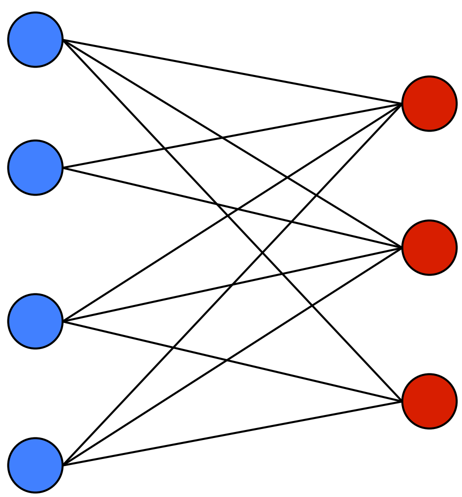
3.4 Testing Bipartiteness

Bipartite Graphs

Def. An undirected graph G = (V, E) is *bipartite (2-colorable)* if the nodes can be colored red or blue such that no edge has both ends the same color.

Applications.

Stable marriage: men = red, women = blue
Scheduling: machines = red, jobs = blue



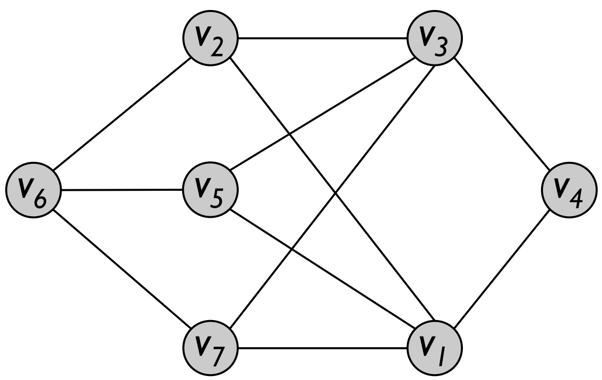
a bipartite graph

"bi-partite" means "two parts." An equivalent definition: G is bipartite if you can partition the node set into 2 parts (say, blue/red or left/right) so that all edges join nodes in different parts/no edge has both ends in the same part.

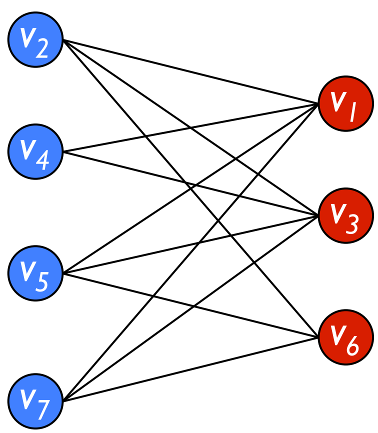
Testing Bipartiteness

Testing bipartiteness. Given a graph G, is it bipartite?

- Many graph problems become:
  - easier if the underlying graph is bipartite (matching)
  - tractable if the underlying graph is bipartite (independent set)
- Before attempting to design an algorithm, we need to understand structure of bipartite graphs.



a bipartite graph G

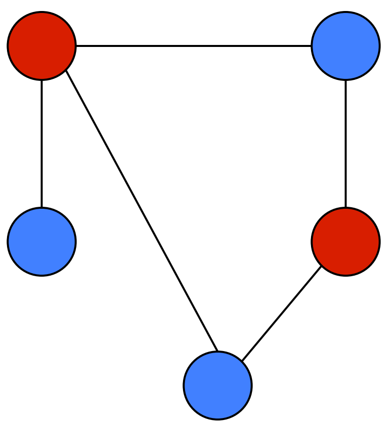


another drawing of G

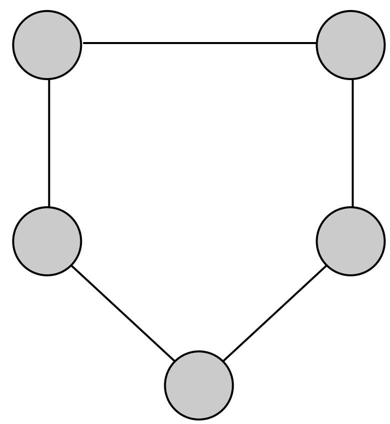
An Obstruction to Bipartiteness

Lemma. If a graph G is bipartite, it cannot contain an odd length cycle.

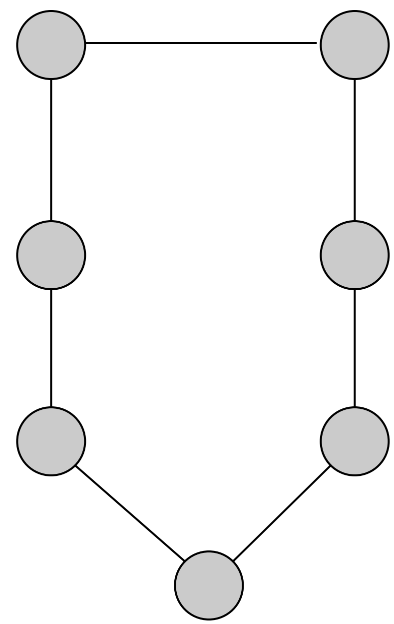
Pf. Impossible to 2-color the odd cycle, let alone G.



bipartite
(2-colorable)



not bipartite
(not 2-colorable)



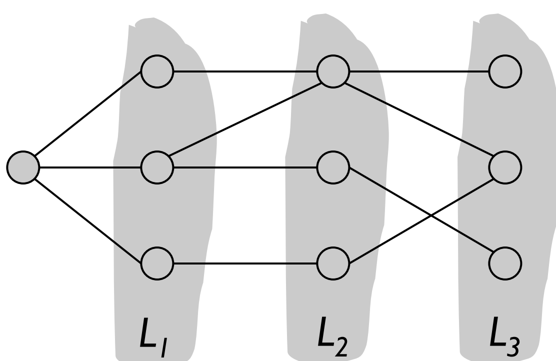
not bipartite
(not 2-colorable)

Bipartite Graphs

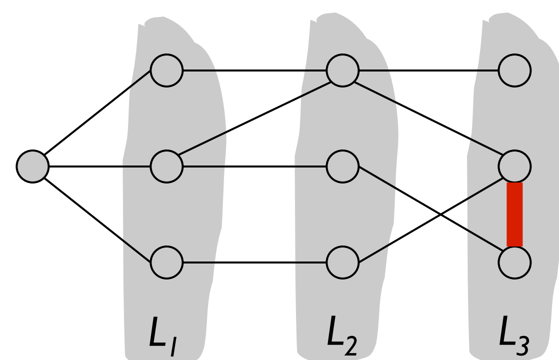
Lemma. Let G be a connected graph, and let L_0, …, L_k be the layers produced by BFS starting at node s. Exactly one of the following holds.

(i) No edge of G joins two nodes of the same layer, and G is bipartite.

(ii) An edge of G joins two nodes of the same layer, and G contains an odd-length cycle (and hence is not bipartite).



Case (i)



Case (ii)

Bipartite Graphs

Lemma. Let G be a connected graph, and let L_0, …, L_k be the layers produced by BFS starting at node s. Exactly one of the following holds.

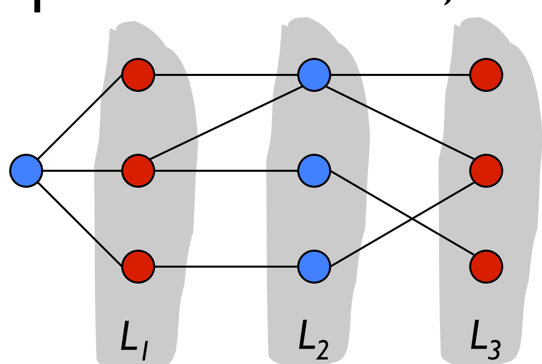
(i) No edge of G joins two nodes of the same layer, and G is bipartite.

(ii) An edge of G joins two nodes of the same layer, and G contains an odd-length cycle (and hence is not bipartite).

Pf. (i)

Suppose no edge joins two nodes in the same layer.

By previous lemma, all edges join nodes on adjacent levels.



Case (i)

Bipartition:

red = nodes on odd levels,

blue = nodes on even levels.

Bipartite Graphs

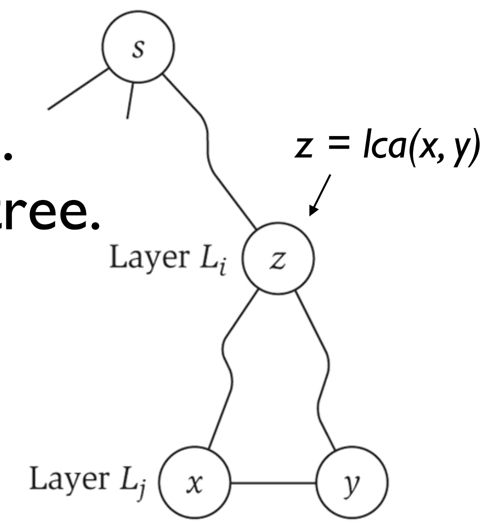
Lemma. Let G be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node s. Exactly one of the following holds.

(i) No edge of G joins two nodes of the same layer, and G is bipartite.

(ii) An edge of G joins two nodes of the same layer, and G contains an odd-length cycle (and hence is not bipartite).

Pf. (ii)

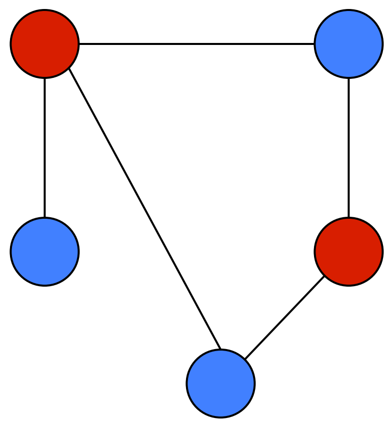
Suppose (x, y) is an edge & x, y in same level Lj.
Let z = their lowest common ancestor in BFS tree.
Let Li be level containing z.
Consider cycle that takes edge from x to y,
then tree from y to z, then tree from z to x.
Its length is $\underbrace{1}_{(x,\,y)} + \underbrace{(j-i)}_{\text{path from } y \text{ to } z} + \underbrace{(j-i)}_{\text{path from } z \text{ to } x}$, which is odd.



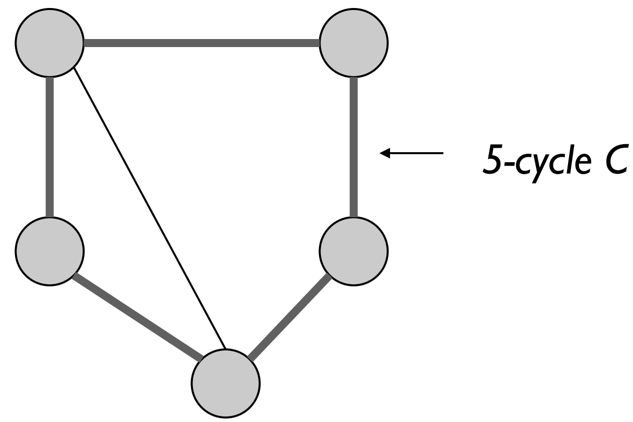
Obstruction to Bipartiteness

Cor: A graph G is bipartite iff it contains no odd length cycle.

NB: the proof is algorithmic–it *finds* a coloring or odd cycle.



bipartite
(2-colorable)



not bipartite
(not 2-colorable)

BFS(s) Implementation

Global initialization: mark all vertices **"undiscovered"**

```
BFS(s)
    mark  s "discovered"
    queue = { s }
    while queue not empty
        u = remove_first(queue)
        for each edge {u,x}
            if (x is undiscovered)
                mark x discovered
                append x on queue
        mark u fully explored
```

Exercise: modify code to determine if the graph is bipartite

3.6 DAGs and Topological Ordering

Precedence Constraints

Precedence constraints. Edge (v_i, v_j) means task v_i must occur before v_j.

Applications

Course prerequisites: course v_i must be taken before v_j

Compilation: must compile module v_i before v_j

Computing workflow: output of job v_i is input to job v_j

Manufacturing or assembly: sand it before you paint it…

Spreadsheet evaluation order: if A7 is "=A6+A5+A4", evaluate them first

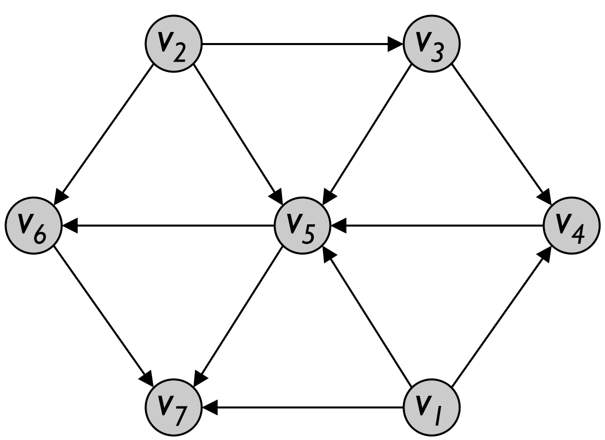
Directed Acyclic Graphs

Def. A DAG is a directed acyclic graph, i.e., one that contains no directed cycles.

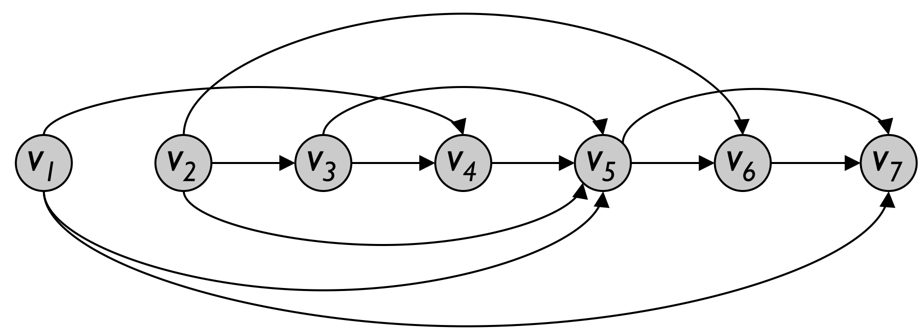
Ex. Precedence constraints: edge (v_i, v_j) means v_i must precede v_j.

Def. A topological order of a directed graph G = (V, E) is an ordering of its nodes as $v_1, v_2, \ldots, v_n$ so that for every edge (v_i, v_j) we have $i < j$.

E.g., $\forall$edge (v_i, v_j), finish v_i before starting v_j



a DAG



a topological ordering of that DAG– all edges left-to-right

Directed Acyclic Graphs

Lemma. If G has a topological order, then G is a DAG.

Directed Acyclic Graphs

Lemma. If G has a topological order, then G is a DAG.

if all edges go L→R, you can't loop back to close a cycle

Pf. (by contradiction)

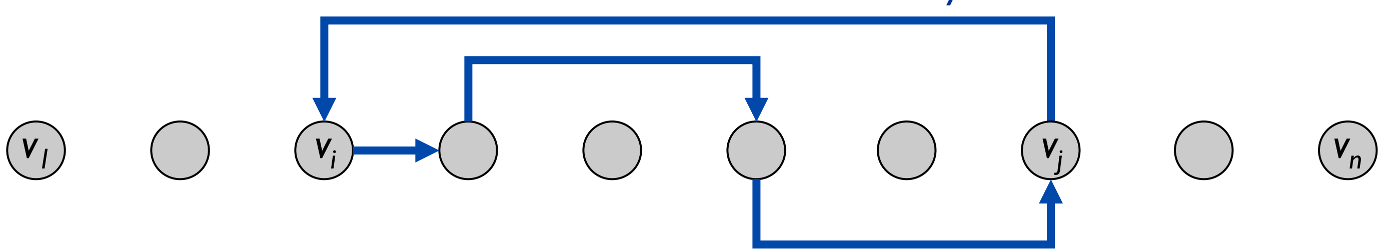
Suppose that G has a topological order $v_1, \ldots, v_n$ and that G also has a directed cycle C.

Let v_i be the lowest-indexed node in C, and let v_j be the node just before v_i; thus (v_j, v_i) is an edge.

By our choice of i, we have $i < j$.

On the other hand, since (v_j, v_i) is an edge and $v_1, \ldots, v_n$ is a topological order, we must have $j < i$, a contradiction.

the directed cycle C



the supposed topological order: $v_1, \ldots, v_n$

Directed Acyclic Graphs

Lemma.
If G has a topological order, then G is a DAG.

Q. Does every DAG have a topological ordering?

Q. If so, how do we compute one?

Directed Acyclic Graphs

Lemma. If G is a DAG, then G has a node with no incoming edges.

Pf. (by contradiction)

Suppose that G is a DAG and every node has at least one incoming edge. Let's see what happens.

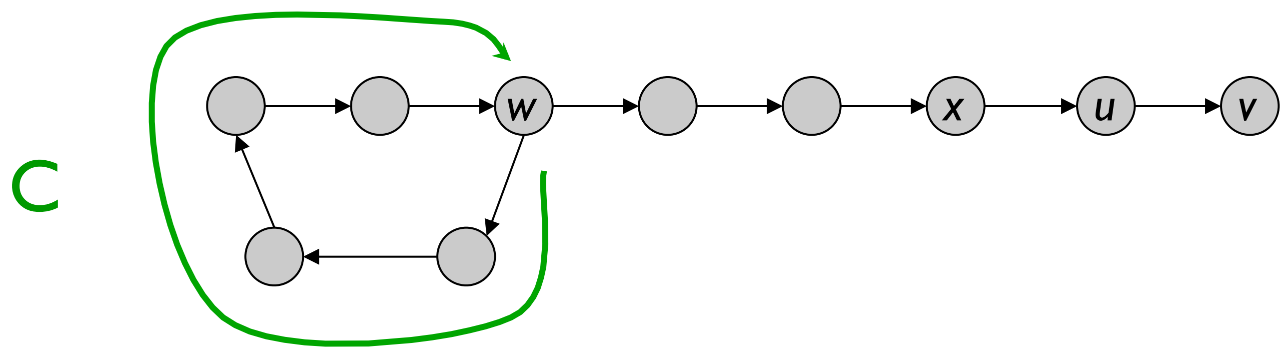
Pick any node v, and begin following edges backward from v. Since v has at least one incoming edge (u, v) we can walk backward to u.

Then, since u has at least one incoming edge (x, u), we can walk backward to x.

Repeat until we visit a node, say w, twice.

Let C be the sequence of nodes encountered between successive visits to w. C is a cycle.

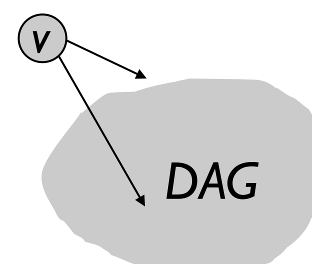
Why must this happen?



Directed Acyclic Graphs

Lemma. If G is a DAG, then G has a topological ordering.

Pf Algorithm?



Directed Acyclic Graphs

Lemma. If G is a DAG, then G has a topological ordering.

Pf. (by induction on n)

Base case: true if n = 1.

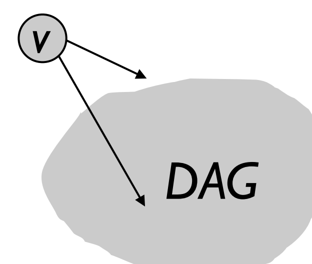
Given DAG on n > 1 nodes, find a node v with no incoming edges.

G - { v } is a DAG, since deleting v cannot create cycles.

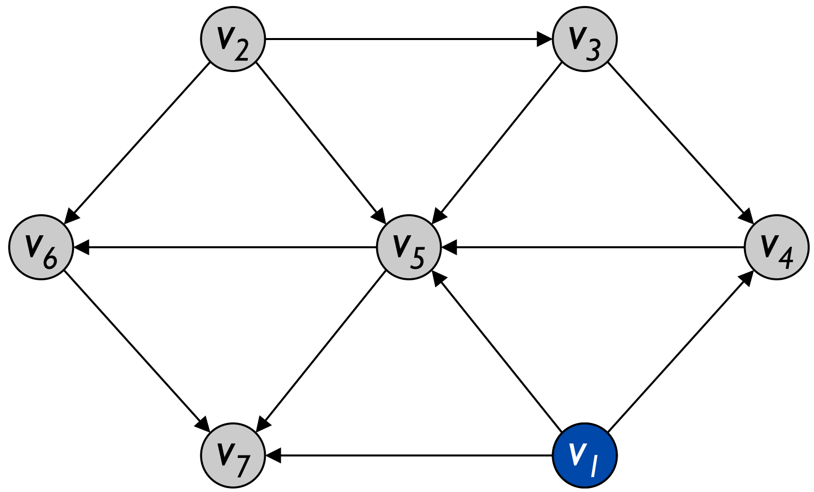
By inductive hypothesis, G - { v } has a topological ordering.

