DFS(v) – Recursive version

```
Global Initialization:
    for all nodes v, v.dfs# = -1         // mark v "undiscovered"
    dfscounter = 0
    for v = 1 to n do
        if state(v) != fully-explored then
        DFS(v):

DFS(v)
    v.dfs# = dfscounter++                // v "discovered", number it
    Mark v ``discovered”.
    for each edge (v,x)
        if (x.dfs# == -1)                // (x previously  undiscovered)
            DFS(x)
        else …
    Mark v “fully-explored”
```

Kinds of edges – DFS on directed graphs

Edge (u,v)

Tree

[u [v v] u]

Forward

[u [v v] u]

Cross

[v v] [u u]

Back

[v [u u] v]

Topological Sort using DFS

```
Global Initialization:
    for all nodes v, v.dfs# = -1          // mark v "undiscovered"
    dfscounter = 0
    current_label = n
    for v = 1 to n do
        if state(v) != fully-explored then
        DFS(v):

DFS(v)
    v.dfs# = dfscounter++                 // v "discovered", number it
    Mark v ``discovered”.
    for each edge (v,x)
        if (x.dfs# == -1)                 // (x previously  undiscovered)
            DFS(x)
        else                              // add check for cycle if needed
    Mark v “fully-explored”
    f(v) = current_label                  // f(v) values give the topological order
    current_label --;
```

Analysis

Running time O(n+m)

Correctness: Need to show that:

if (u,v) is an edge then f(u) < f(v)

Case 1: DFS(u) called before DFS(v), so DFS(v) finishes first, which means f(v) > f(u).

Case 2: DFS(v) called before DFS(u). But there cannot be a directed path from v to u, so recursive call to DFS(v) will finish before recursive call to DFS(u) starts, so f(v) > f(u)

A simple problem on trees

Given: tree T, a value L(v) defined for every vertex v in T

Goal: find M(v), the min value of L(v) anywhere in the subtree rooted at v (including v itself).

How? Depth first search, using:

$$M(v) = \begin{cases} L(v) & \text{if } v \text{ is a leaf} \\ \min(L(v),\ \min_{w \text{ a child of v}} M(w)) & \text{otherwise} \end{cases}$$

