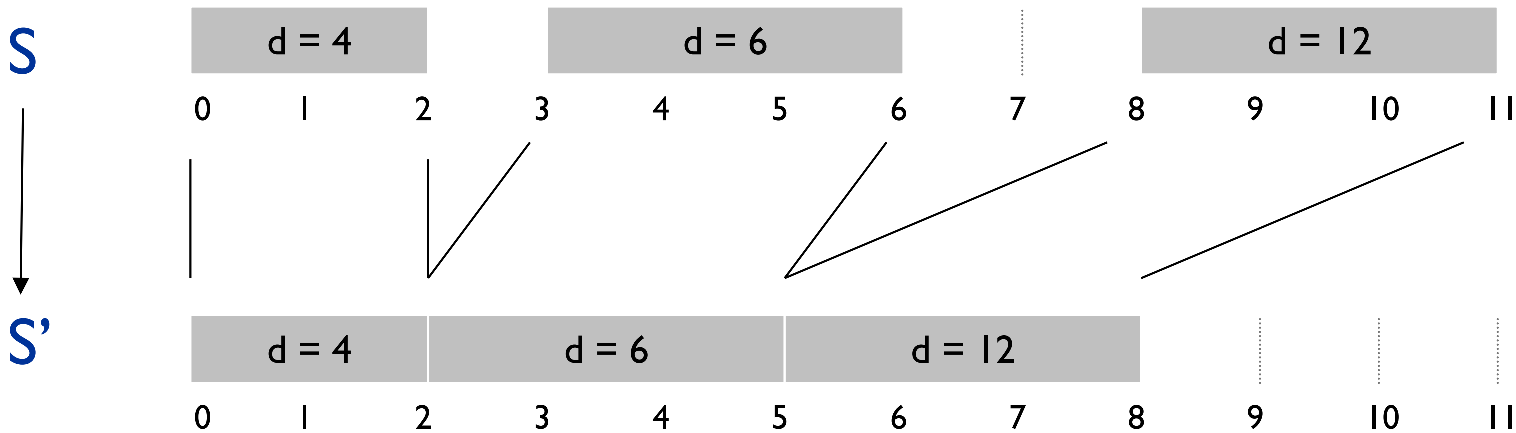
## New (simpler?) proof.

The 7 slides above summarize the correctness proof for the "earliest deadline first" greedy algorithm for "minimizing lateness" as presented in our text and in my lecture on 1/25/19.

The 6 slides below outline an alternative proof that I think is a little simpler. It uses the same core "exchange argument" idea, while avoiding the correct but slightly tangential discussion of multiple jobs with the same deadline. It also feels a little more algorithm-oriented in that it shows how to turn an arbitrary schedule into exactly the greedy schedule.

Students are not required to study this, but may find it useful.

Claim 1: There is an optimal schedule with no idle time.




No job ends later in S' than S, so max lateness in not increased

Henceforth, assume all schedules are idle-free

# Proof Strategy

A *schedule* is an ordered list of jobs. (No idle; only order matters)

Suppose $S_1$ is any schedule & let G be the the greedy algorithm's schedule

To show: Lateness($S_1$) ≥ Lateness(G)

Idea: find simple changes that successively transform $S_1$ into other schedules increasingly like G, each better (or at least no worse) than the last, until we reach G. I.e.

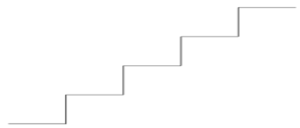
$$\text{Lateness}(S_1) \geq \text{Lateness}(S_2) \geq \text{Lateness}(S_3) \geq \ldots \geq \text{Lateness}(G)$$

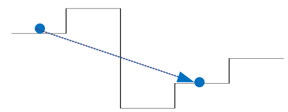
If it works for *any* $S_1$, it will work for an *optimal* $S_1$, so G is optimal

HOW?: *exchange* pairs of jobs

# Minimizing Lateness: Inversions

(Re-)number the jobs in the order that Greedy schedules them. Then a "Schedule" is just permutation of 1..n. E.g.:





G: 1 2 3 4 5 S: 4 5 1 2 3

Def. An *inversion* in schedule S is a pair of jobs i and j s.t. greedy did i before j (i.e., i < j), but S does j before i.

E.g., (4,2) in S above; also (4,1), (5,3), …

Claim 2: If schedule S has an inversion, it has an *adjacent* inversion, i.e., a pair of inverted jobs scheduled consecutively.

Ex: (4,2) are not adjacent, but (5,1) is an adjacent inversion

Pf: An adjacent pair x,y in S is an adjacent *inversion* if y is smaller than x. The sublist of S from j to i must have such an adjacent pair since i is smaller than j. "A walk from high to low must have a 1st step down."