Place v first in topological ordering; then append nodes of G - { v } in topological order. This is valid since v has no incoming edges. ▪

```
To compute a topological ordering of G:
Find a node v with no incoming edges and order it first
Delete v from G
Recursively compute a topological ordering of G−{v}
  and append this order after v
```

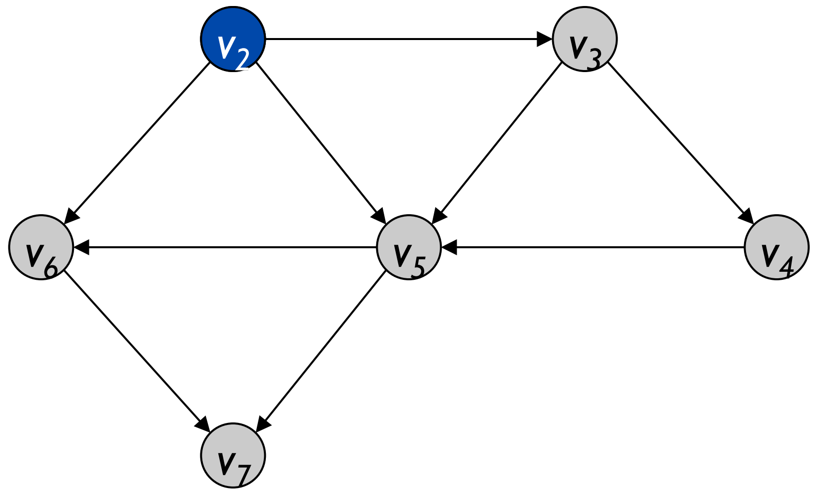


Topological Ordering Algorithm: Example



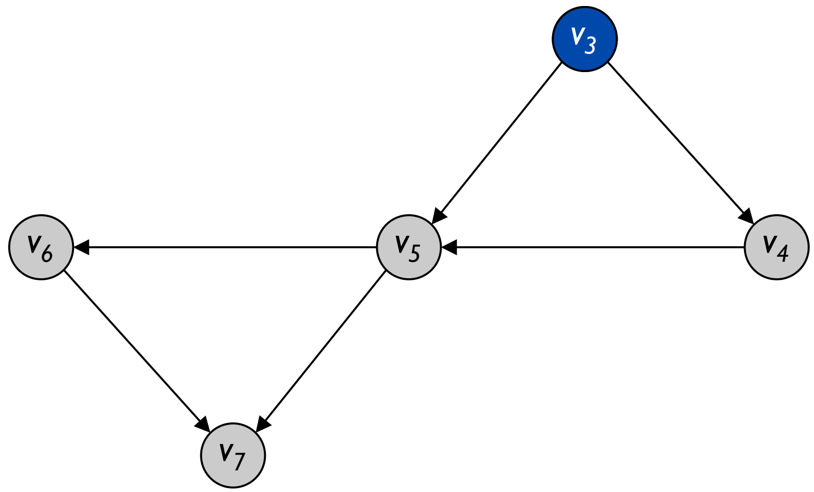
Topological order:

Topological Ordering Algorithm: Example



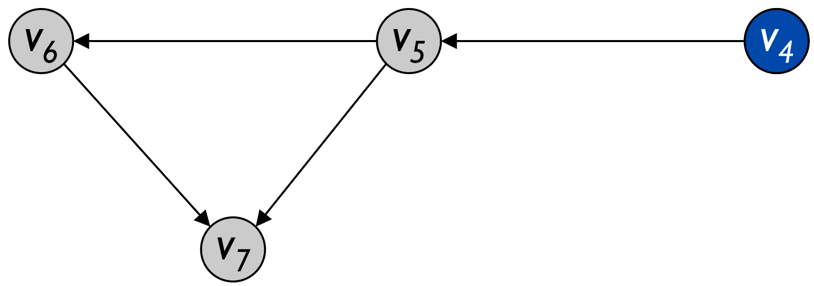
Topological order: v_1

Topological Ordering Algorithm: Example



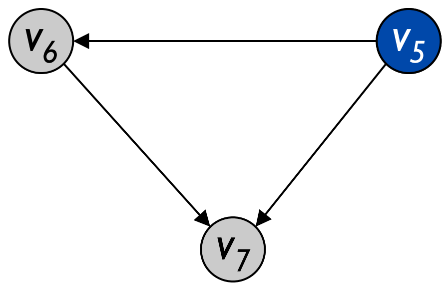
Topological order: v_1, v_2

Topological Ordering Algorithm: Example



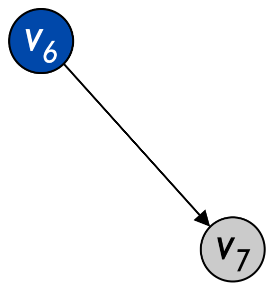
Topological order: v_1, v_2, v_3

Topological Ordering Algorithm: Example



Topological order: v_1, v_2, v_3, v_4

Topological Ordering Algorithm: Example

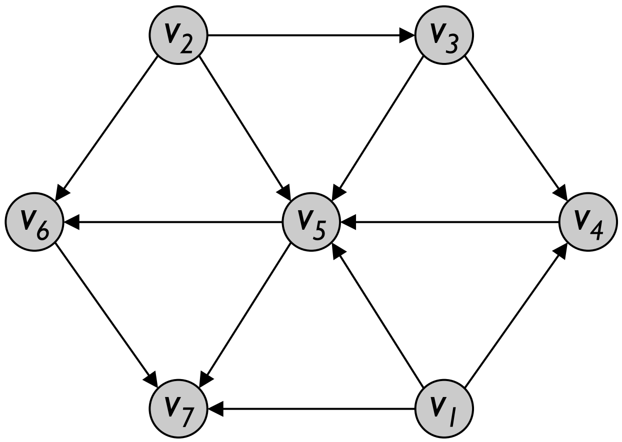


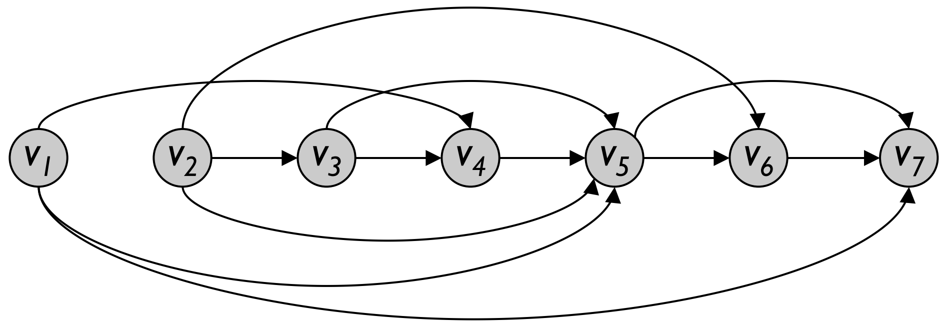
Topological order: v_1, v_2, v_3, v_4, v_5

Topological Ordering Algorithm: Example

Topological order: $v_1, v_2, v_3, v_4, v_5, v_6$

Topological Ordering Algorithm: Example





Topological order: $v_1, v_2, v_3, v_4, v_5, v_6, v_7$.

Topological Sorting Algorithm

Linear time implementation?

Topological Sorting Algorithm

Maintain the following:

count[w] = (remaining) number of incoming edges to node w

S = set of (remaining) nodes with no incoming edges

Initialization:

count[w] = 0 for all w

count[w]++ for all edges (v,w)

S = S ∪ {w} for all w with count[w]==0

} O(m + n)

Main loop:

while S not empty

remove some v from S

make v next in topo order

for all edges from v to some w

decrement count[w]

add w to S if count[w] hits 0

} O(1) per node, O(1) per edge

Correctness: clear, I hope

Time: O(m + n) (assuming edge-list representation of graph)

Depth-First Search

Follow the first path you find as far as you can go

Back up to last unexplored edge when you reach a dead end, then go as far you can

Naturally implemented using recursive calls or a stack

DFS(v) – Recursive version

```
Global Initialization:
    for all nodes v, v.dfs# = -1   // mark v "undiscovered"
    dfscounter = 0                 // (global variable)
    DFS(s);                        // start DFS at node s;

DFS(v)
    v.dfs# = dfscounter++          // v "discovered", number it
    for each edge (v,x)
        if (x.dfs# = -1)           // tree edge (x previously  undiscovered)
            DFS(x)
```

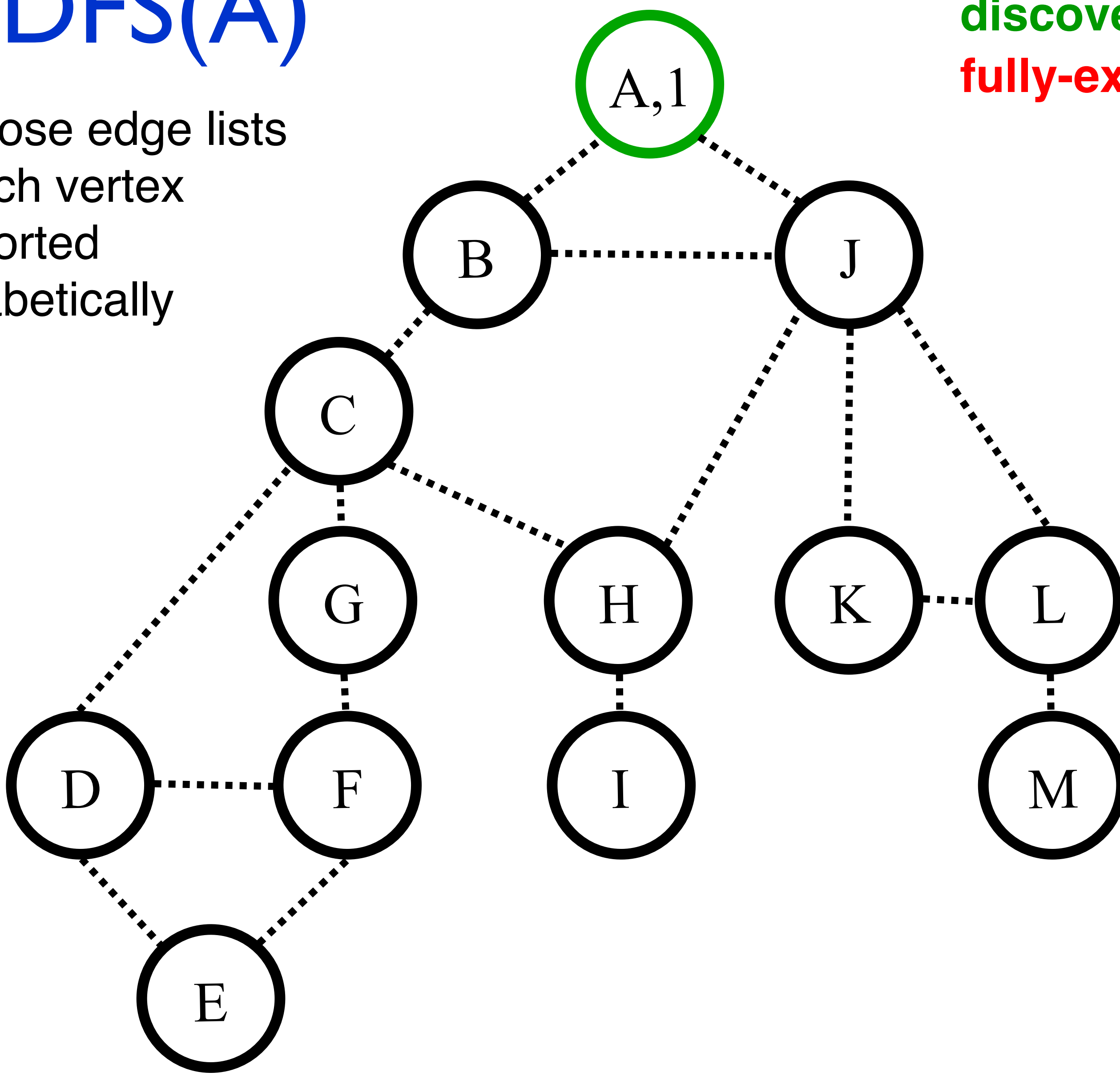
Why fuss about trees (again)?

BFS tree $\neq$ DFS tree, but, as with BFS, DFS has found a tree in the graph s.t. non-tree edges are "simple" – only descendant/ ancestor

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically

Color code:
undiscovered
discovered
fully-explored



Call Stack (Edge list):
A (B,J)

DFS(A)

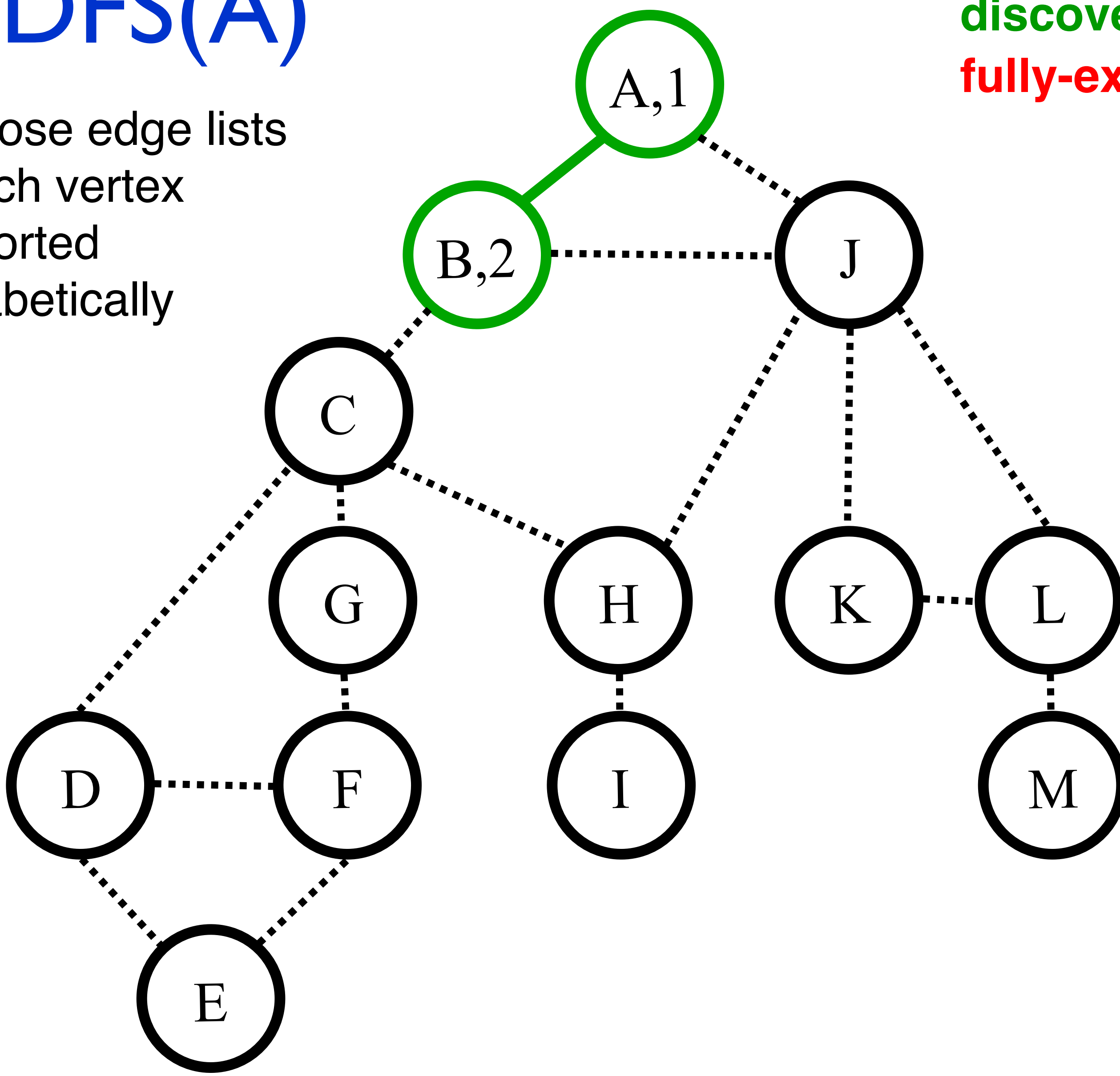
Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

discovered

fully-explored



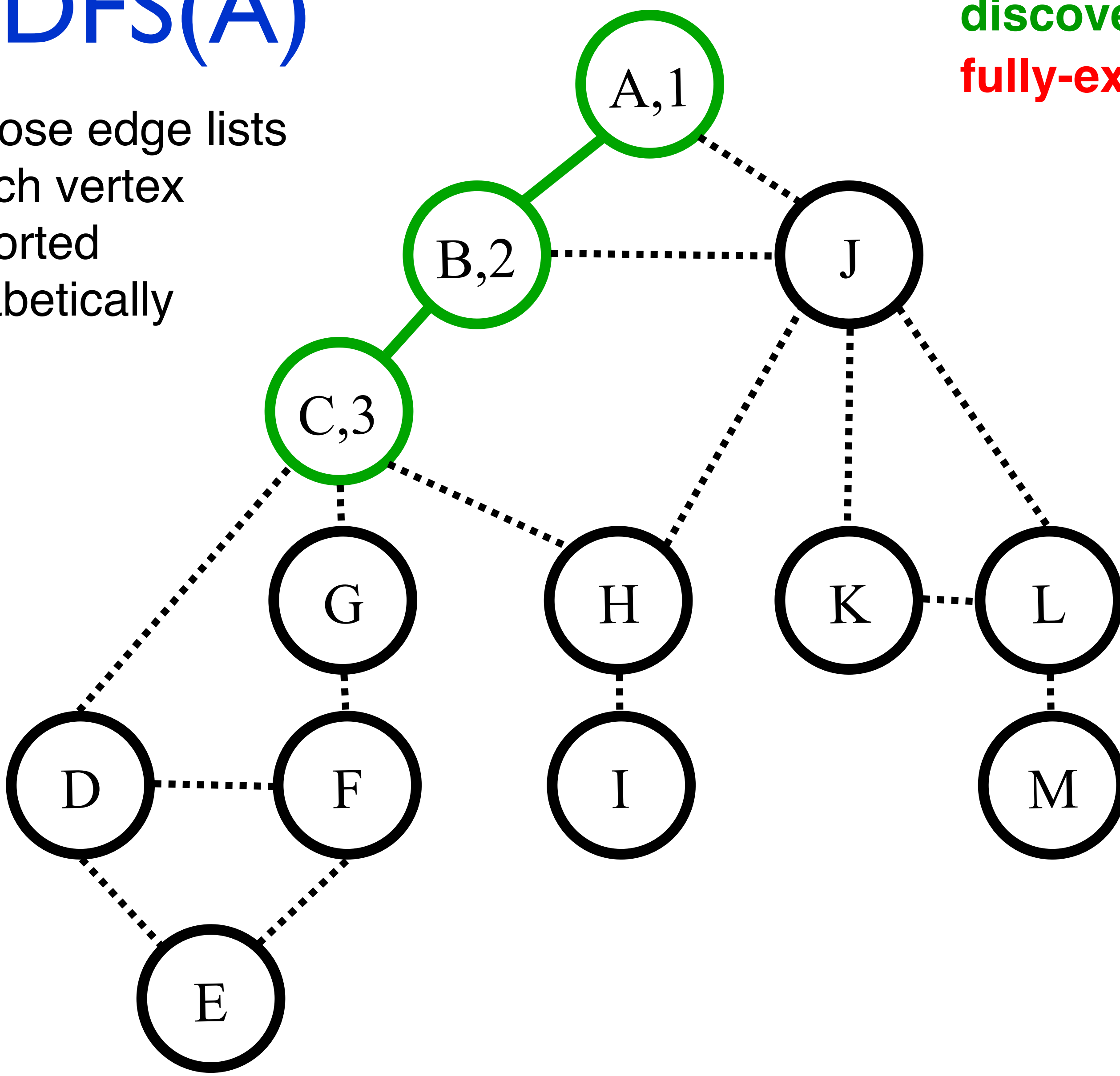
Call Stack:
(Edge list)

A (~~B~~,J)
B (A,C,J)

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically

Color code:
undiscovered
discovered
fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (B,D,G,H)

DFS(A)