DFS(v) – Recursive version

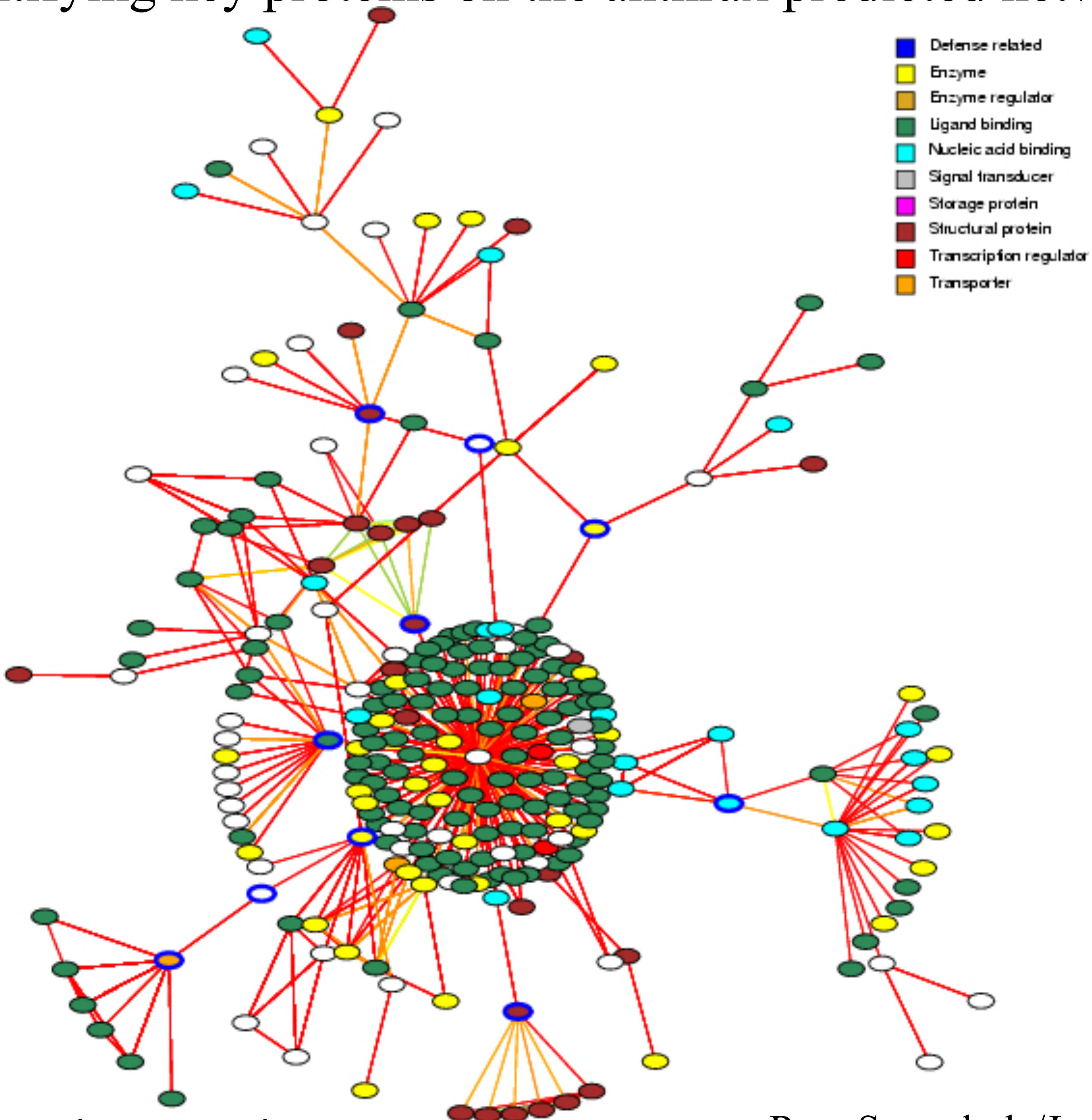
```
Global Initialization:
    for all nodes v, v.dfs# = -1   // mark v "undiscovered"
    dfscounter = 0                 // (global variable)
    DFS(s);                        // start DFS at node s;

DFS(v)
    v.dfs# = dfscounter++          // v "discovered", number it
    for each edge (v,x)
        if (x.dfs# = -1)           // tree edge (x previously  undiscovered)
            DFS(x)
```

Application: Articulation Points

A node in an undirected graph is an **articulation point** iff removing it disconnects the graph

articulation points represent vulnerabilities in a network – single points whose failure would split the network into 2 or more disconnected components