Ex: 4 5 1 2

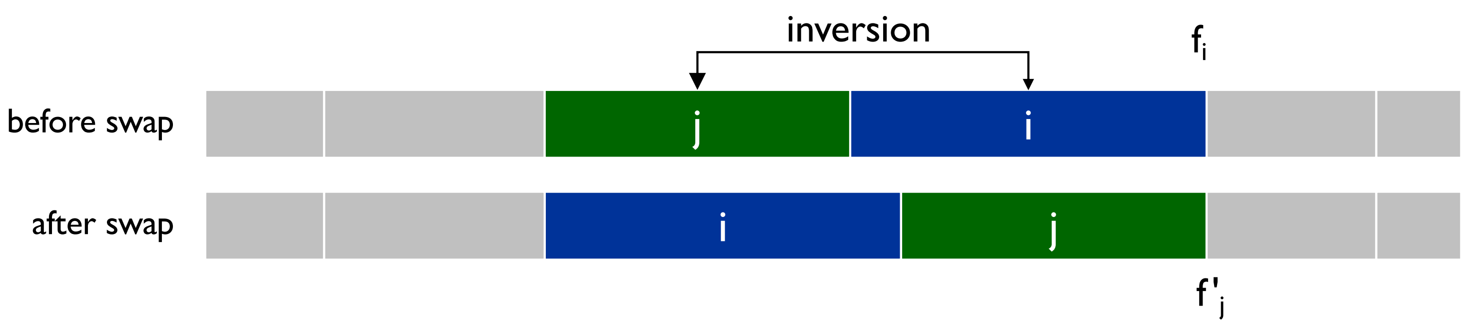
Claim 3: Swapping an *adjacent* inversion reduces the total number of inversions by 1 (exactly)

Pf: To be clear about the defn, since S is just a list of the numbers between 1 and n, in some order, for any p≠q in 1..n, p,q is an inversion ⇔ the larger precedes the smaller in list S. Let i, j be an adjacent inversion. Inversion status of any pair p,q is unchanged by i↔j swap unless {p,q} = {i,j}, and the i,j inversion is removed by that swap. In more detail, if neither p nor q is either i or j, then neither p nor q moves, so status is unchanged. If one of p,q is i or j, say, p≠i, and q=j, then since j is moved *only one position* in the list (i & j are adjacent), it can't move to the other side of p, and again status is unchanged.

# Minimizing Lateness: Inversions

**Def.** An *inversion* in schedule S is a pair of jobs i and j s.t. greedy did i before j (i.e., i < j), but S does j before i.

**Claim 4.** *Swapping* two adjacent, inverted jobs does not increase the max lateness.




(j had later or equal deadline, so is not tardier after swap than i was before swap)

**Pf.** Let $\ell$ / $\ell'$ be the lateness before / after swap, resp.

- $\ell'_k = \ell_k$ for all $k \neq i, j$
- $\ell'_i \leq \ell_i$
- If job j is now late:

$$\begin{aligned}
\ell'_j &= f'_j - d_j && \text{(definition)} \\
&= f_i - d_j && (j \text{ finishes at time } f_i) \\
&\leq f_i - d_i && (d_i \leq d_j) \\
&= \ell_i && \text{(definition)}
\end{aligned}$$

only j moves later, but it's no later than i was, so max not increased

## Theorem. Greedy schedule G is optimal

Pf. Let $S_1$ be an optimal schedule. If $S_1$ has idle time, by claim 1, we can remove it to form $S_2$ without increasing lateness. If $S_2$ has any inversions, by claim 2 it has an adjacent inversion, and by claims 3 & 4, we can swap to form $S_3$ which has fewer inversions and no greater maximum lateness. Repeating this produces an idle-free, inversion-free schedule, which is exactly the greedy schedule G, without ever having increased lateness. Hence Lateness(G) ≤ Lateness($S_1$), and so is optimal.

## A slightly tidier way to say this:

Among all optimal schedules, let S* be one with the fewest inversions, and, by claim 1, no idle time. If S* has inversions, it has adjacent inversions (claim 2); swapping one decreases the number of inversions (claim 3) without increasing maximum lateness (claim 4), contradicting choice of S*. So, S* has no inversions nor idle time. But that's exactly schedule G, hence G is optimal.

# Optional Exercise

Here's an outline for a third proof, that is, in my opinion, even simpler. You might enjoy fleshing this out as an exercise.

Defn: two vectors $(u_1, u_2, \ldots u_n)$ and $(v_1, v_2, \ldots, v_n)$ are *lexicographically ordered*

$u \prec v$ if for some i, $u_1=v_1$, $u_2=v_2$, …, $u_{i-1}=v_{i-1}$, and $u_i < v_i$

I.e., they are identical in their first i-1 positions, and u is smaller in the $i^{th}$ , the first position where they differ.