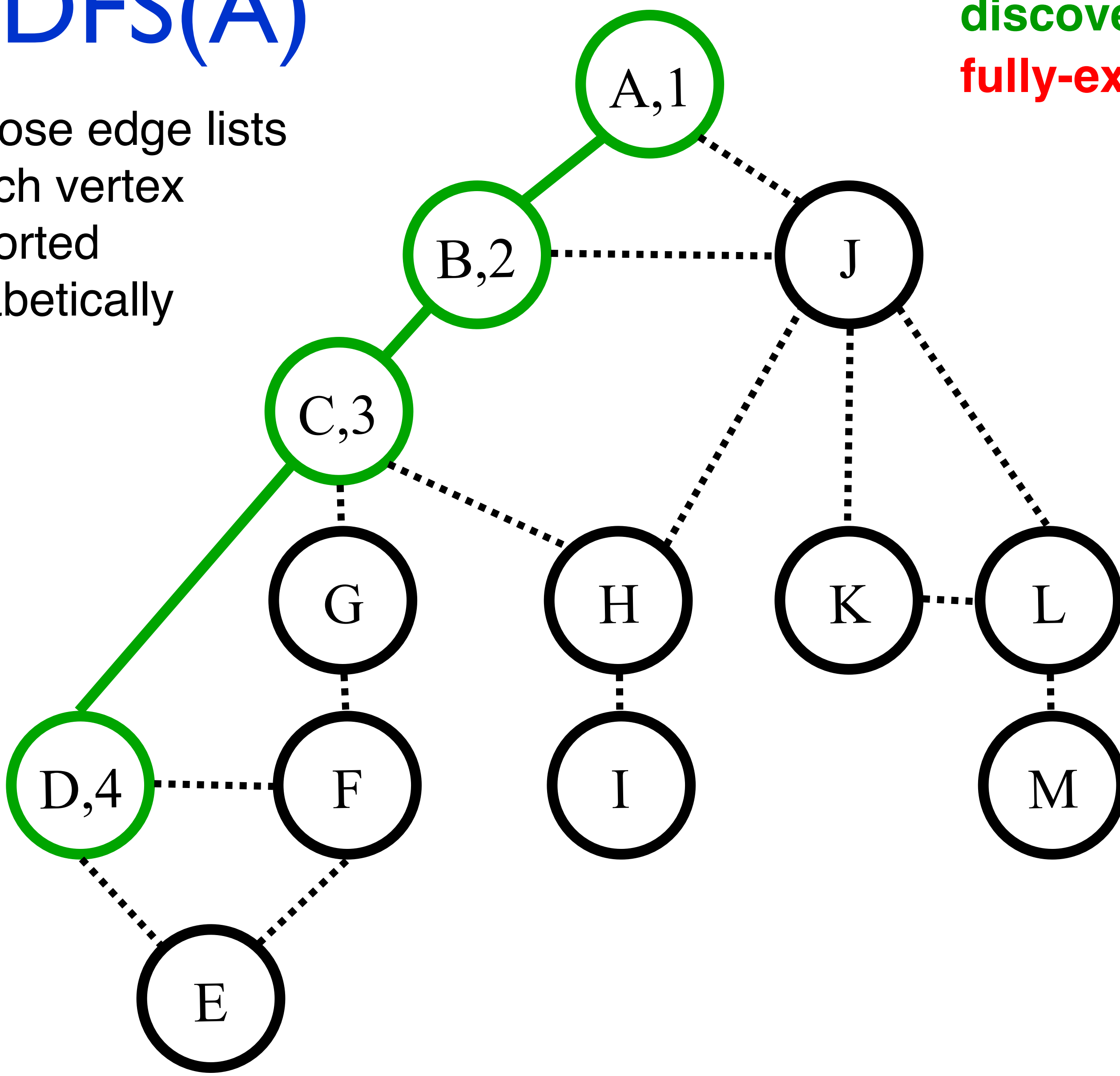
Color code:

undiscovered

discovered

fully-explored

Suppose edge lists at each vertex are sorted alphabetically



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (C,E,F)

Color code:

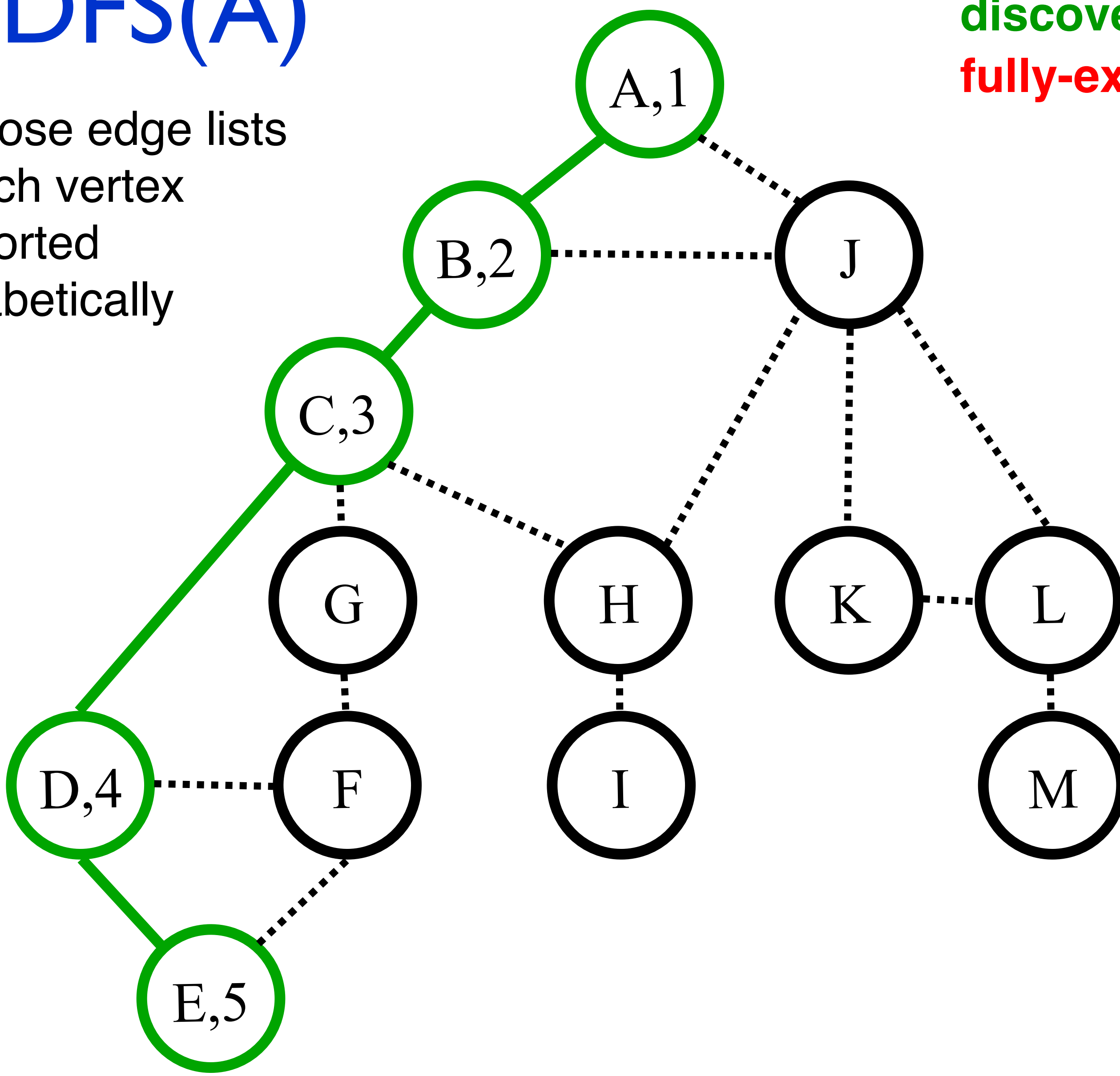
undiscovered

discovered

fully-explored

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (D,F)

DFS(A)

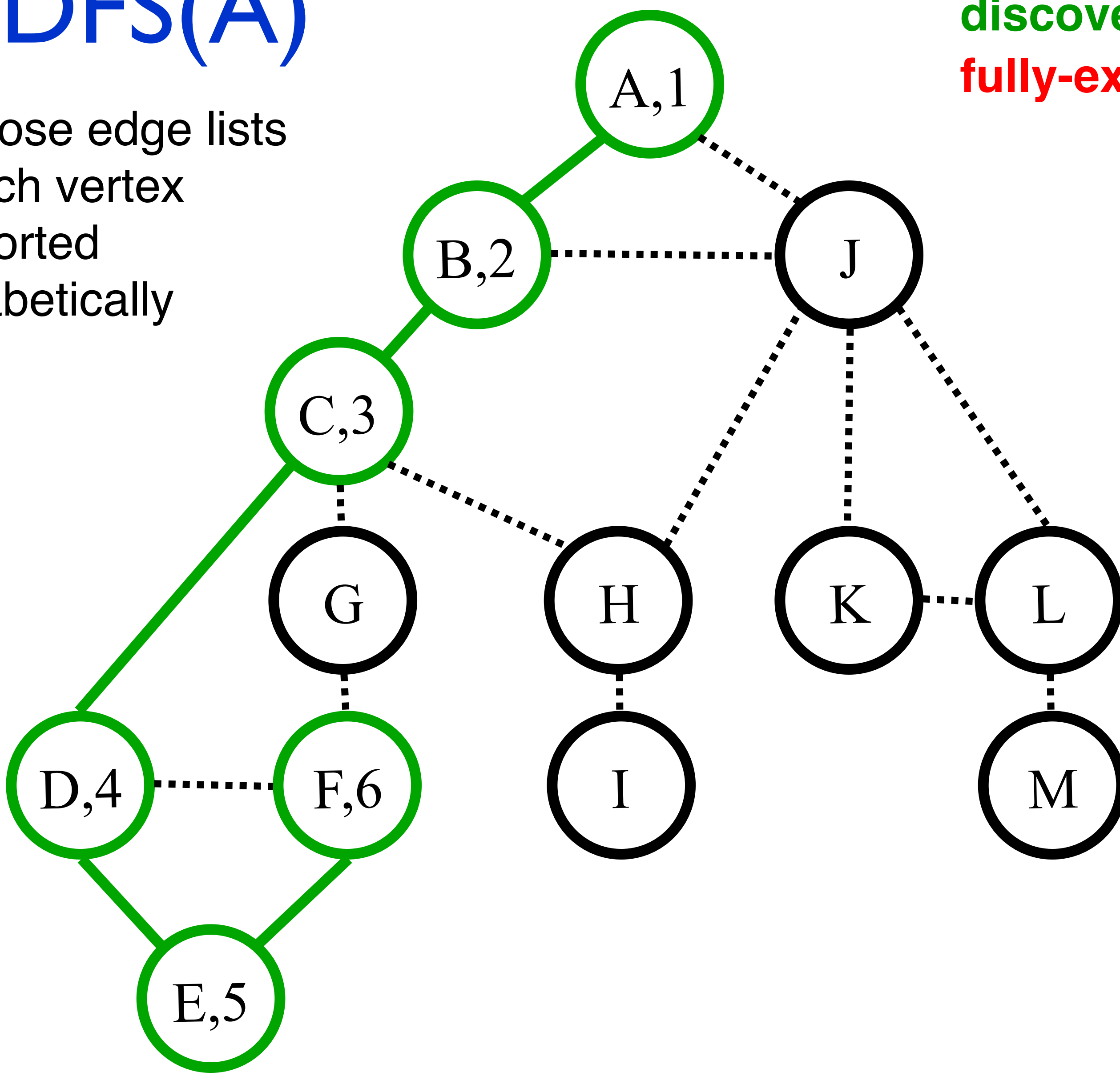
Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)
F (D,E,G)

DFS(A)

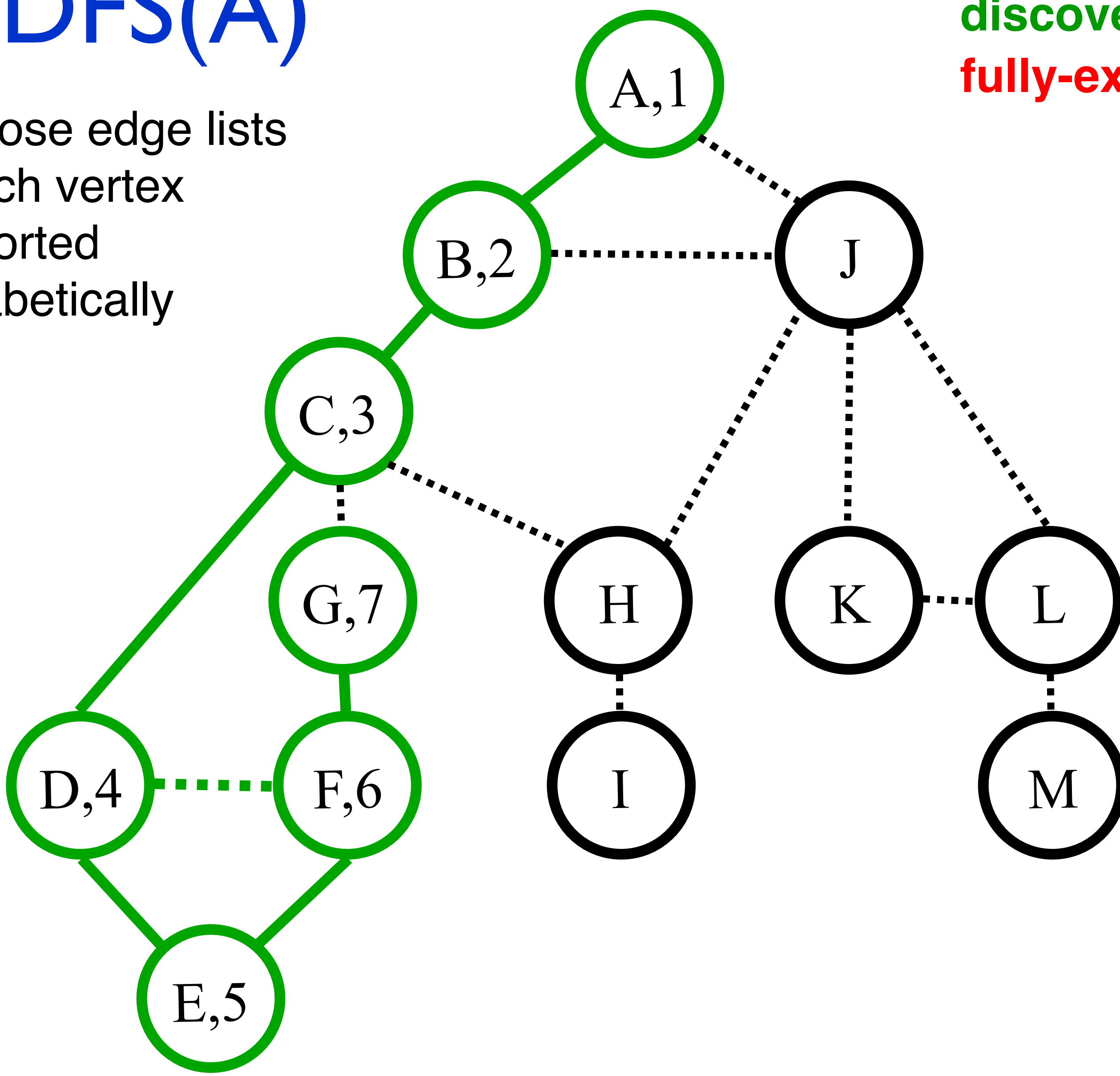
Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

discovered

fully-explored

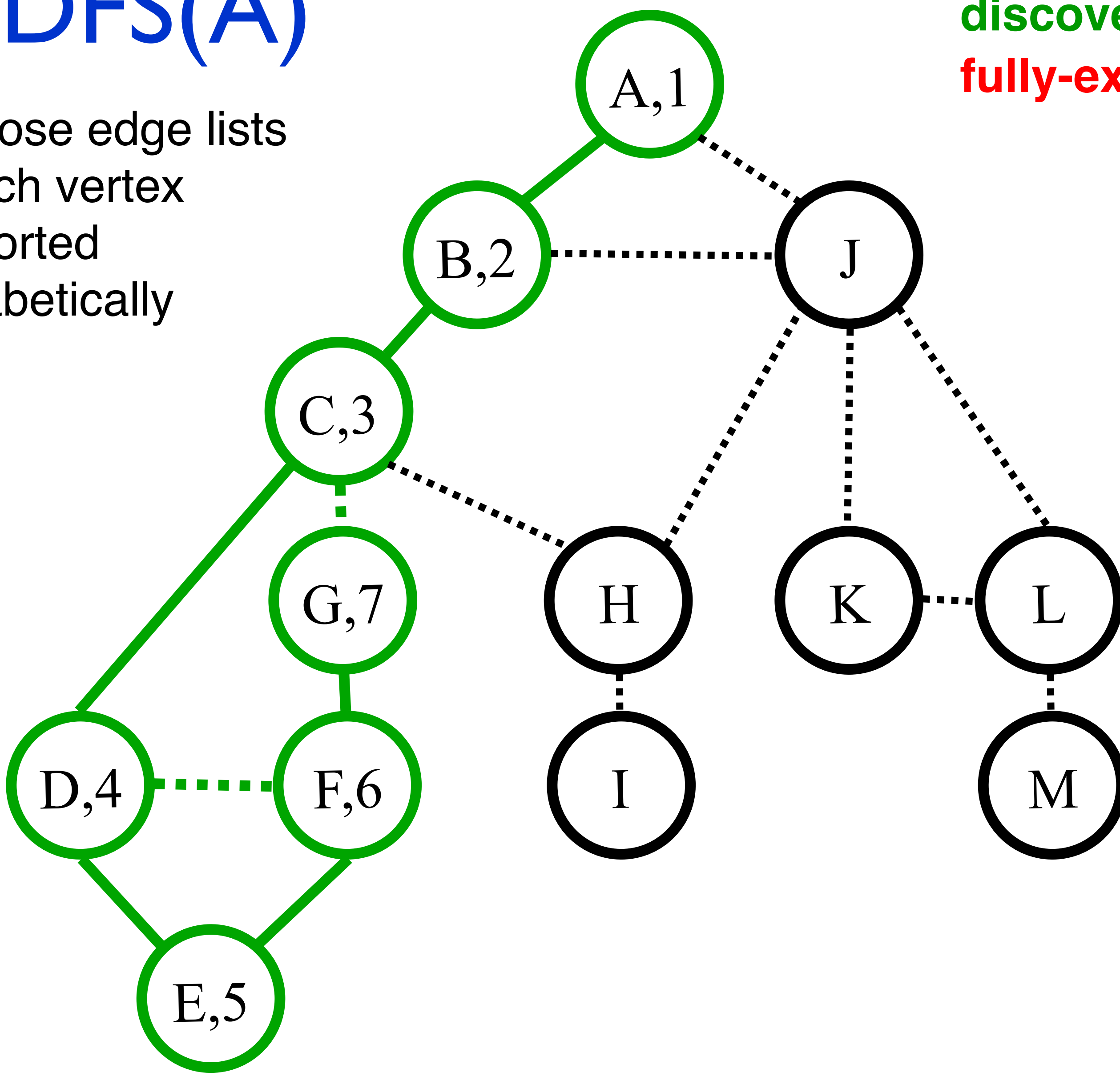


Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)
F (~~D~~,~~E~~,~~G~~)
G (C,F)

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Color code:

undiscovered

discovered

fully-explored

Call Stack: (Edge list)
A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)
F (~~D~~,~~E~~,~~G~~)
G (~~C~~,~~F~~)