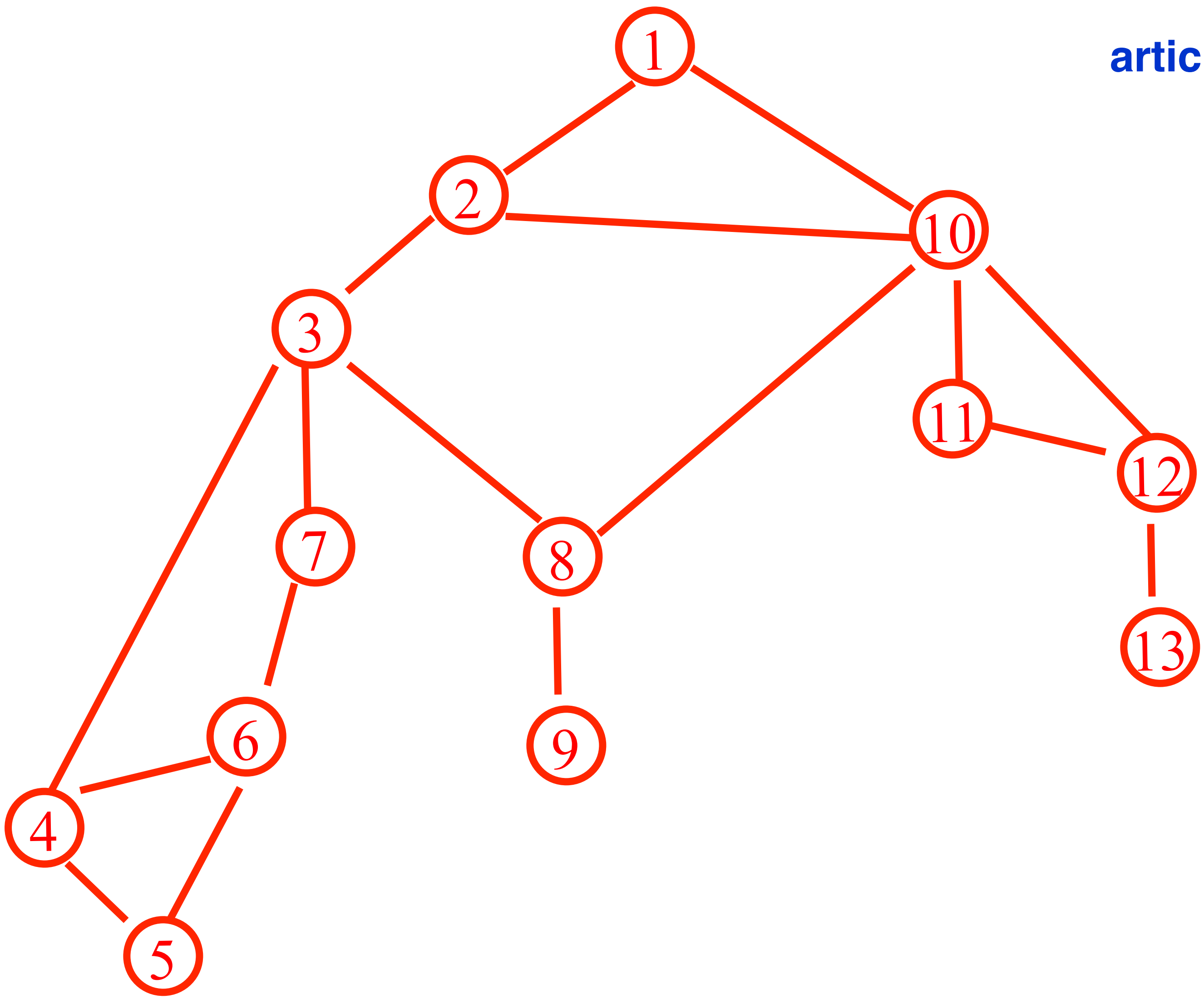
Identifying key proteins on the anthrax predicted network