Ex: the 6 permutations of 1,2,3 in lex order: $123 \prec 132 \prec 213 \prec 231 \prec 312 \prec 321$

Proof Outline: Let S* be the lexicographically first idle-free optimal schedule. Argue by contradiction that S* = G, since otherwise, letting i be the 1$^{st}$ position where they differ, S* looks like (1, 2, 3, …, i-2, i-1, x, y, …, z, i, …) where x≠i. But z must be larger than i (why?), so z,i is an adjacent inversion; flipping it gives a lexicographically *smaller* sequence of no larger max lateness, contradicting choice of S*. (This uses claims 1 & 4 above; claims 2 & 3 are no longer needed.)

# Greedy Analysis Strategies

Greedy algorithm *stays ahead.* Show that after each step of the greedy algorithm, its solution is at least as "good" as any other algorithm's. (Part of the cleverness is deciding what's "good.")

*Structural.* Discover a simple "structural" bound asserting that every possible solution must have a certain value. Then show that your algorithm always achieves this bound. (Cleverness here is usually in finding a useful structural characteristic.)

*Exchange argument.* Gradually transform any solution to the one found by the greedy algorithm without hurting its quality. (Cleverness usually in choosing which pair to swap.)

(In all 3 cases, proving these claims may require cleverness, too.)

# 4.4 Shortest Paths in a Graph

You've seen this in prerequisite courses, so this section and next two on min spanning tree are review. I won't lecture on them, but you should review the material. Both, but especially shortest paths, are common problems, having many applications. (And, hint, hint, *very* frequent fodder for job interview questions…)

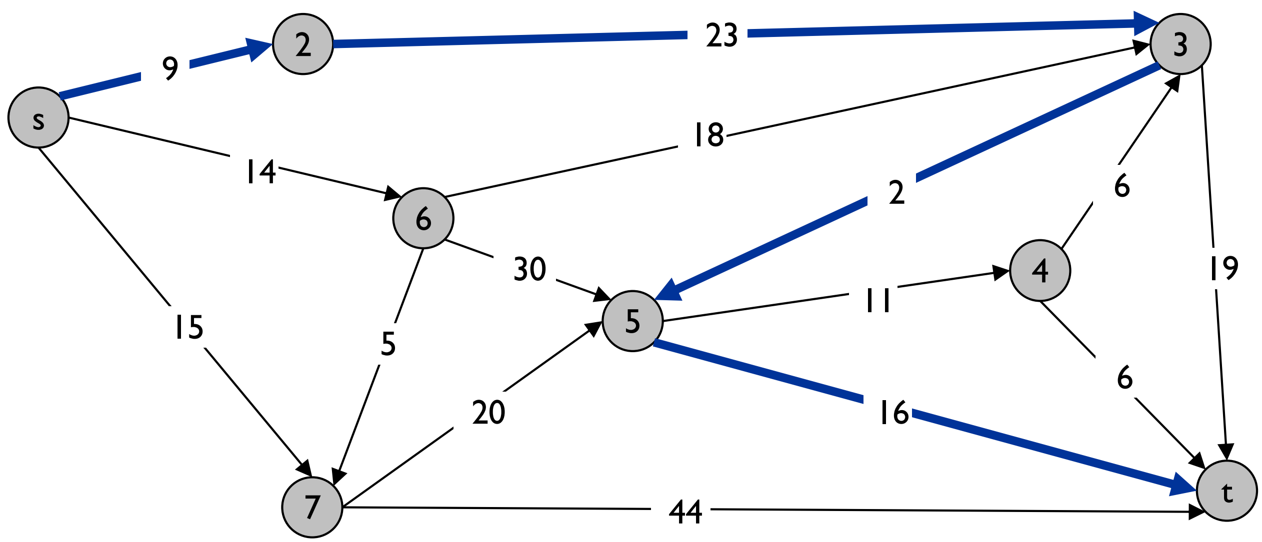
# Shortest Path Problem

Shortest path network.

- Directed graph G = (V, E).
- Source s, destination t.
- Length $\ell_e$ = length of edge e.

Shortest path problem: find shortest directed path from s to t.

cost of path = sum of edge costs in path




Cost of path s-2-3-5-t
= 9 + 23 + 2 + 16
= 48.