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

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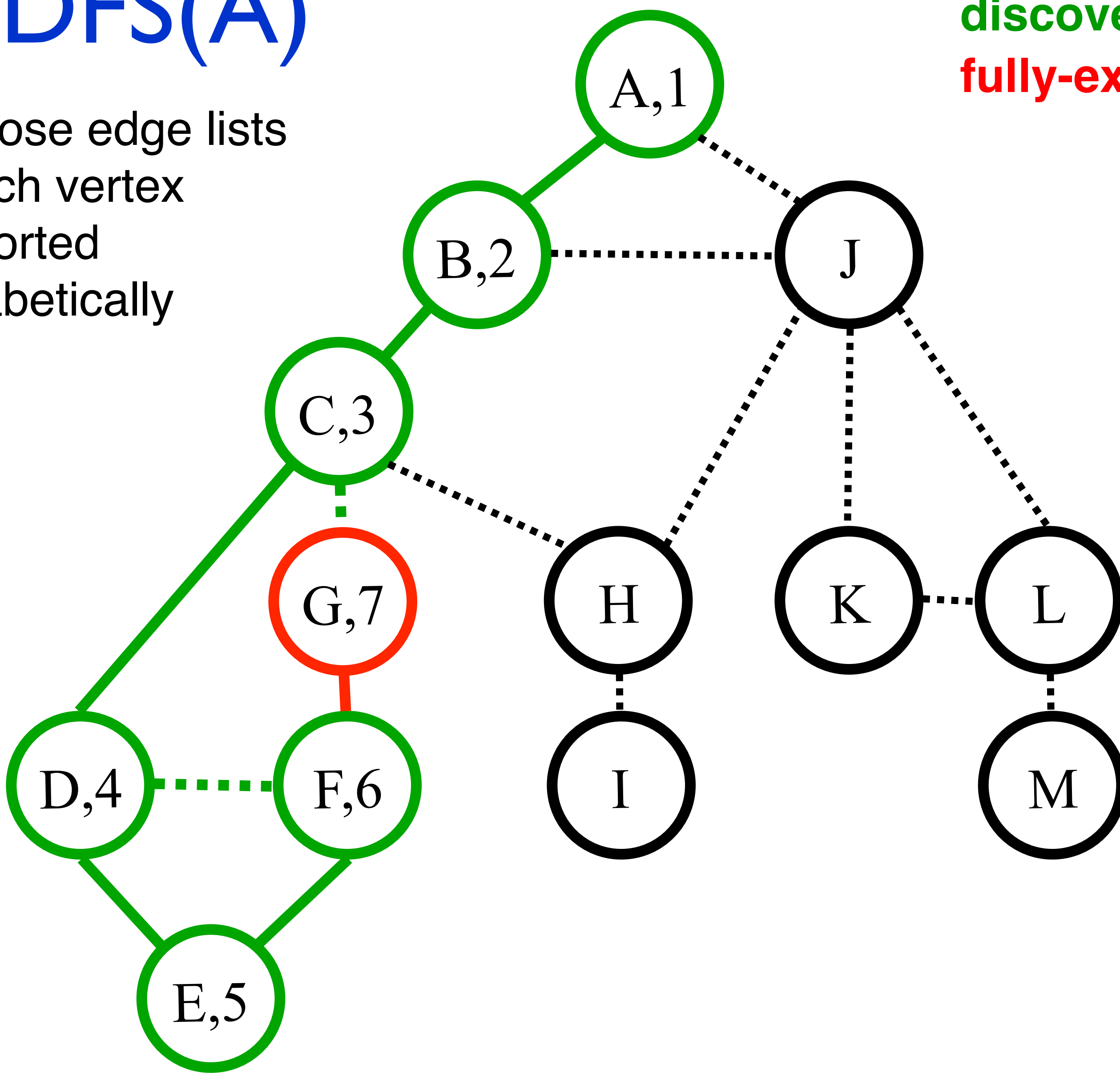


Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)
F (~~D~~,~~E~~,~~G~~)

DFS(A)

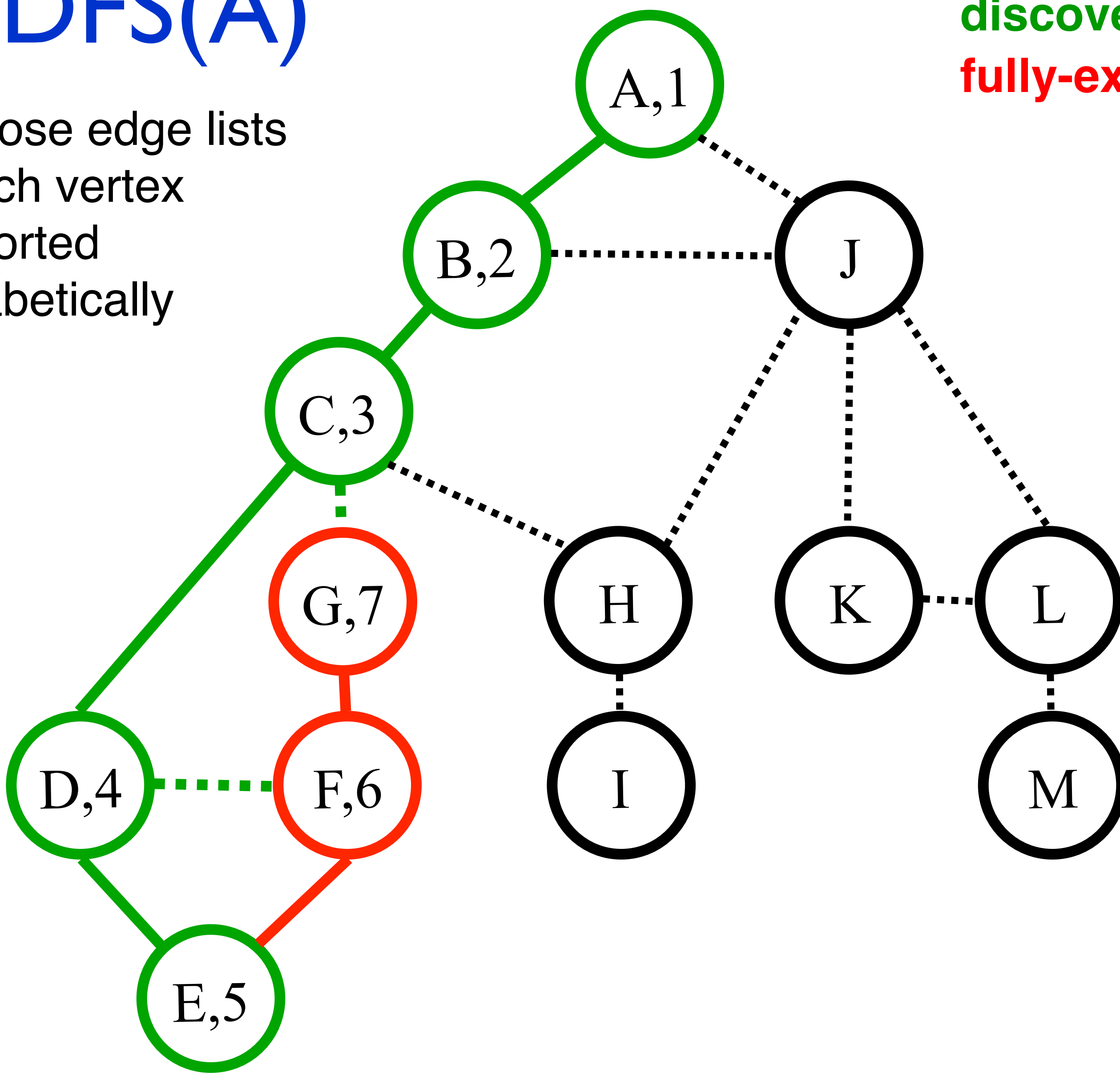
Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

discovered

fully-explored

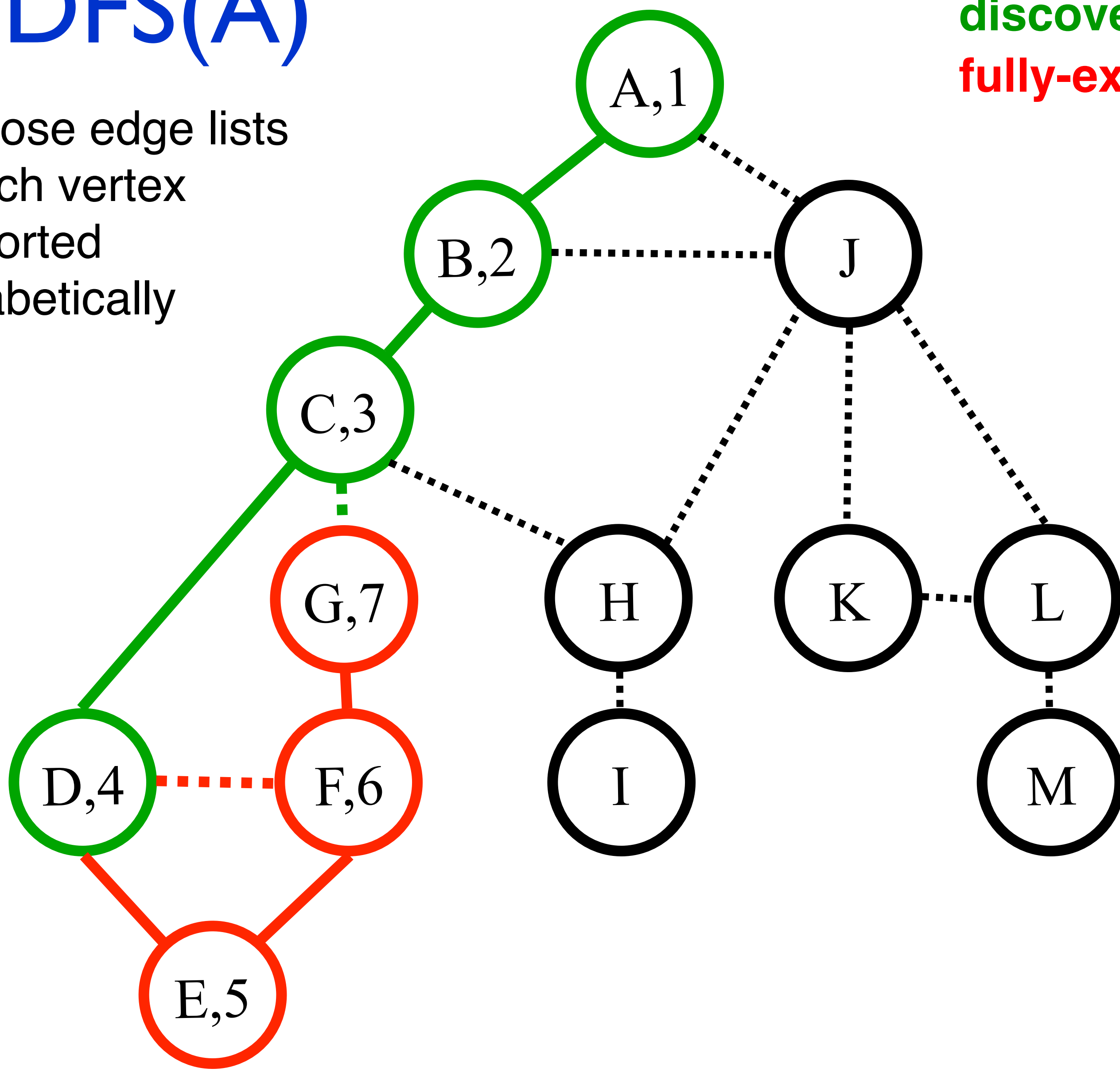


Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Color code:

undiscovered

discovered

fully-explored

Call Stack: (Edge list)
A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,~~F~~)

DFS(A)

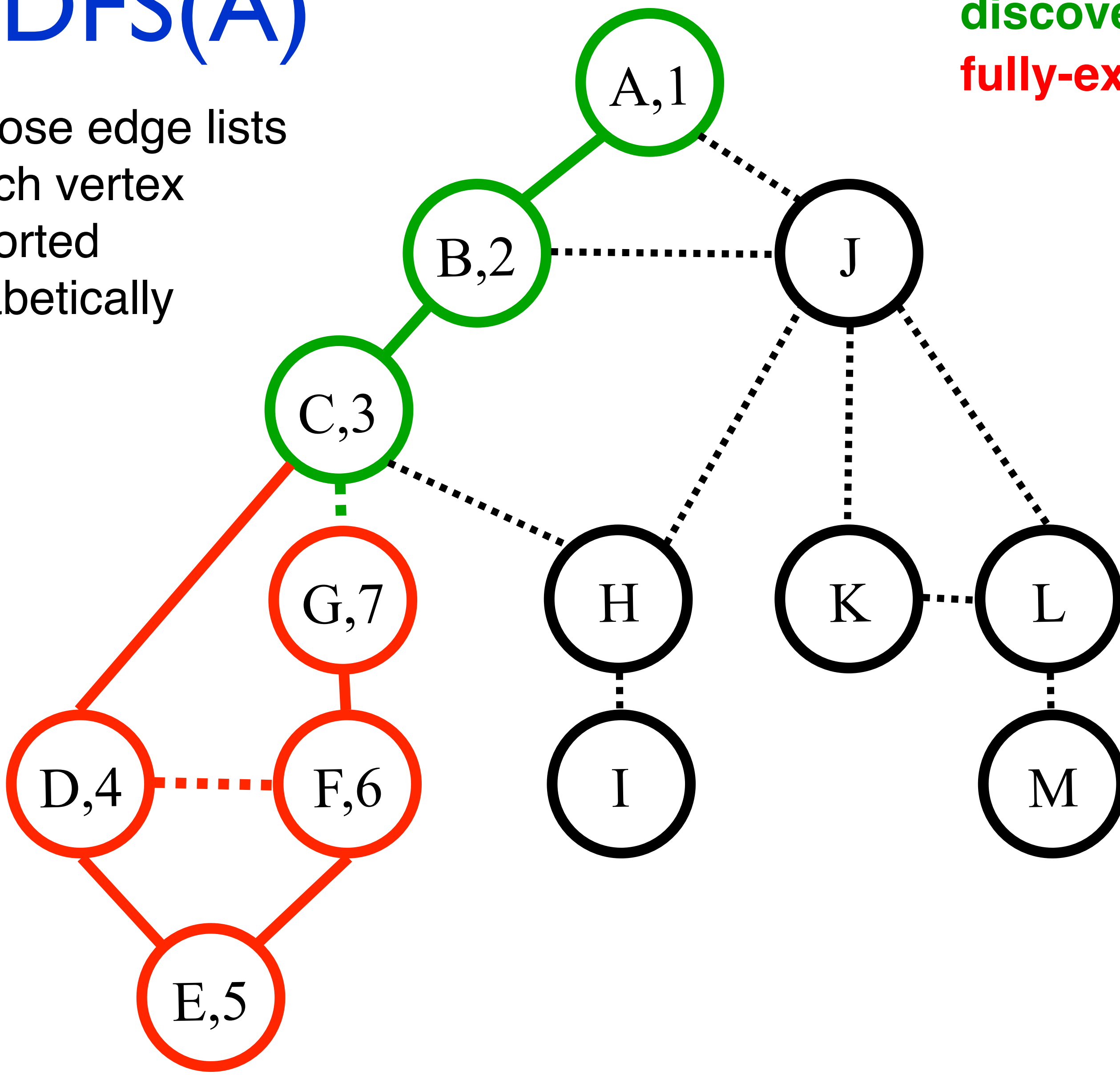
Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

discovered

fully-explored

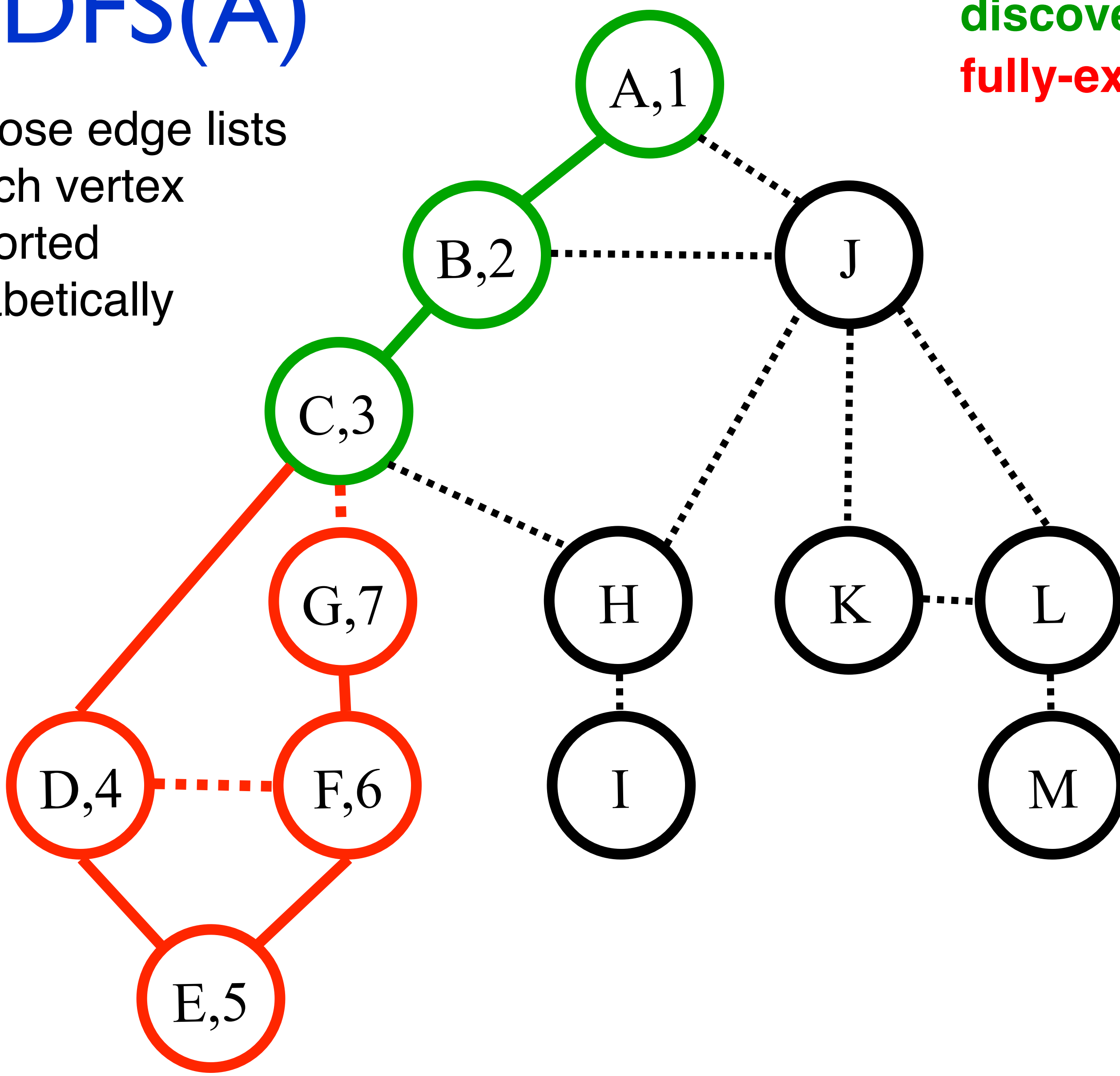


Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Color code:

undiscovered

discovered

fully-explored

Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,H)

DFS(A)

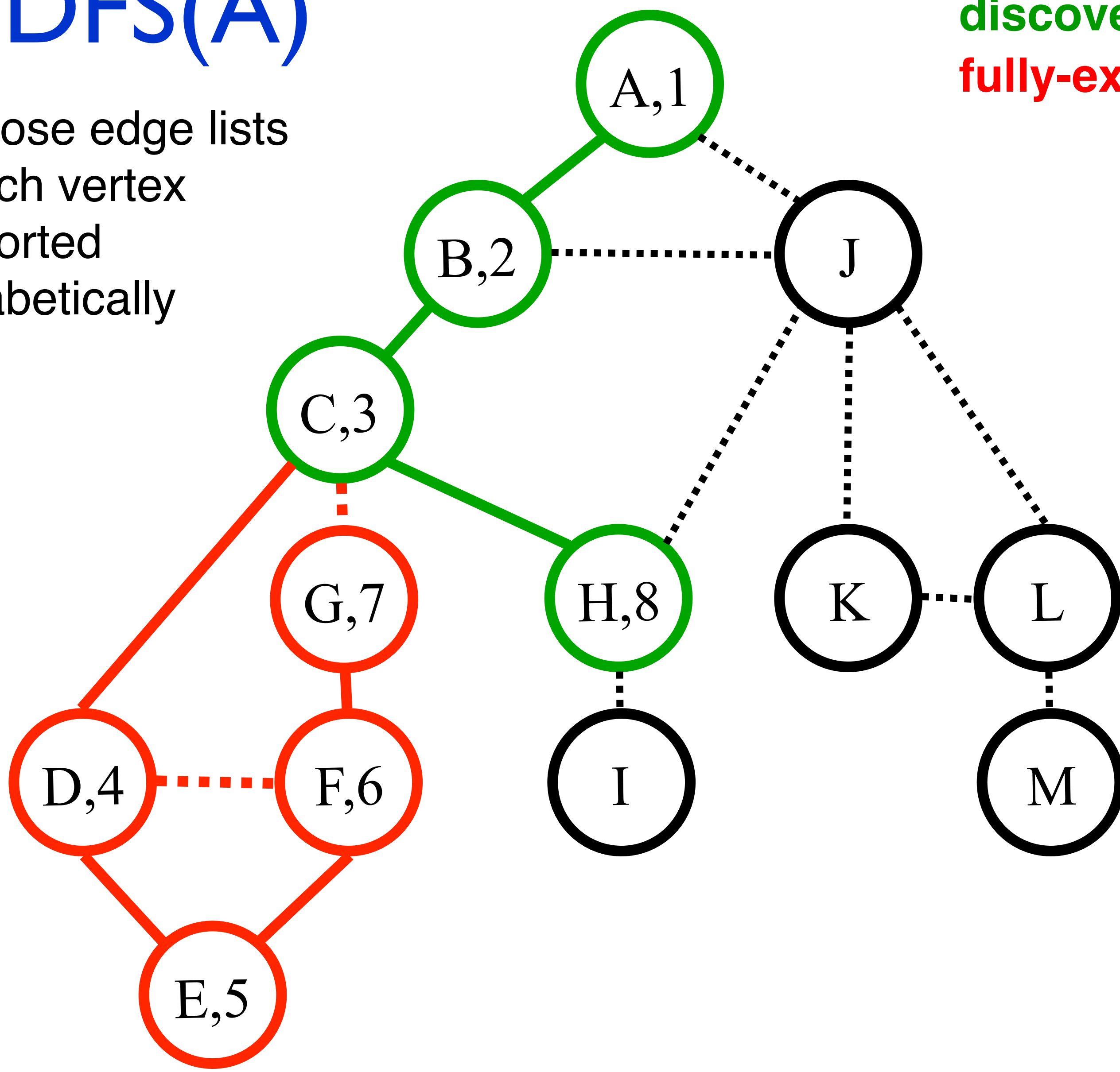
Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (C,I,J)