Articulation point proteins

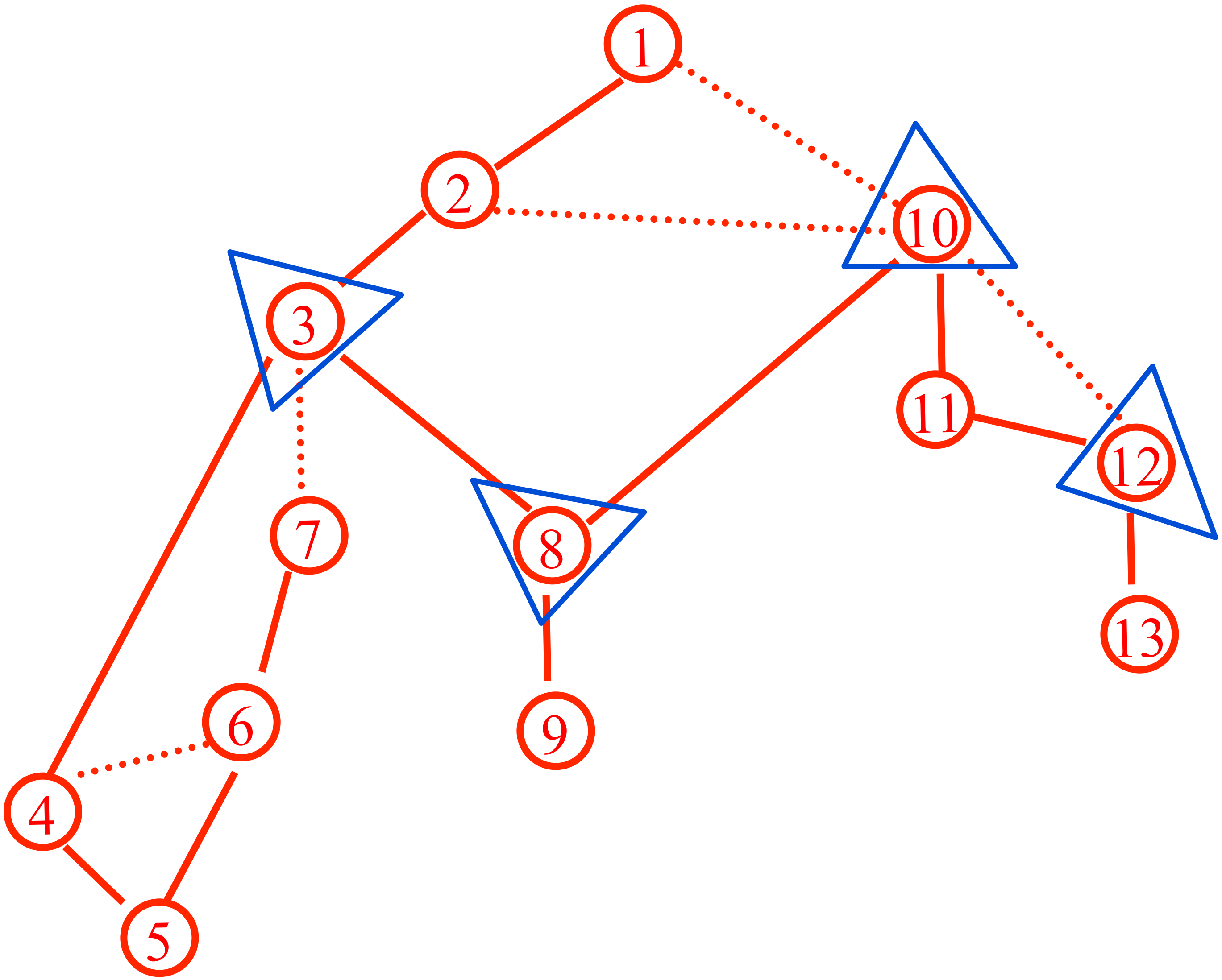
Ram Samudrala/Jason McDermott

Articulation Points



articulation point
iff its removal
disconnects
the graph

Articulation Points



Simple Case: Artic. Pts in a tree

Which nodes in a rooted tree are articulation points?

Simple Case: Artic. Pts in a tree

Leaves – never articulation points

Internal nodes – always articulation points

Root – articulation point if and only if two or more children

Non-tree: extra edges remove some articulation points (which ones?)

Recall: all edges either tree edges or back edges in DFS on undirected graph

Consider edge (u,v).

If u discovered first, then edge (u,v) will be explored before DFS(u) completes.

If at the time it is explored v is undiscovered, the edge will become a tree edge.

If v is already discovered, then since DFS(v) was called after DFS(u), it completes before DFS(u) completes,

So v is a descendent of u.