# Dijkstra's Algorithm

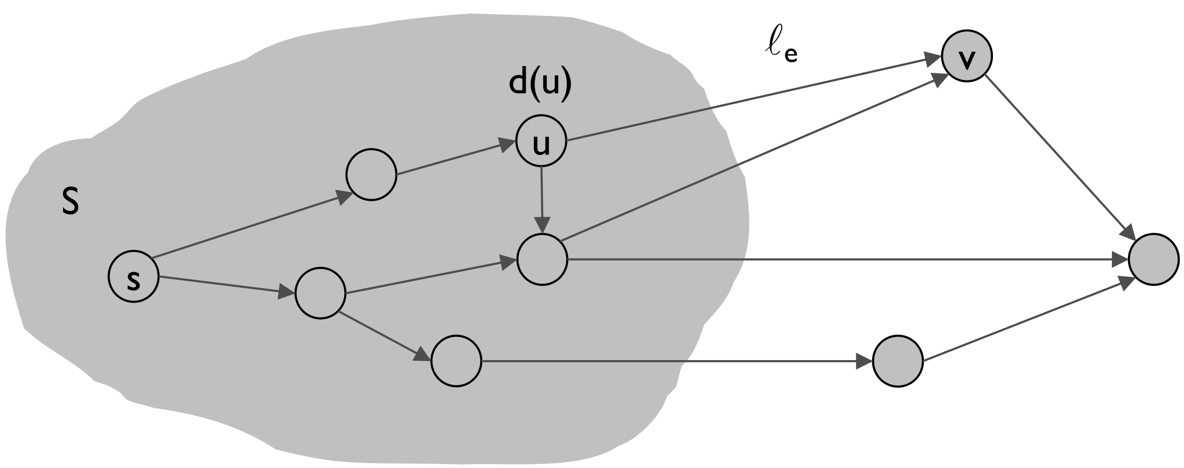
Dijkstra's algorithm.

- Maintain a set of explored nodes S for which we have determined the shortest path distance d(u) from s to u.
- Initialize S = { s }, d(s) = 0.
- Repeatedly choose unexplored node v which minimizes

$$\pi(v) = \min_{e = (u,v) : u \in S} d(u) + \ell_e ,$$

shortest path to some u in explored part, followed by a single edge (u, v)

add v to S, and set d(v) = π(v).



# Dijkstra's Algorithm

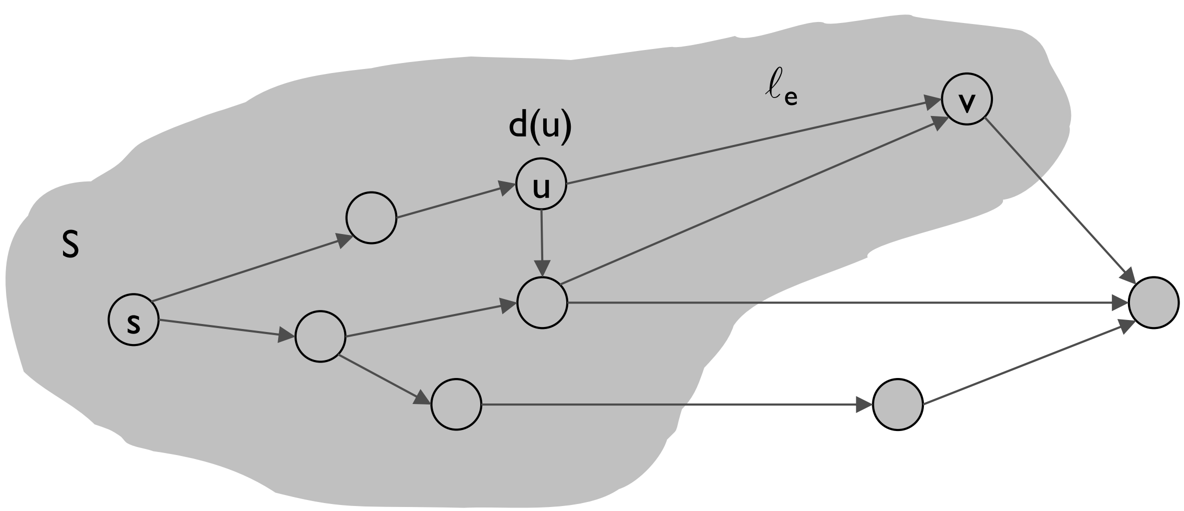
Dijkstra's algorithm.

- Maintain a set of explored nodes S for which we have determined the shortest path distance d(u) from s to u.
- Initialize S = { s }, d(s) = 0.
- Repeatedly choose unexplored node v which minimizes

$$\pi(v) = \min_{e = (u,v) : u \in S} d(u) + \ell_e ,$$

shortest path to some u in explored part, followed by a single edge (u, v)

add v to S, and set d(v) = $\pi$(v).



# Summary

"Greedy" algorithms are natural, often intuitive, tend to be simple and efficient

But seductive – often incorrect!

E.g., "Change making," depending on the available denominations

So, we look at a few examples, each useful in its own right, but emphasize *correctness*, and various approaches to reasoning about these algorithms

Interval Scheduling – greedy stays ahead

Interval Partitioning – greedy matches structural lower bound

Minimizing Lateness – exchange arguments

Next: Huffman codes and another exchange argument

Also: This is a good time to review shortest paths and min spanning trees