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

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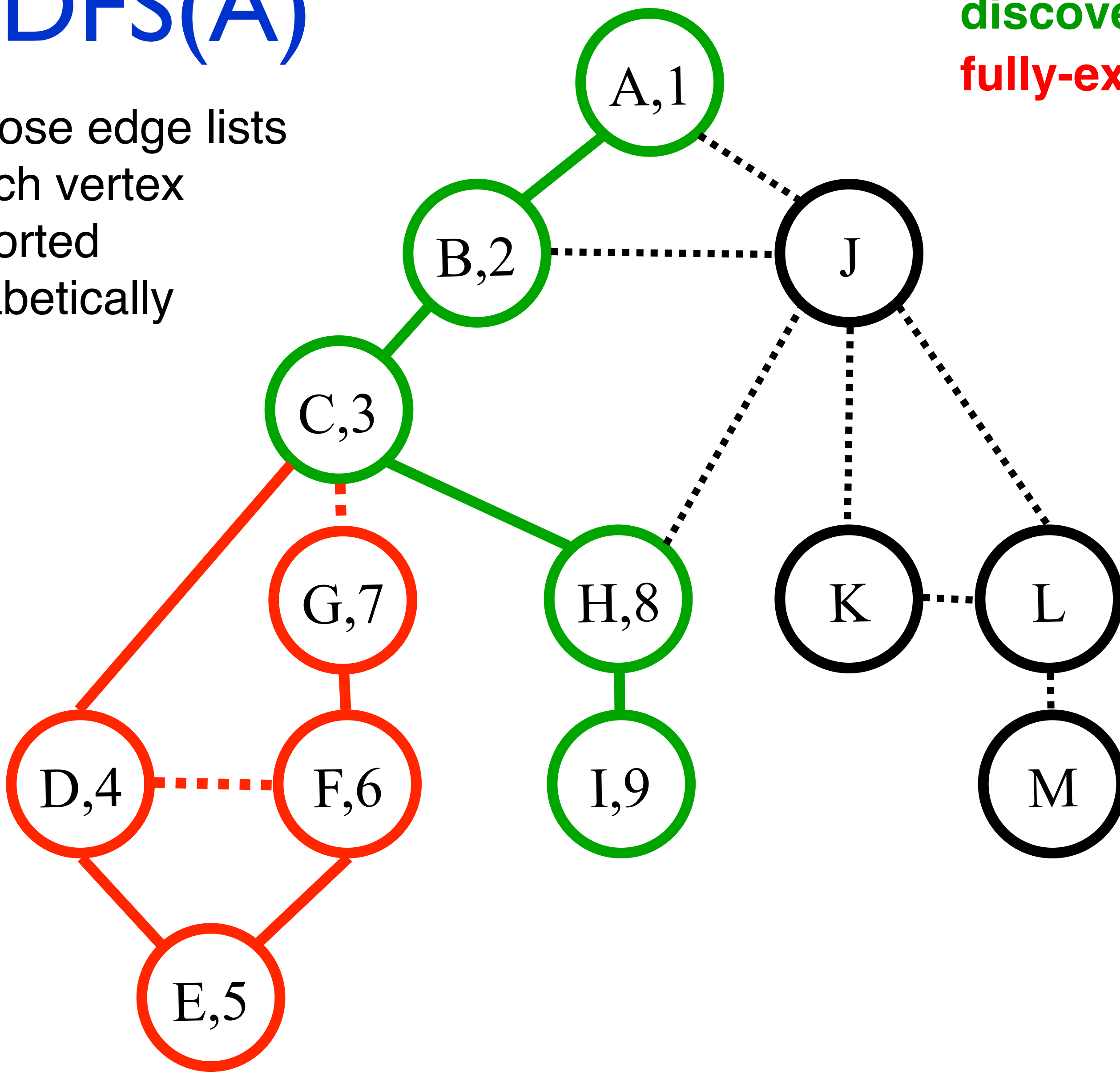
fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,J)
I (H)

DFS(A)

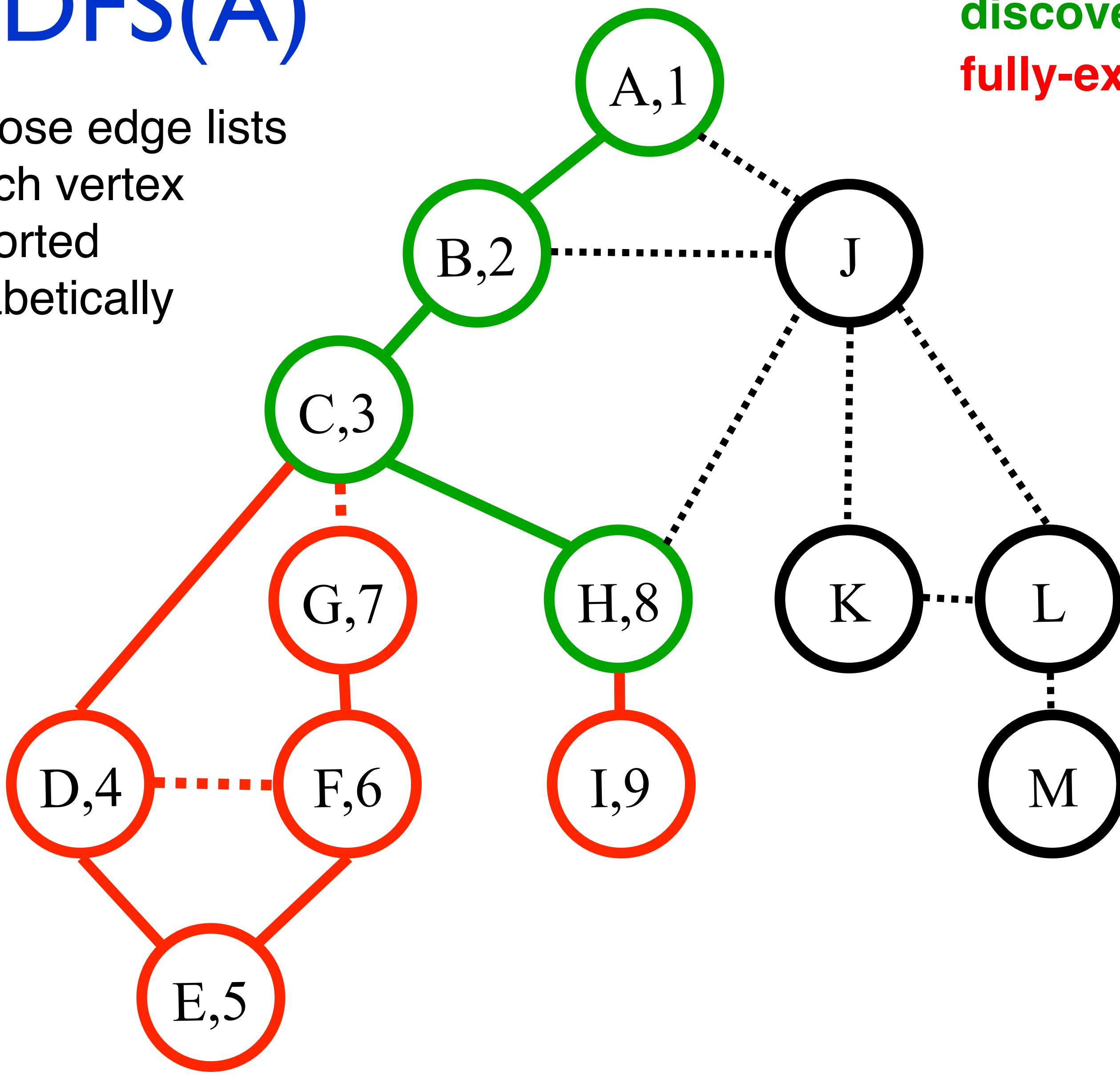
Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

discovered

fully-explored

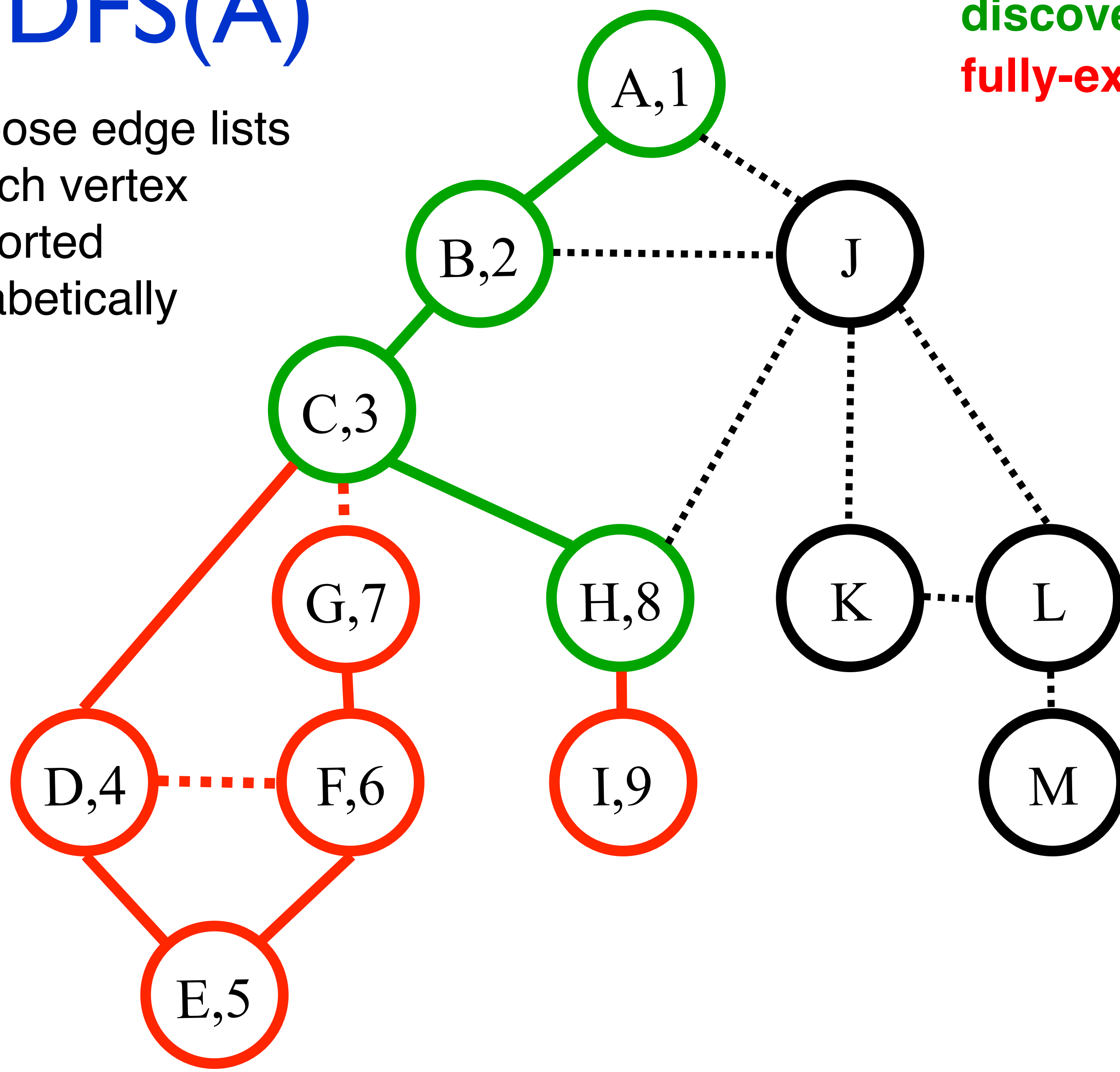


Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,J)
I (~~H~~)

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Color code:

undiscovered

discovered

fully-explored

Call Stack:
(Edge list)

A (B,J)
B (A,C,J)
C (B,D,G,H)
H (C,I,J)

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically

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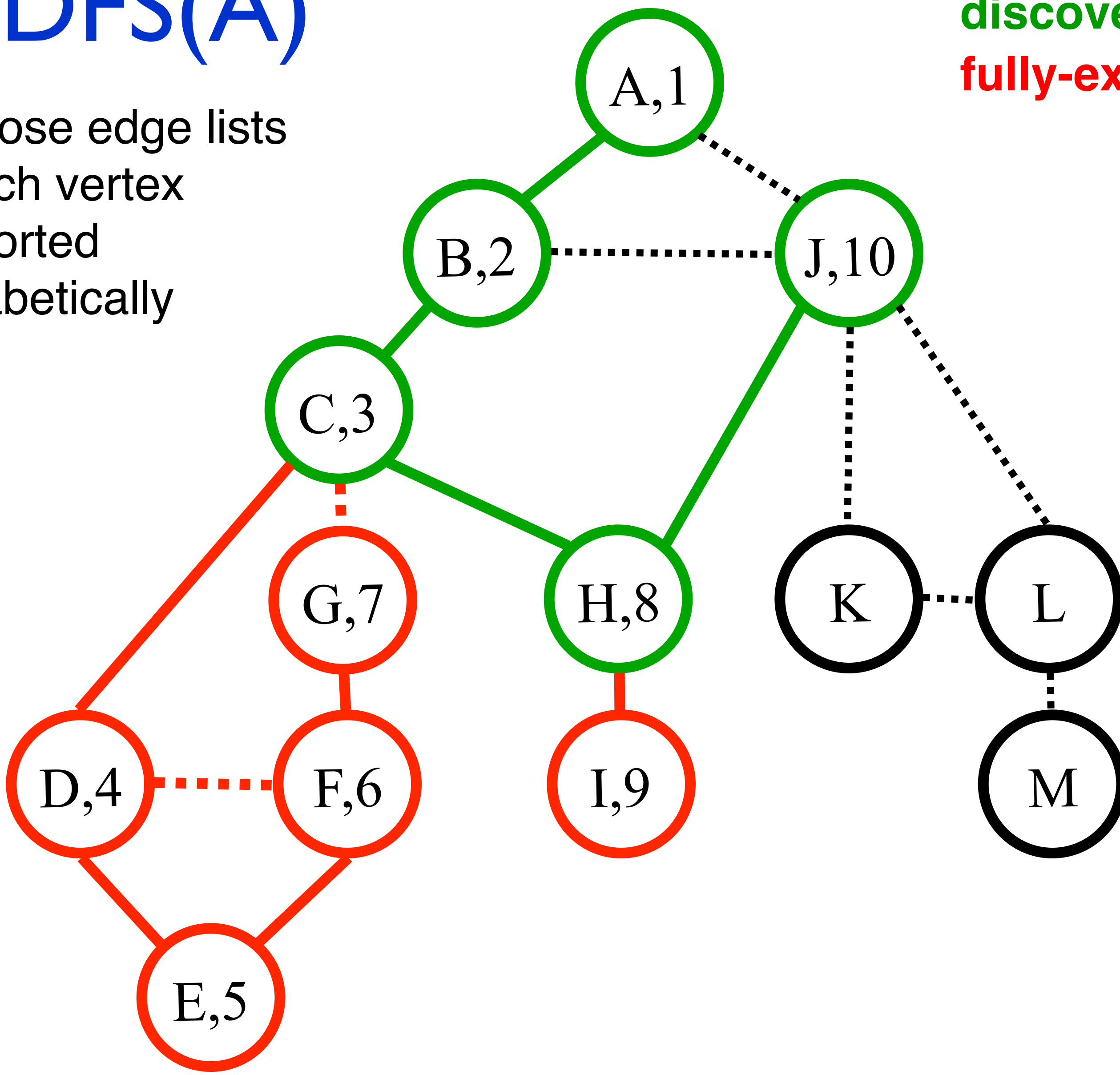
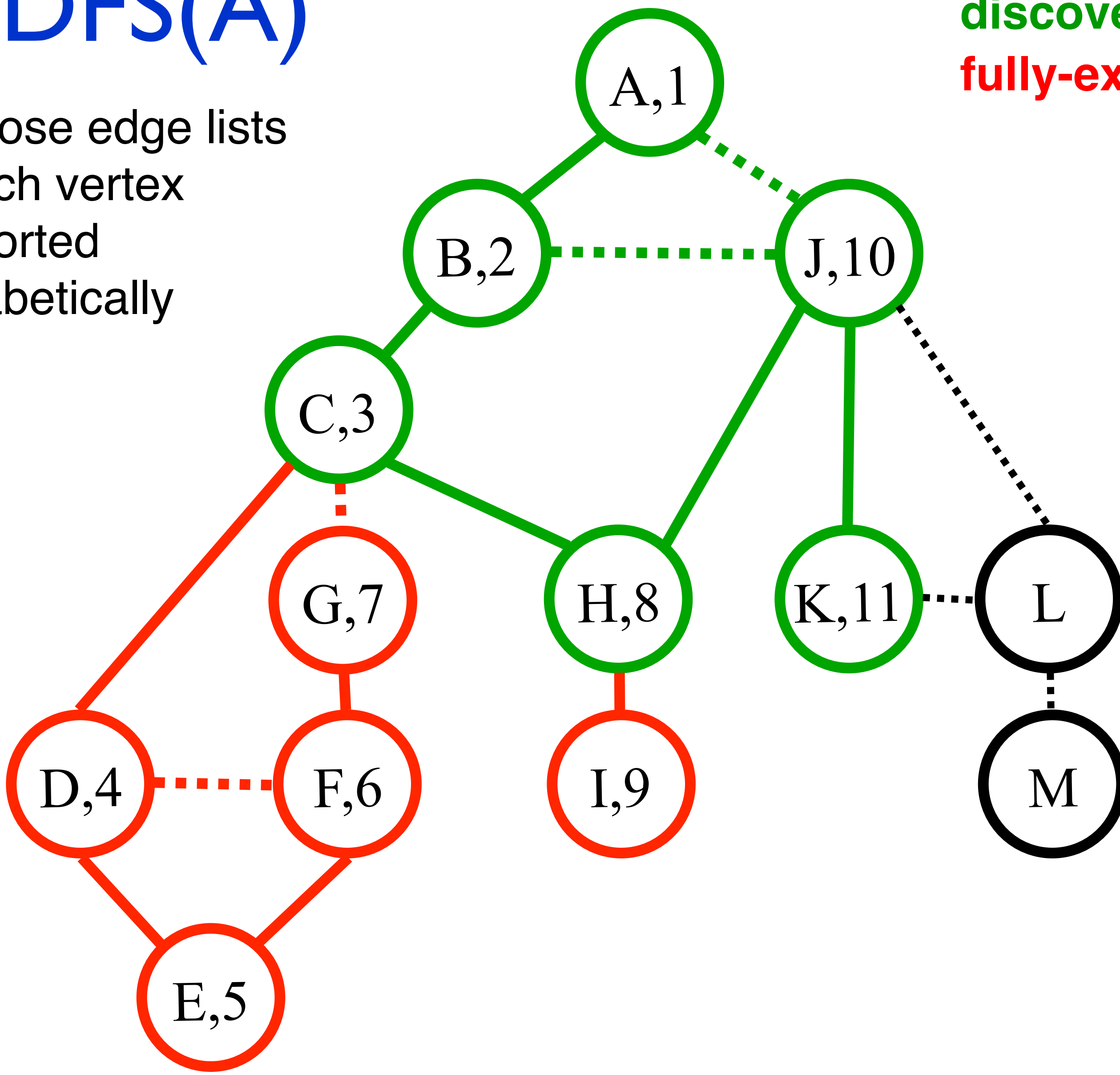


Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (A,B,H,K,L)

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Color code:

undiscovered

discovered

fully-explored

Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (~~A~~,~~B~~,~~H~~,~~K~~,L)
K (J,L)

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically

Color code:

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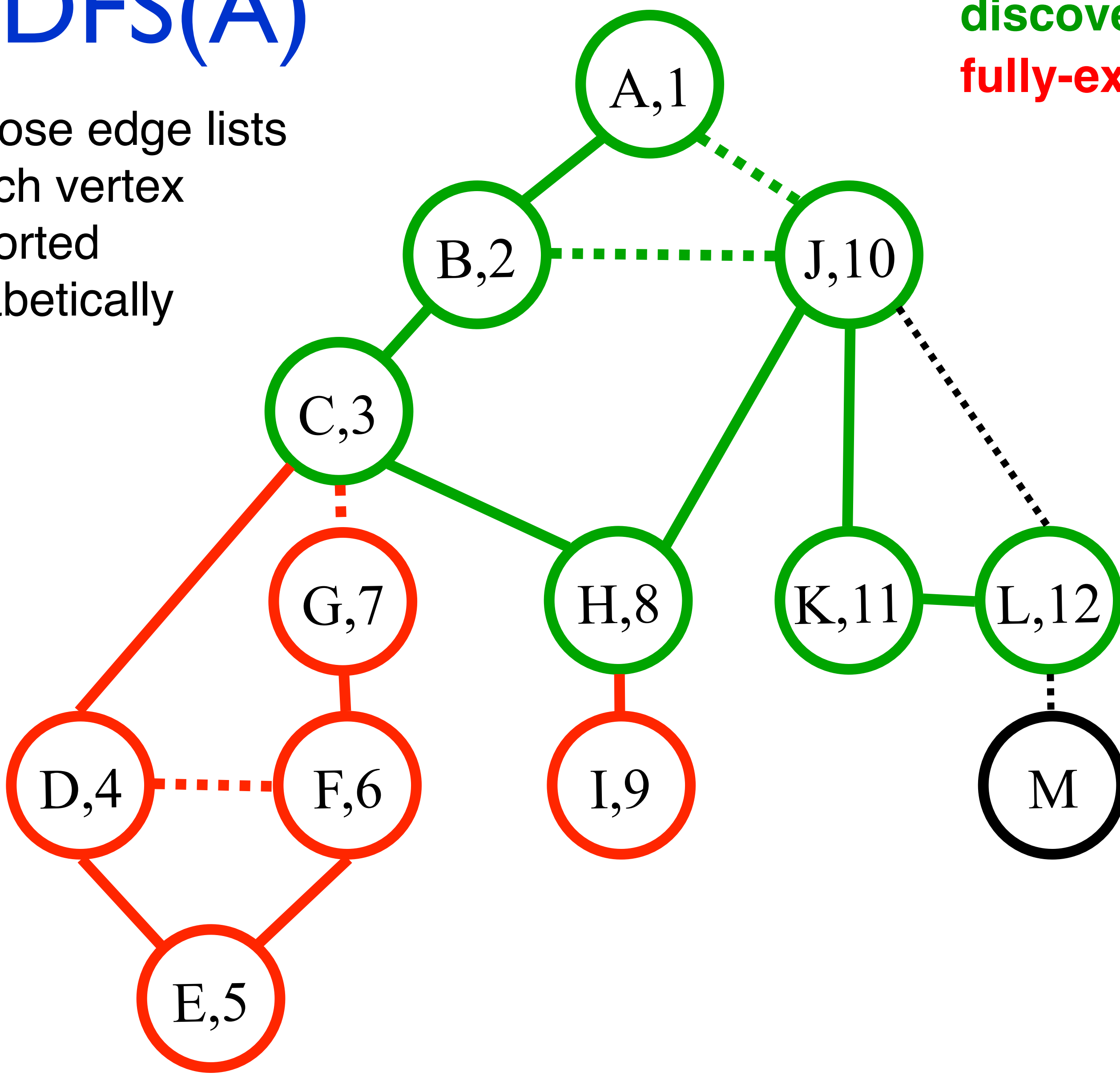
discovered

fully-explored



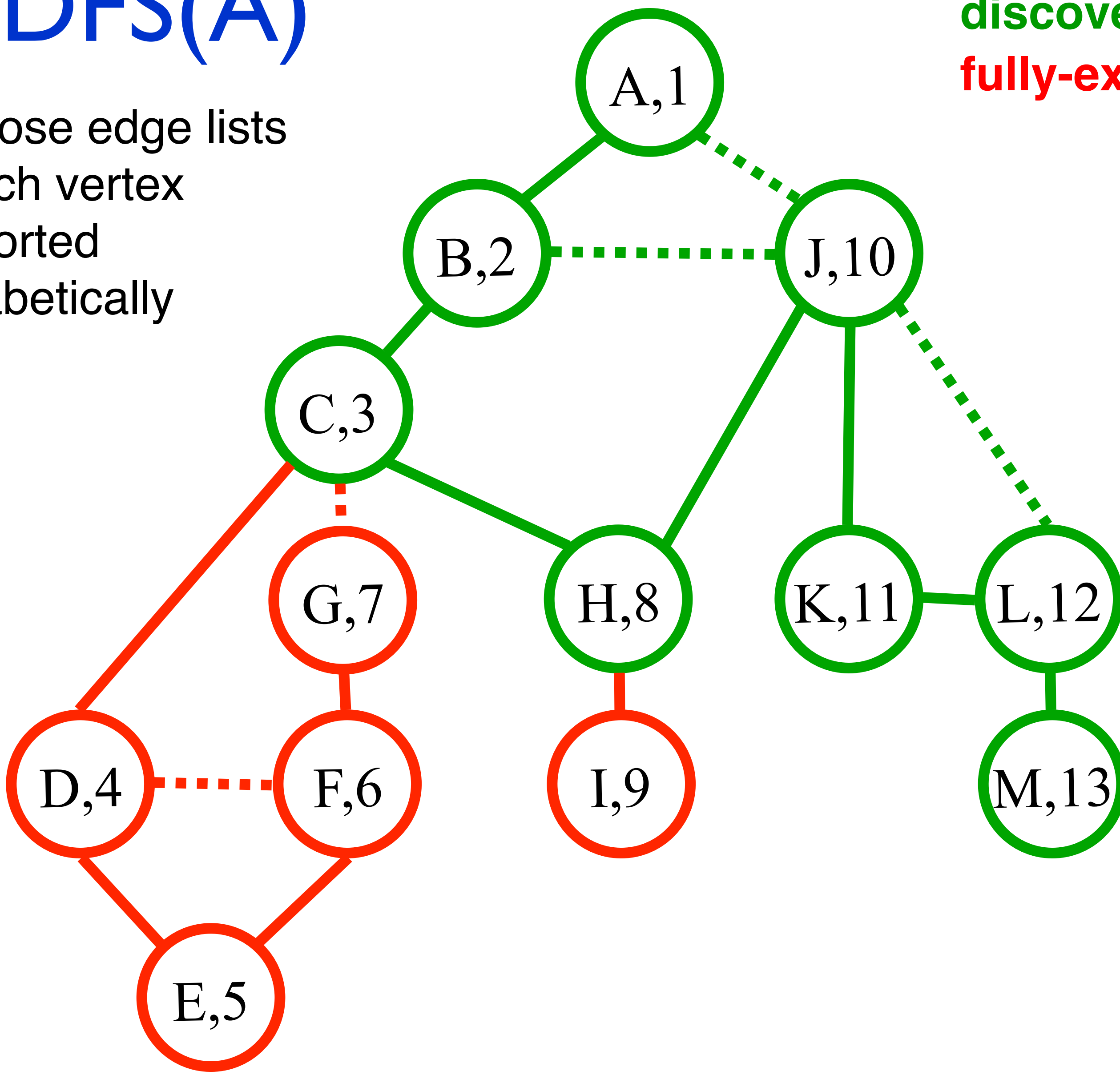
Call Stack:
(Edge list)

A (B,J)
B (A,C,J)
C (B,D,G,H)
H (C,I,J)
J (A,B,H,K,L)
K (J,L)
L (J,K,M)

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically

Color code:
undiscovered
discovered
fully-explored



Call Stack:
(Edge list)

A (B,J)
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C (B,D,G,H)
H (C,I,J)
J (A,B,H,K,L)
K (J,L)
L (J,K,M)
M (L)

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically

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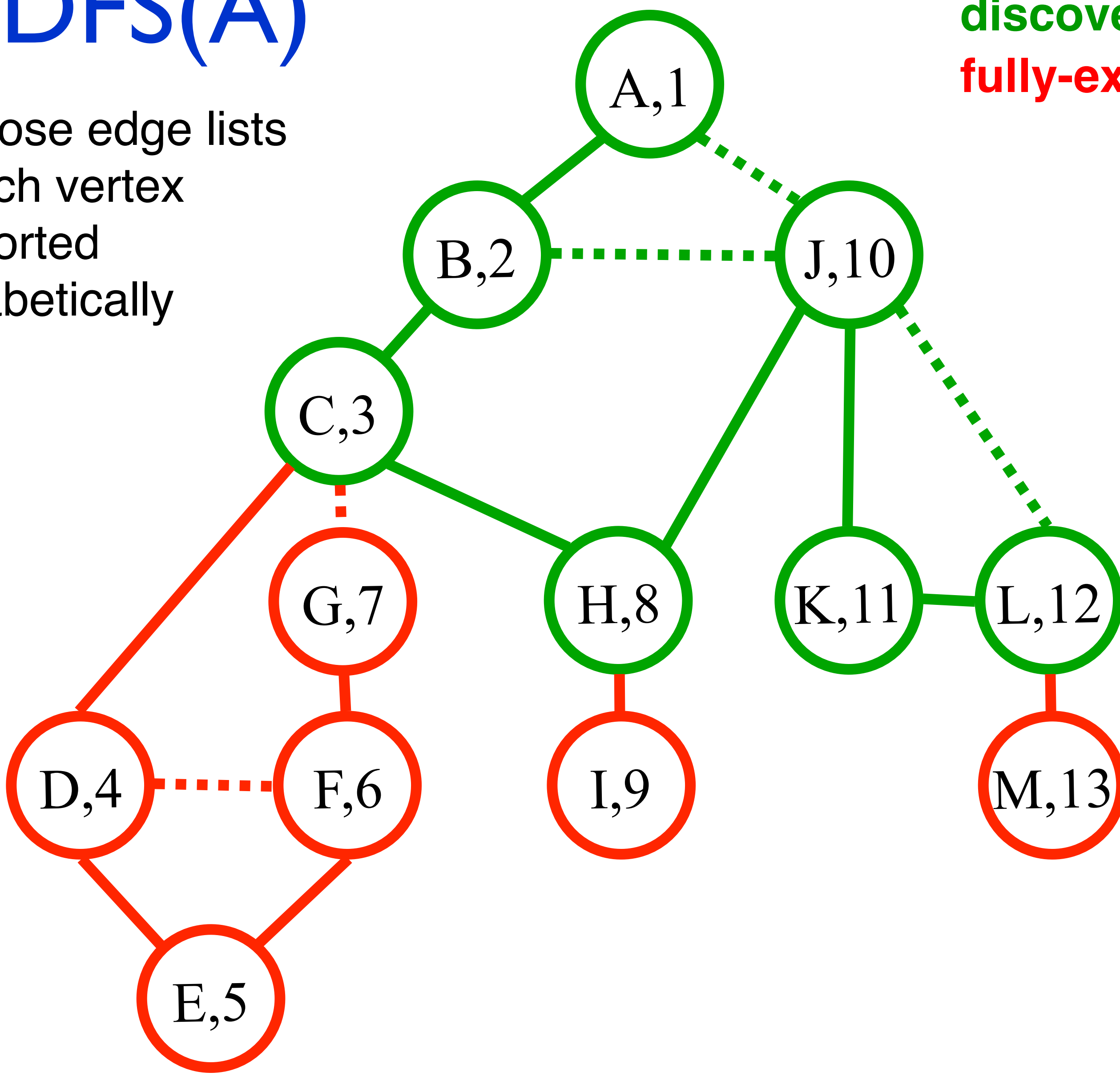

Call Stack: (Edge list)
A (B,J)
B (A,C,J)
C (B,D,G,H)
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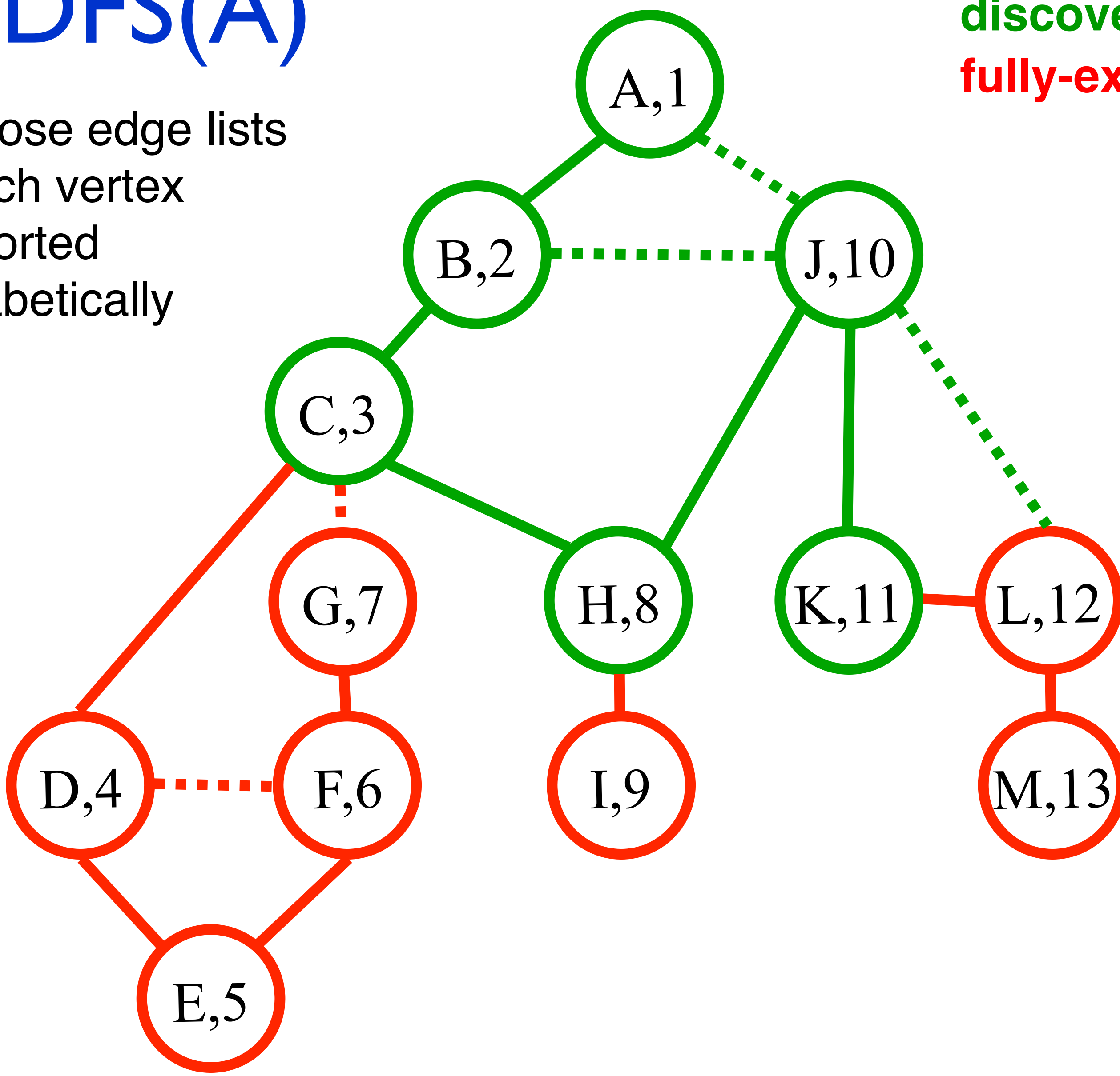

Call Stack:
(Edge list)

A (B,J)
B (A,C,J)
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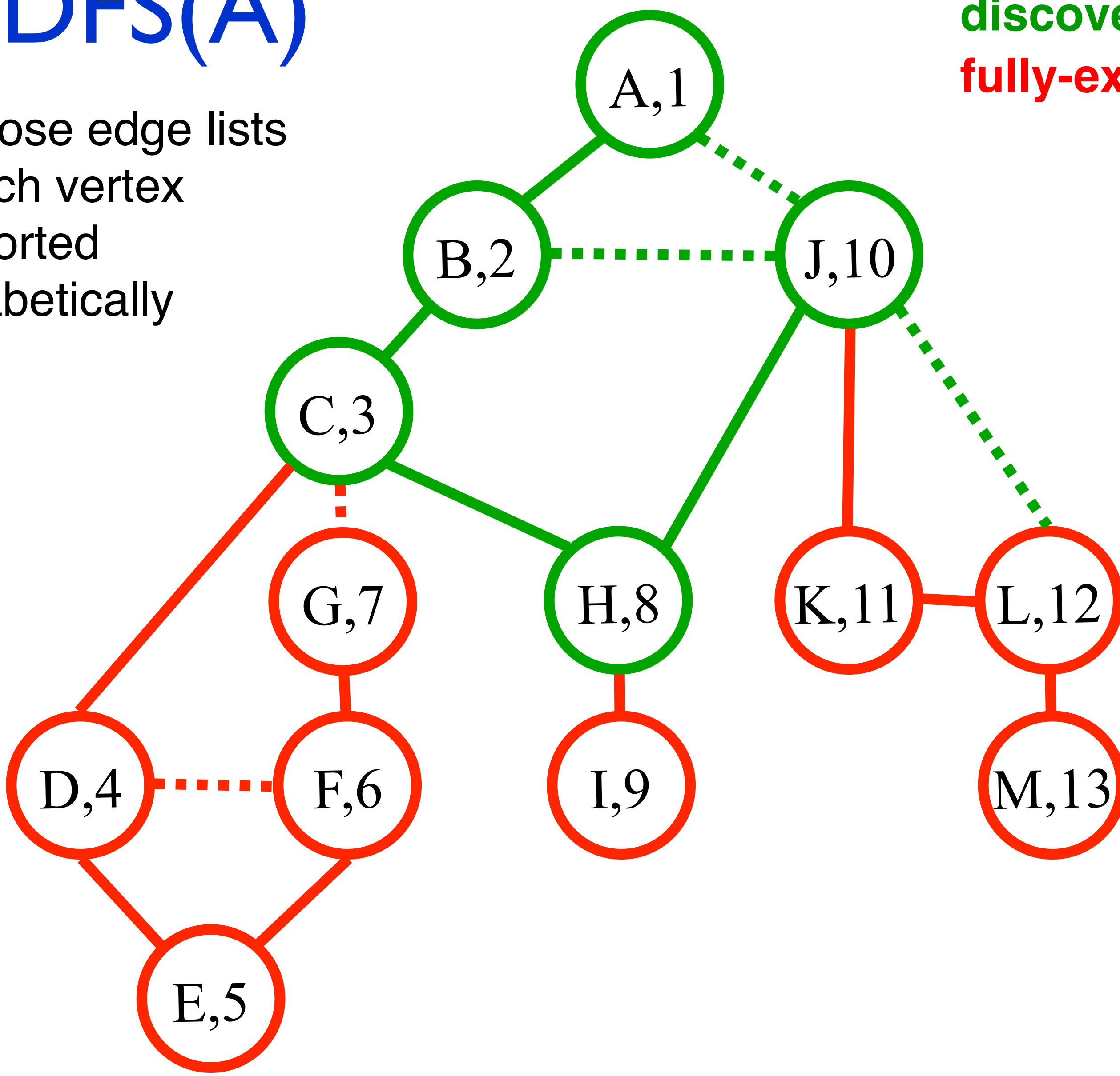
DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Color code:

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fully-explored

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A (B,J)
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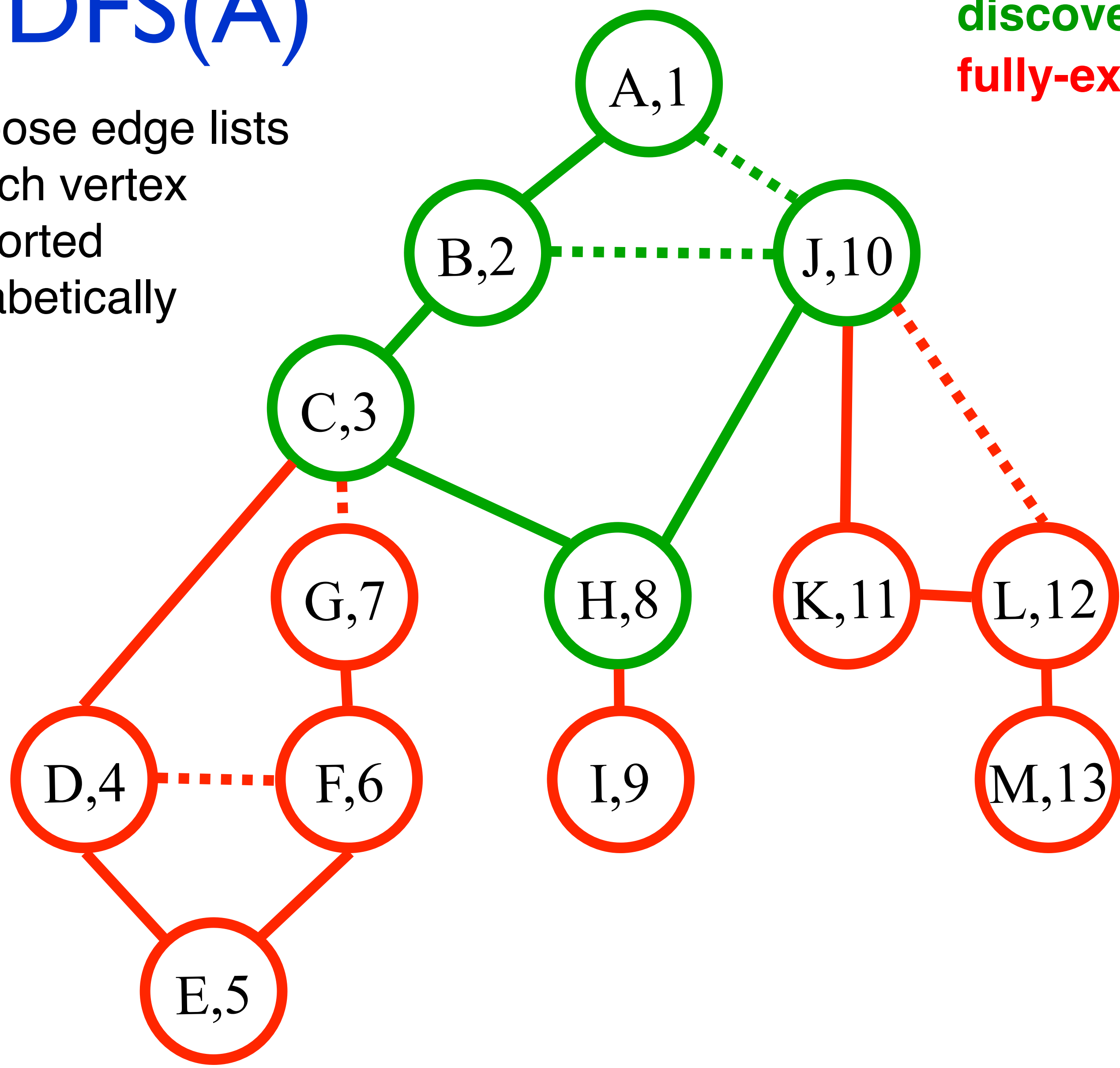
DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Color code:

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B (A,C,J)
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DFS(A)

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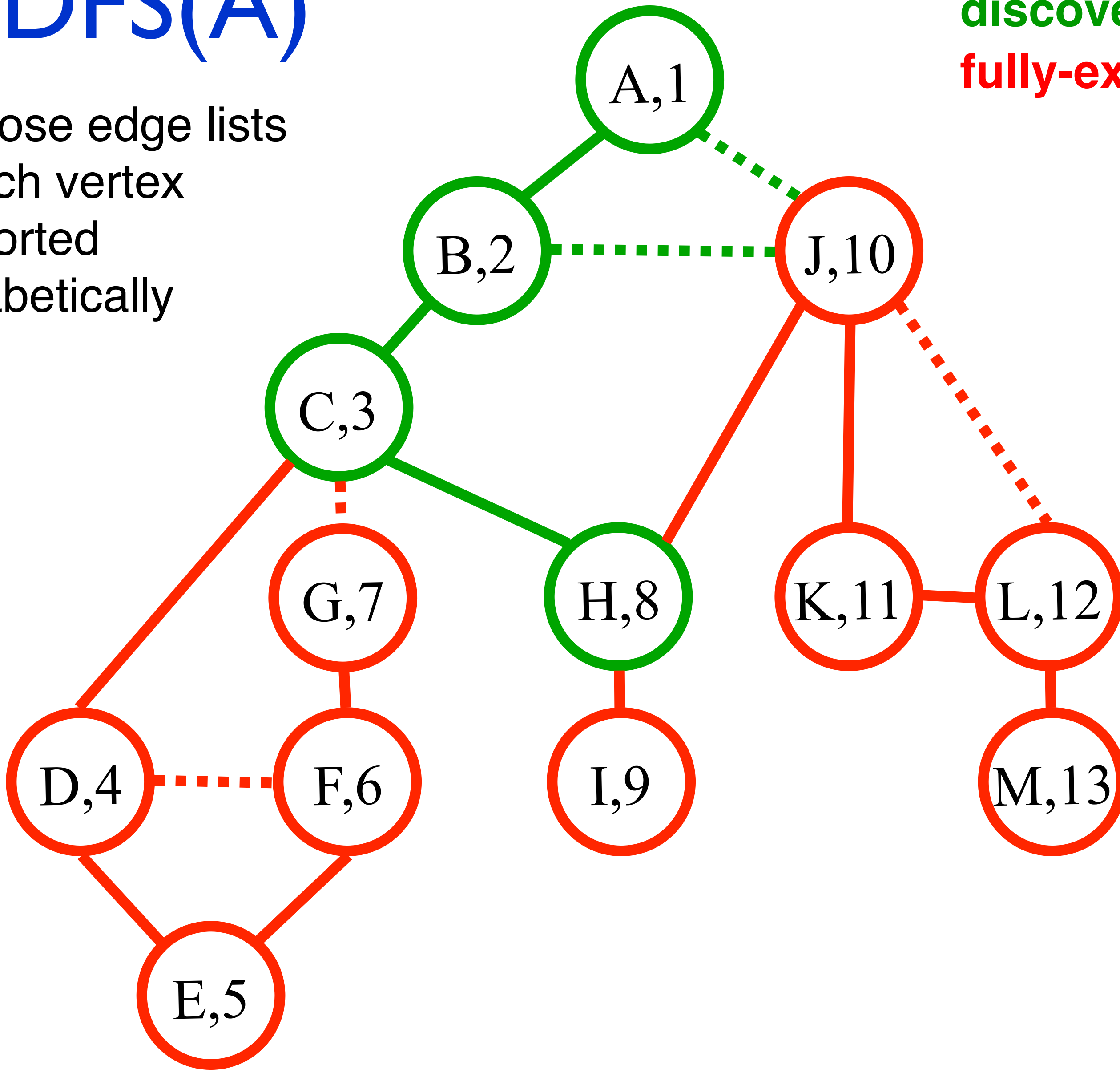

Call Stack:
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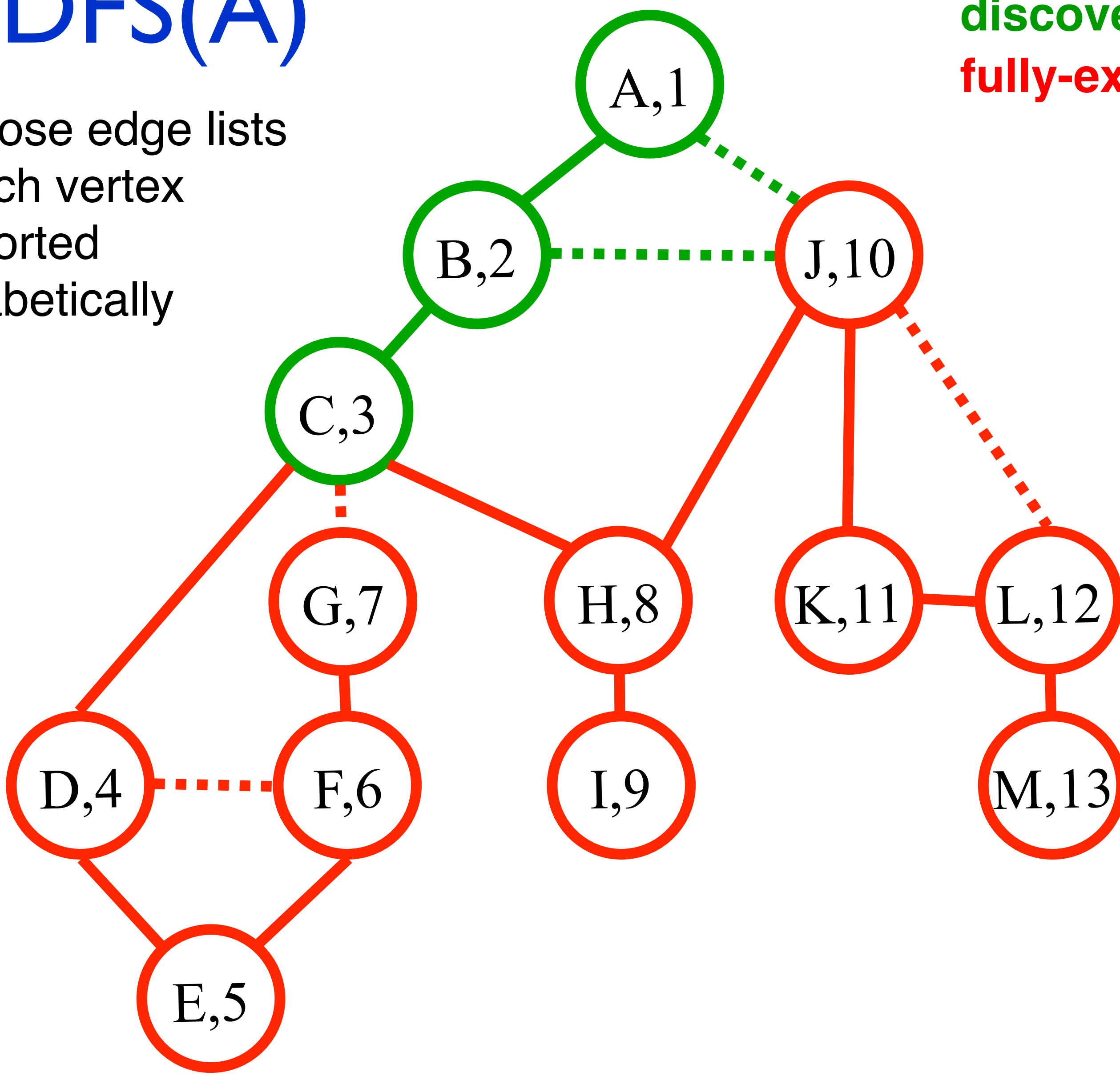
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Color code:

undiscovered

discovered

fully-explored

Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)

DFS(A)

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at each vertex
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alphabetically

Color code:
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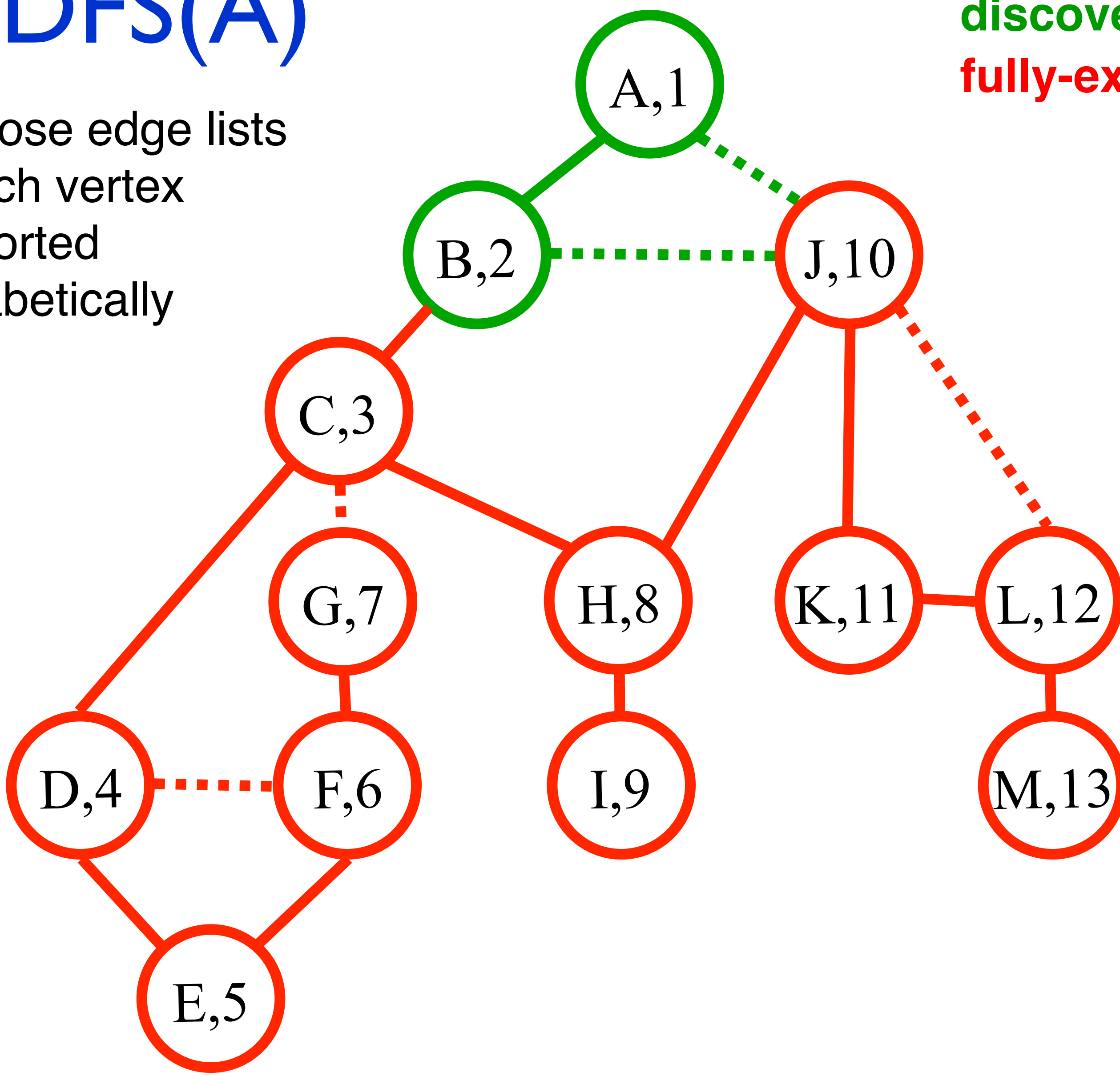
fully-explored



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(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically

Color code:

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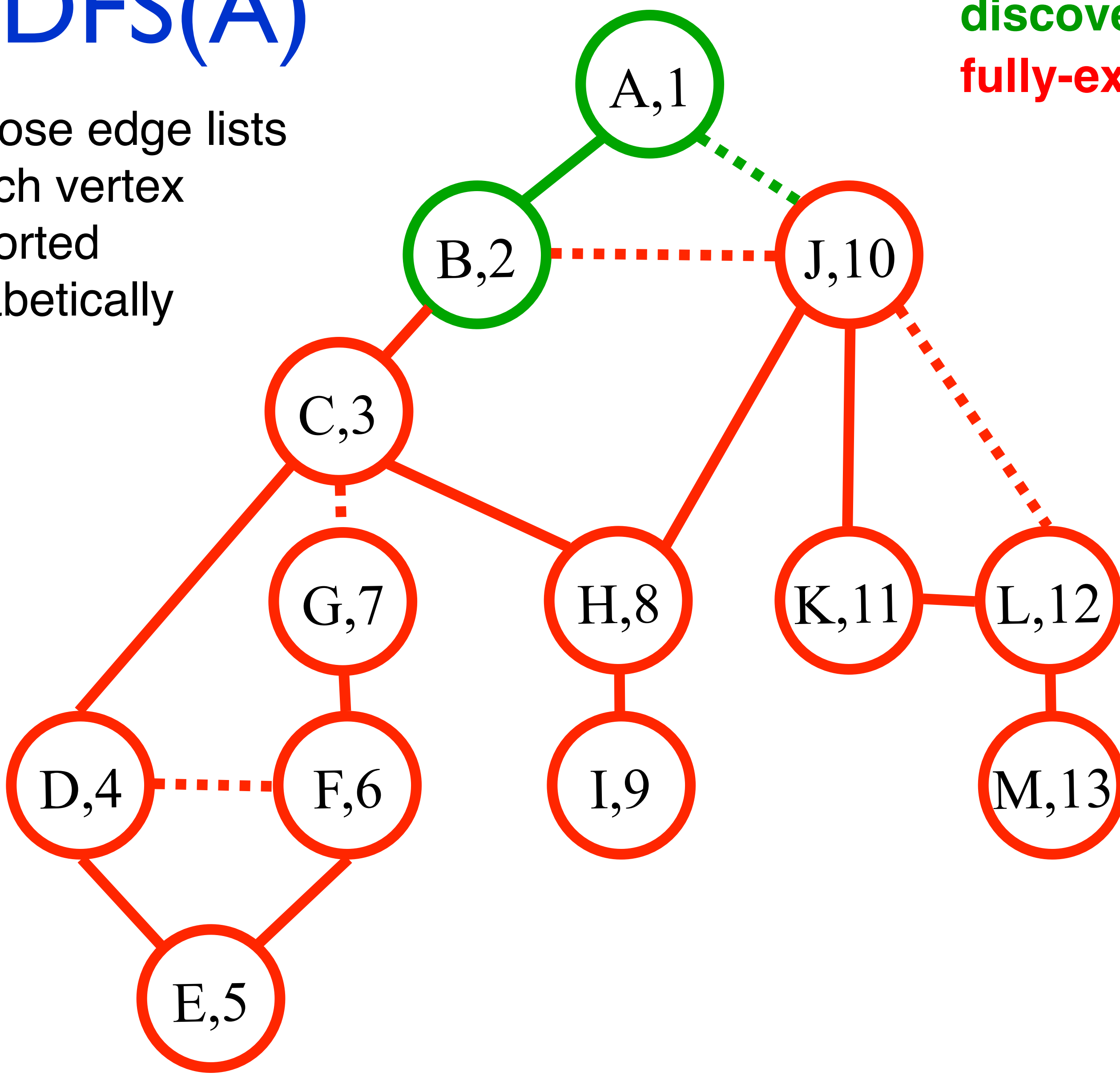

Call Stack:
(Edge list)

A (B,J)
B (A,C,J)

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically

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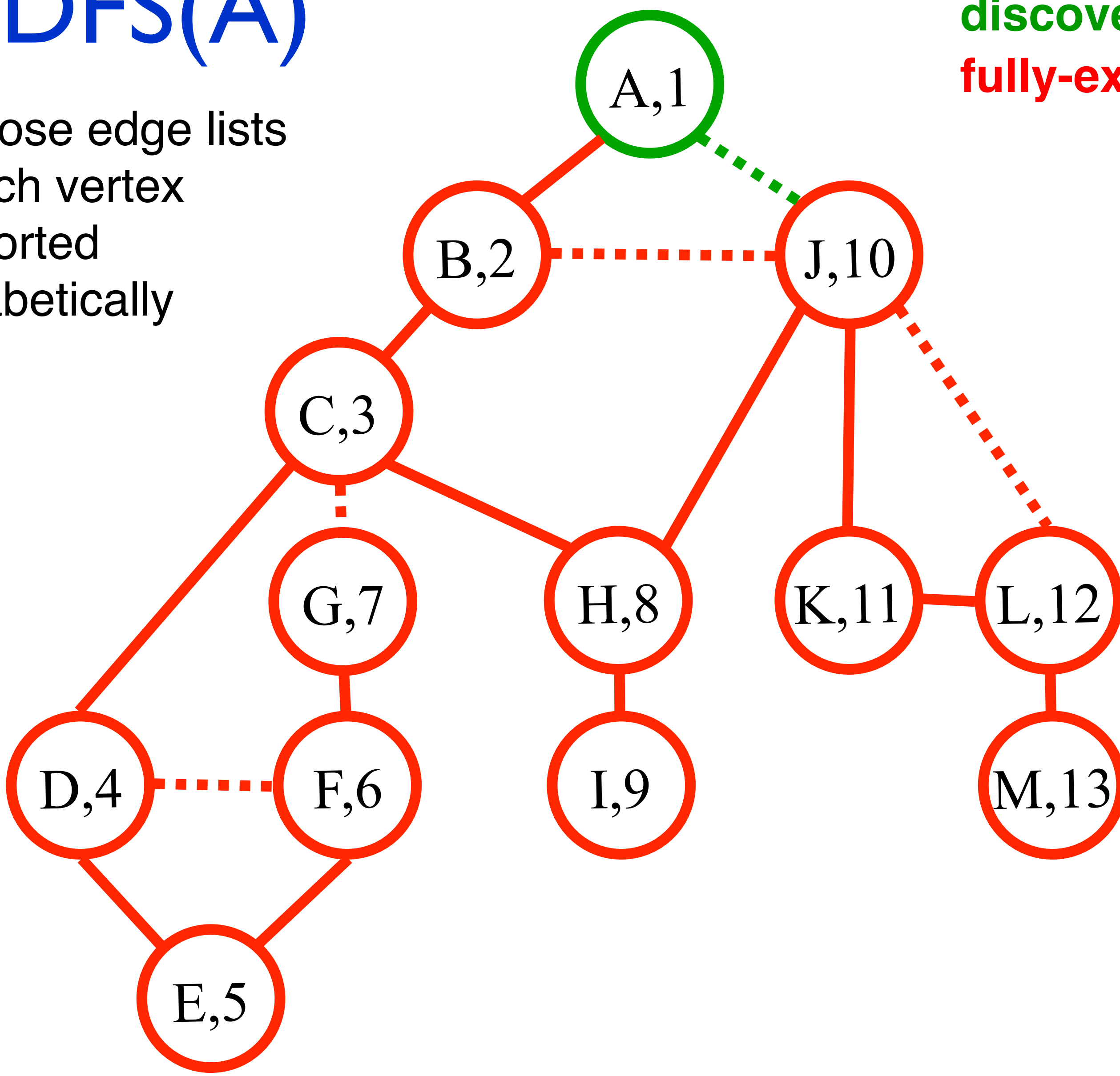