Recall: all edges either tree edges or back edges in DFS on undirected graph

If u is an ancestor of v, then

dfs# of u is lower than dfs# of v

Simple Case: Artic. Pts in a tree

Leaves – never articulation points

Internal nodes – always articulation points

Root – articulation point if and only if two or more children

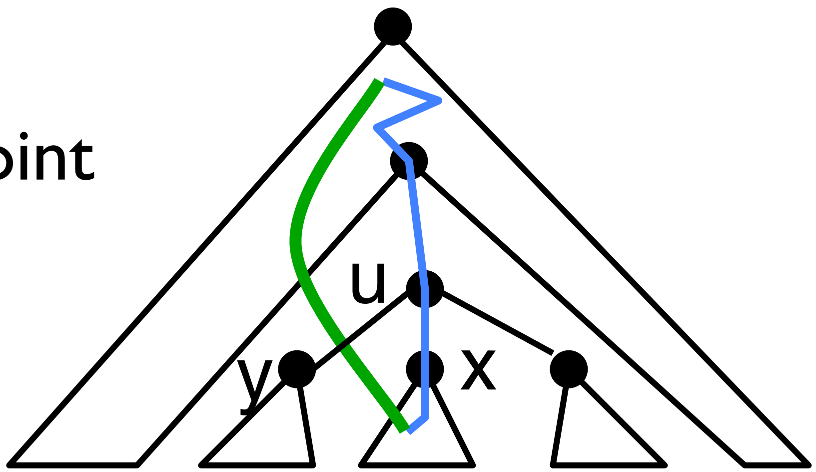
Non-tree: extra edges remove some articulation points (which ones?)

Articulation Points from DFS

Root node is an articulation point iff ….

Leaf is never an articulation point

non-leaf, non-root node u is an articulation point



Articulation Points from DFS

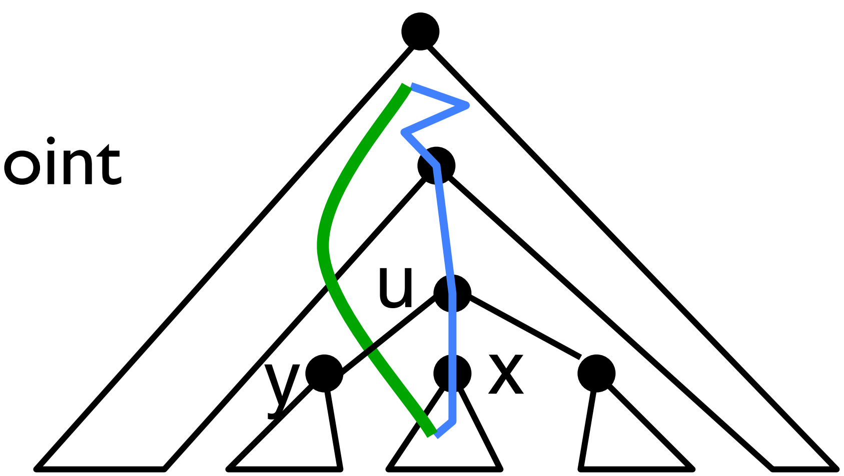
Root node is an articulation point iff it has more than one child

Leaf is never an articulation point

non-leaf, non-root node u is an articulation point

⇕

∃ some child y of u s.t. no non-tree edge goes above u from y or below



If removal of u does NOT separate x, there must be an exit from x's subtree. How? Via back edge.

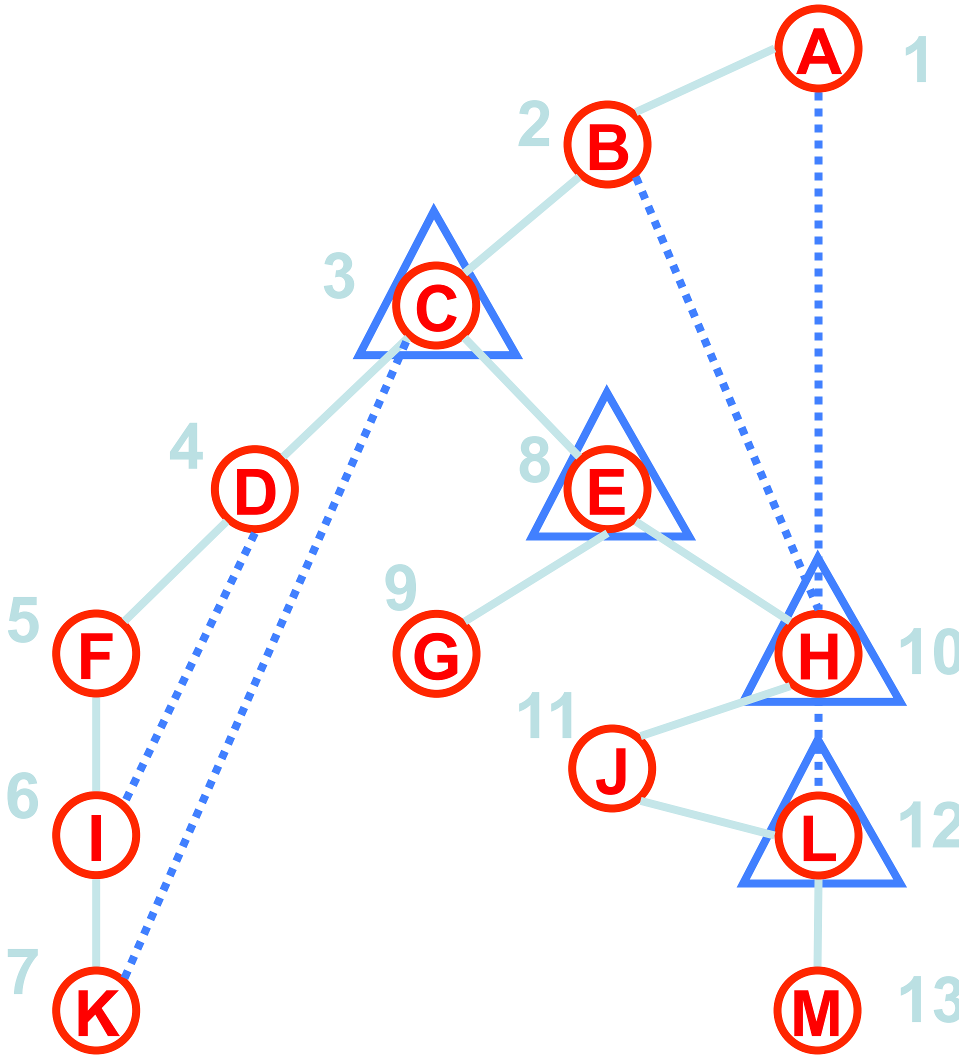
Articulation Points: the "LOW" function

Definition: LOW(v) is the lowest dfs# of any vertex that is either in the dfs subtree rooted at v (including v itself) or connected to a vertex in that subtree by a back edge.

trivial

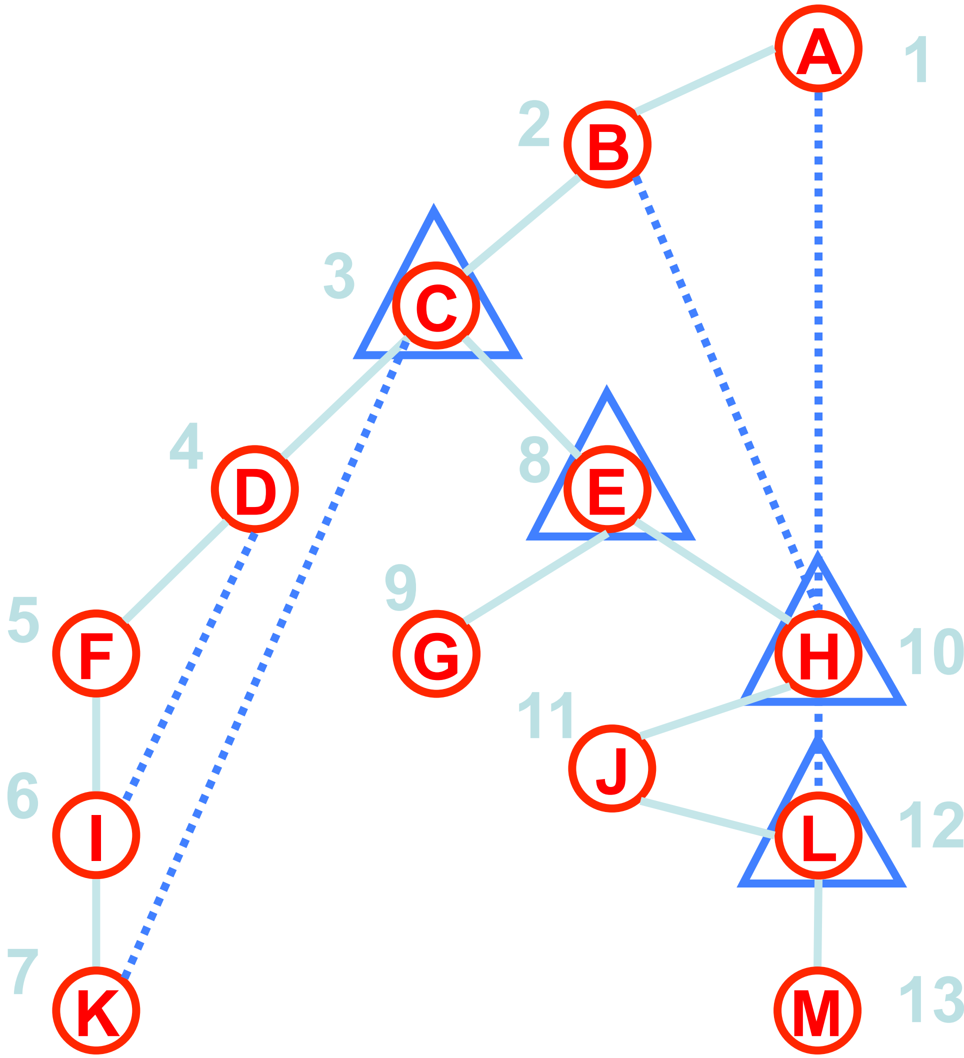
critical

LOW(v) is the lowest dfs# of any vertex that is either in the dfs subtree rooted at v (including v itself) or connected to a vertex in that subtree by a back edge.



Vertex	DFS #	Low
A	1	
B	2	
C	3	
D	4	
E	8	
F	5	
G	9	
H	10	
I	6	
J	11	
K	7	
L	12	
M	13	

Articulation Points



Vertex	DFS #	Low
A	1	1
B	2	1
C	3	1
D	4	3
E	8	1
F	5	3
G	9	9
H	10	1
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L	12	10
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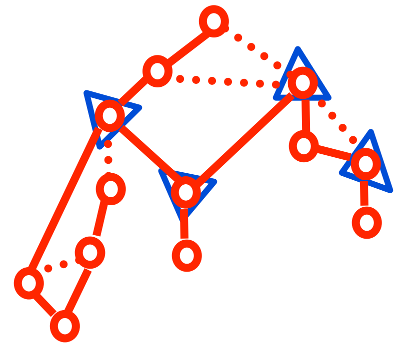
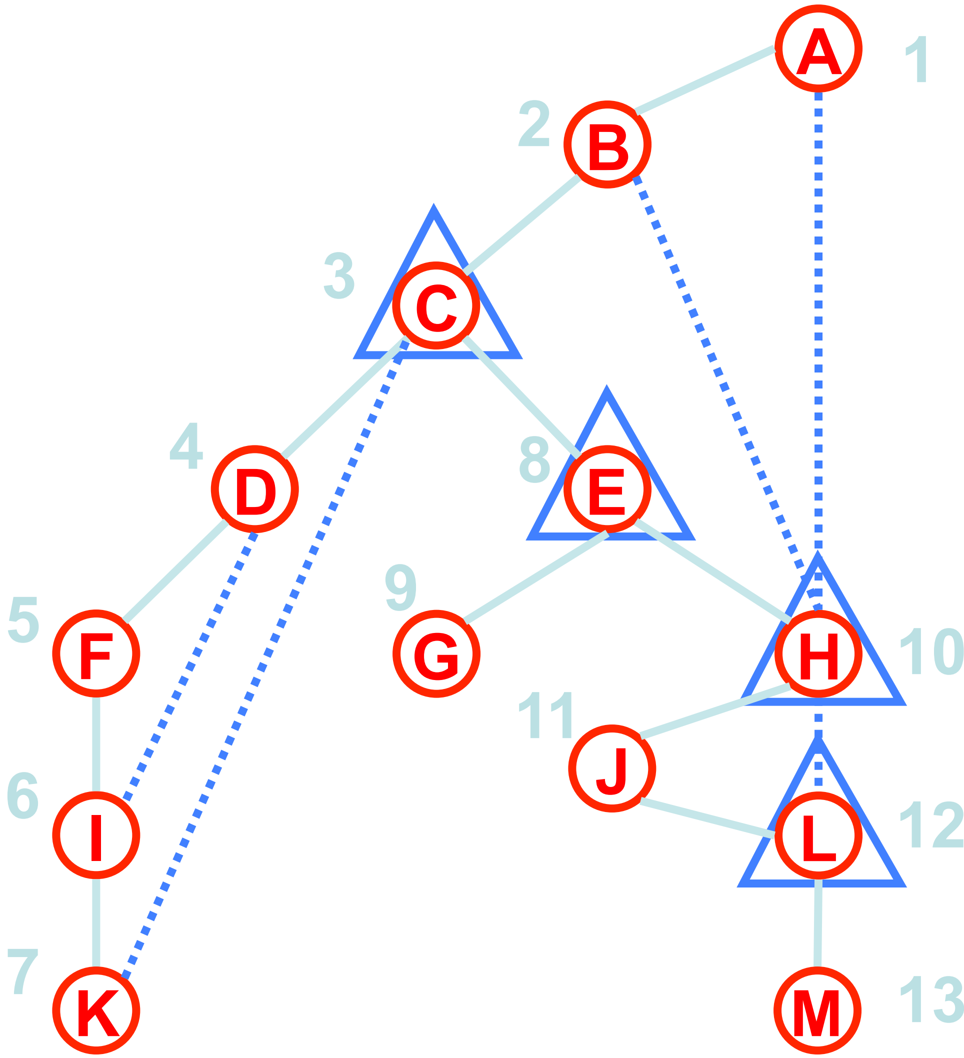
Articulation Point





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Articulation Points: the "LOW" function

trivial

Definition: LOW(v) is the lowest dfs# of any vertex that is either in the dfs subtree rooted at v (including v itself) or connected to a vertex in that subtree by a back edge.

critical

v articulation point iff…

Articulation Points: the "LOW" function

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v (non-root) articulation point iff some child x of v has LOW(x) ≥ dfs#(v)

Articulation Points: the "LOW" function

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critical

v (nonroot) articulation point iff some child x of v has LOW(x)) ≥ dfs#(v)

LOW(v) =

min ({dfs#(v)} ∪ {LOW(w) | w a child of v } ∪ { dfs#(x) | {v,x} is a back edge from v })

DFS(v) for Finding Articulation Points

```
Global initialization: v.dfs# = -1 for all v.
DFS(v)
 v.dfs# = dfscounter++
 v.low = v.dfs#                    // initialization
 for each edge {v,x}
      if (x.dfs# == -1)  // x is undiscovered
          DFS(x)
          v.low = min(v.low, x.low)
          if (x.low >= v.dfs#)
              print "v is art. pt., separating x"
      else if (x is not v's parent)
          v.low = min(v.low, x.dfs#)
```

Except for root. Why?

Equiv: "if({v,x} is a back edge)" Why?

Summary

Graphs –abstract relationships among pairs of objects

Terminology – node/vertex/vertices, edges, paths, multi-edges, self-loops, connected

Representation – edge list, adjacency matrix

Nodes vs Edges – $m = O(n^2)$, often less

BFS – Layers, queue, shortest paths, all edges go to same or adjacent layer

DFS – recursion/stack; all edges ancestor/descendant

Algorithms – connected components, bipartiteness, topological sort, articulation points