Call Stack:
(Edge list)

A (B,J)

DFS(A)

Suppose edge lists
at each vertex
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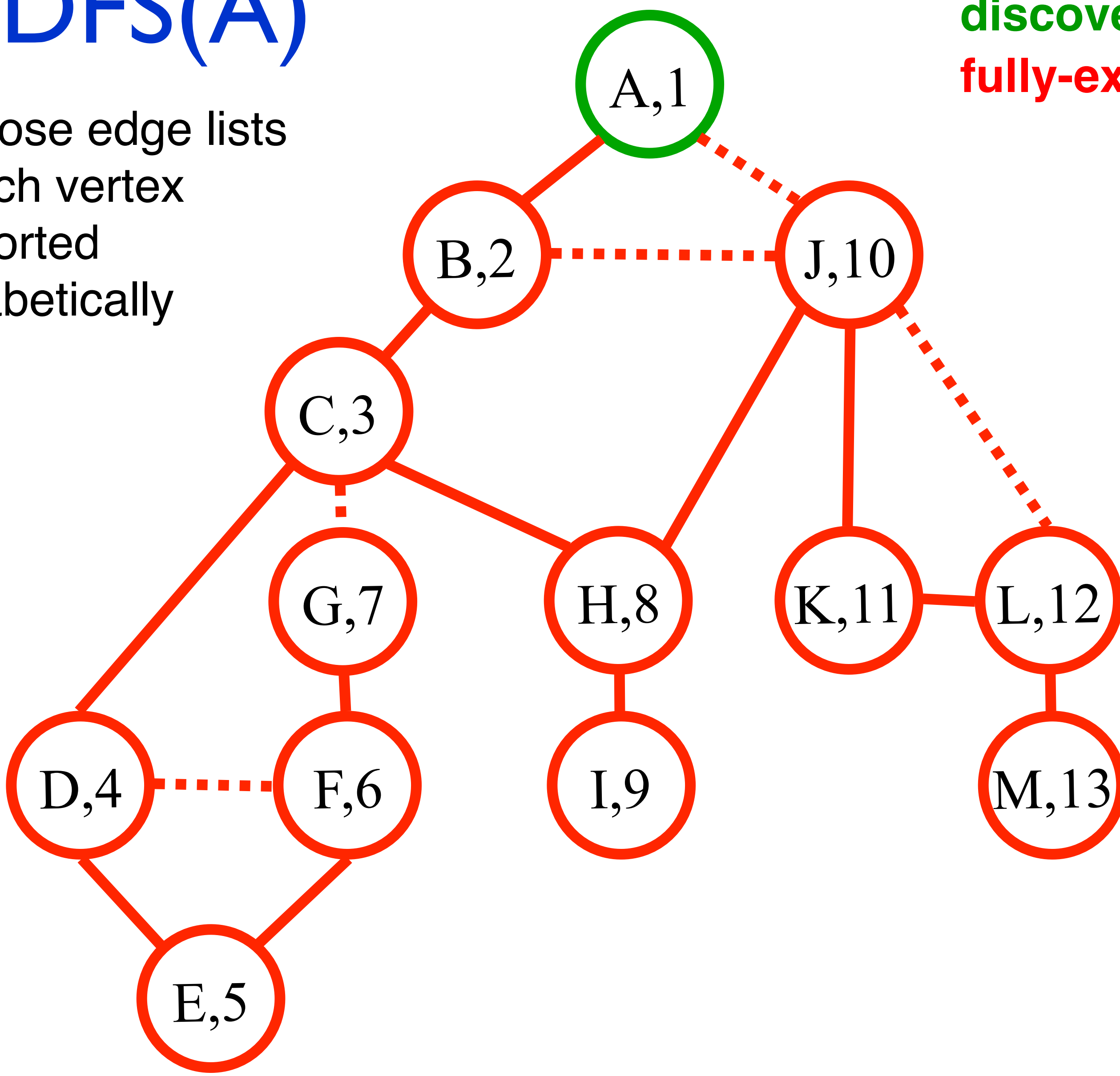
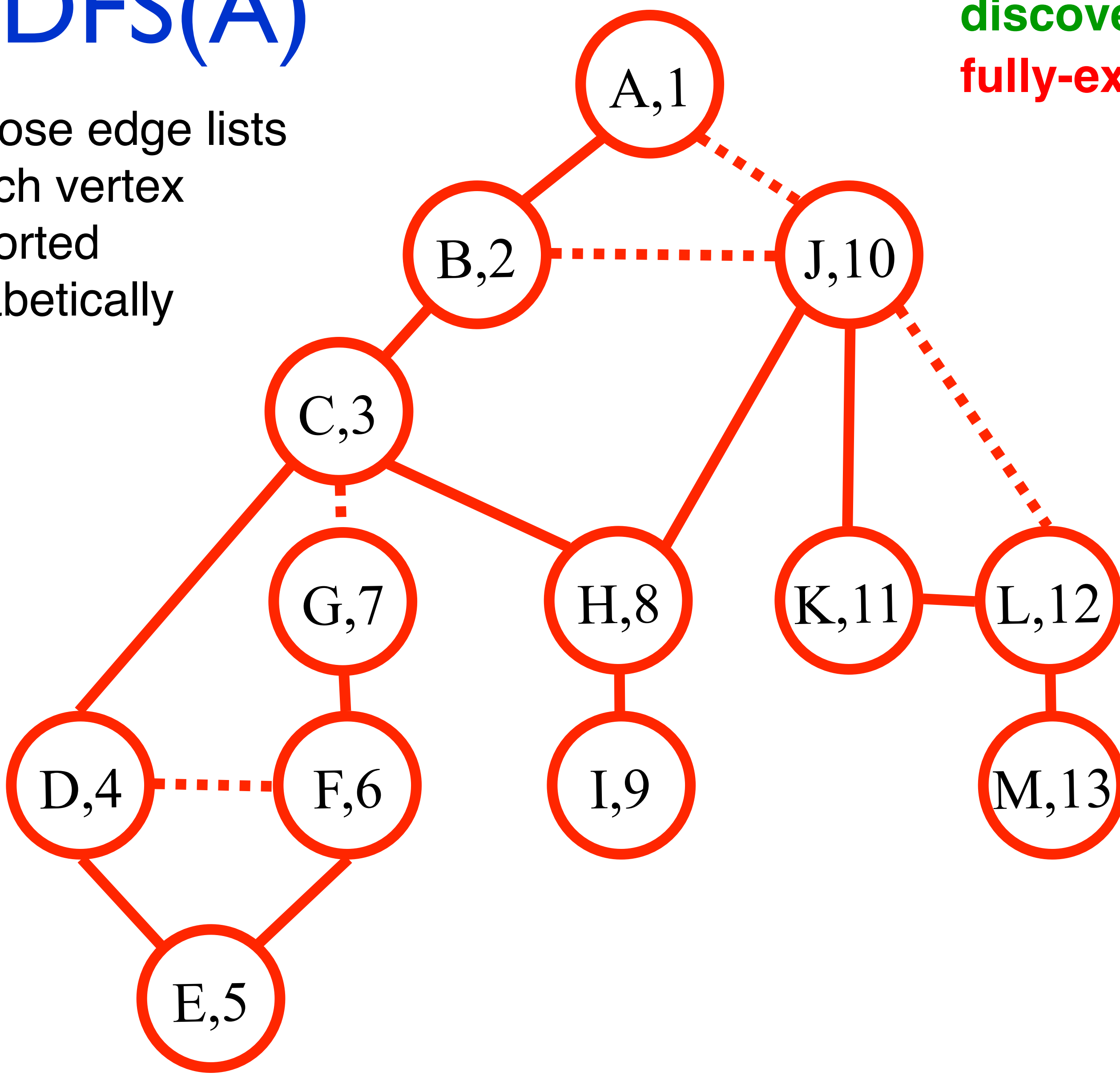


Call Stack:
(Edge list)

A (~~B~~,~~J~~)

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Color code:

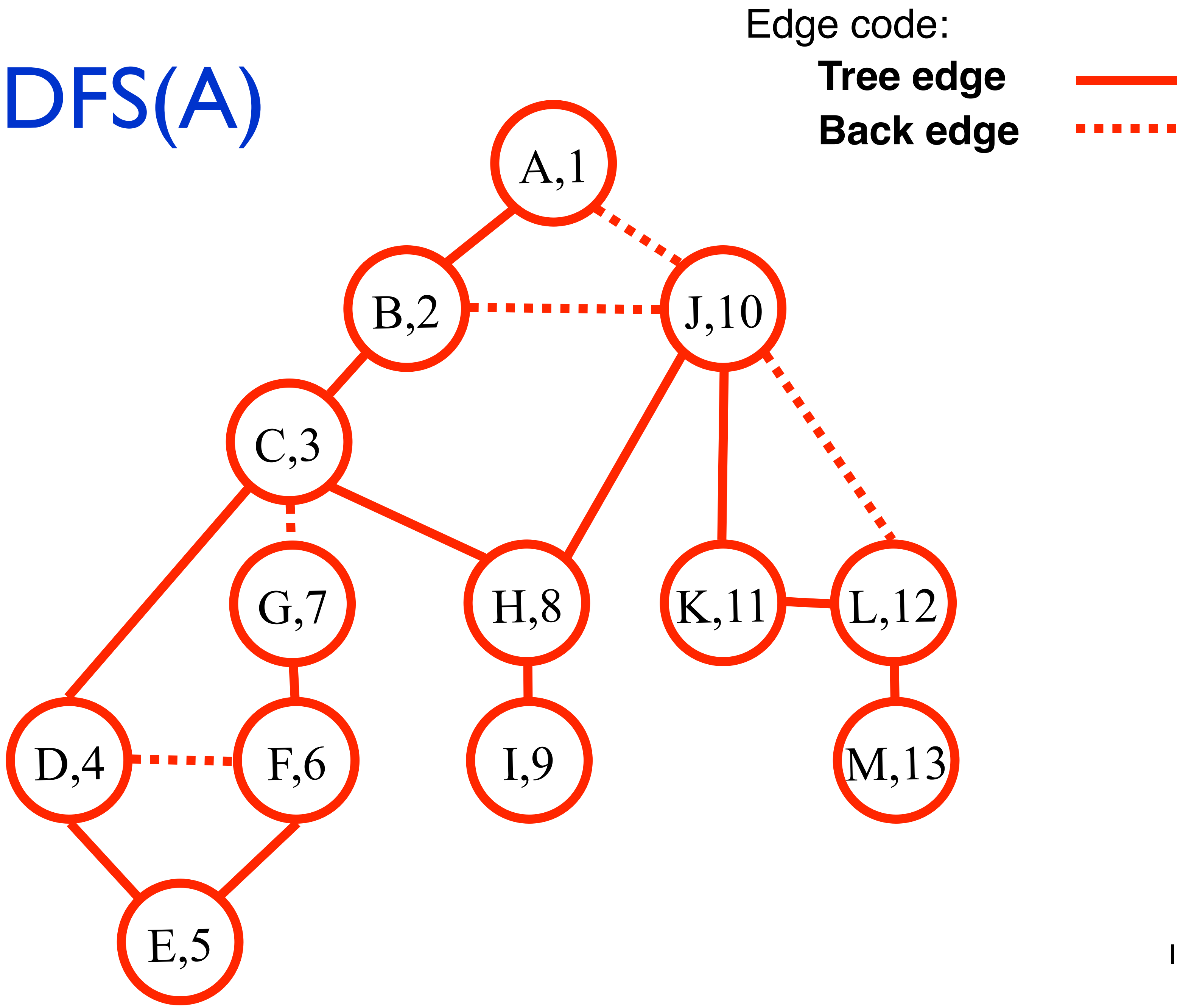
undiscovered

discovered

fully-explored

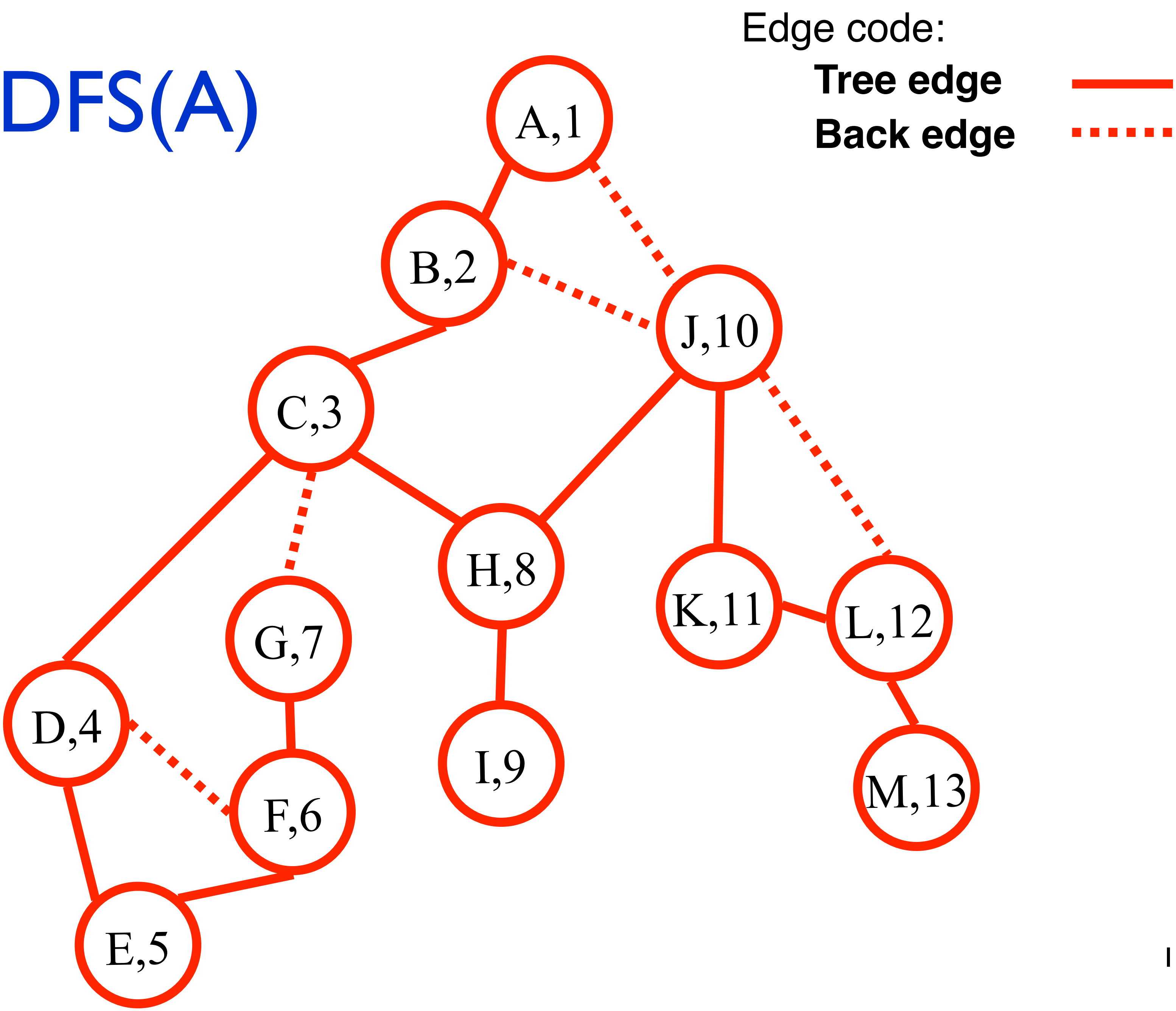
Call Stack: (Edge list)
TA-DA!!

DFS(A)



Edge code:
Tree edge
Back edge
A,1
B,2
J,10
C,3
G,7
H,8
K,11
L,12
D,4
F,6
I,9
M,13
E,5

DFS(A)



Edge code:
Tree edge
Back edge
A,1
B,2
C,3
D,4
E,5
F,6
G,7
H,8
I,9
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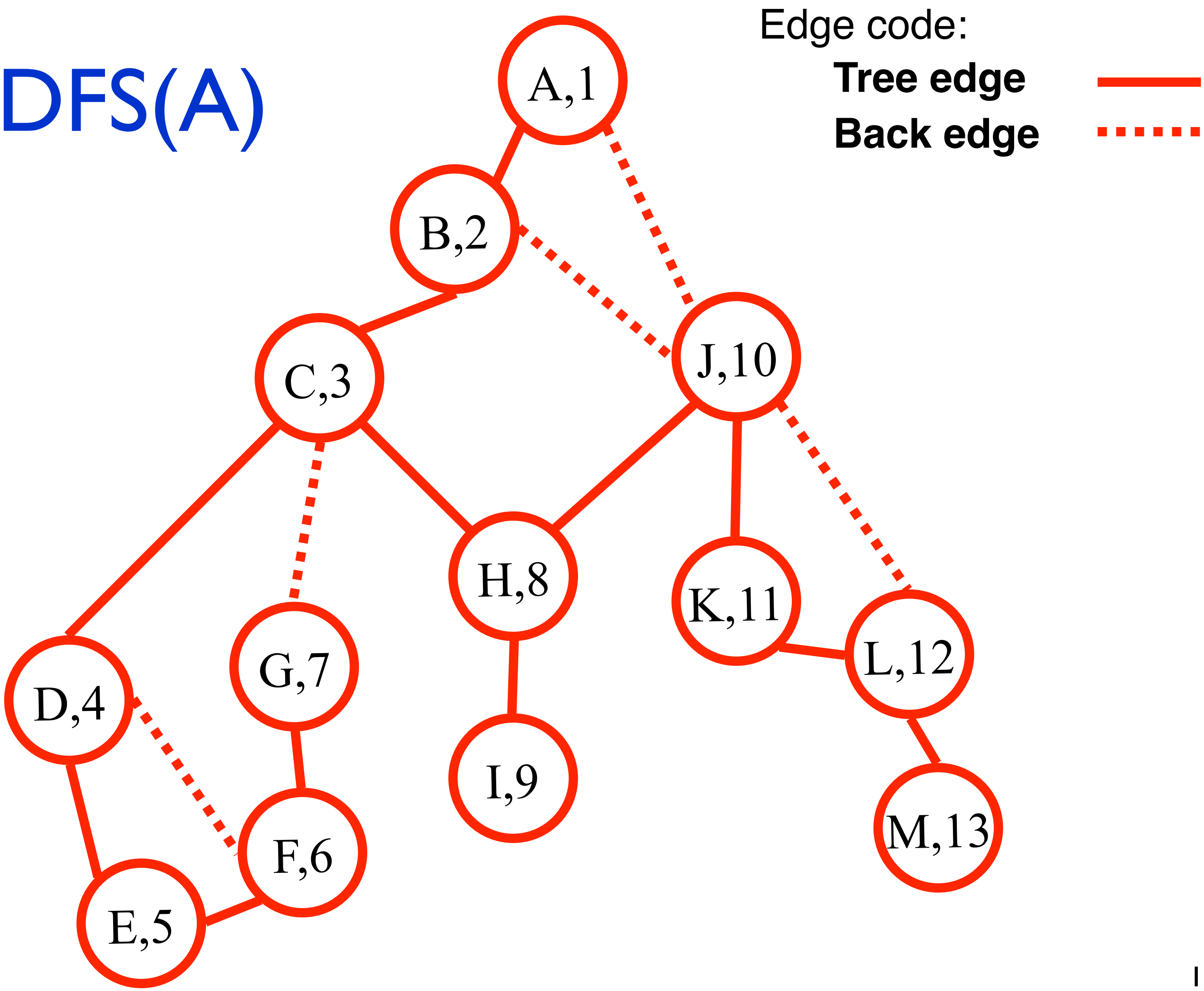
K,11
L,12
M,13

DFS(A)



Edge code:
Tree edge
Back edge
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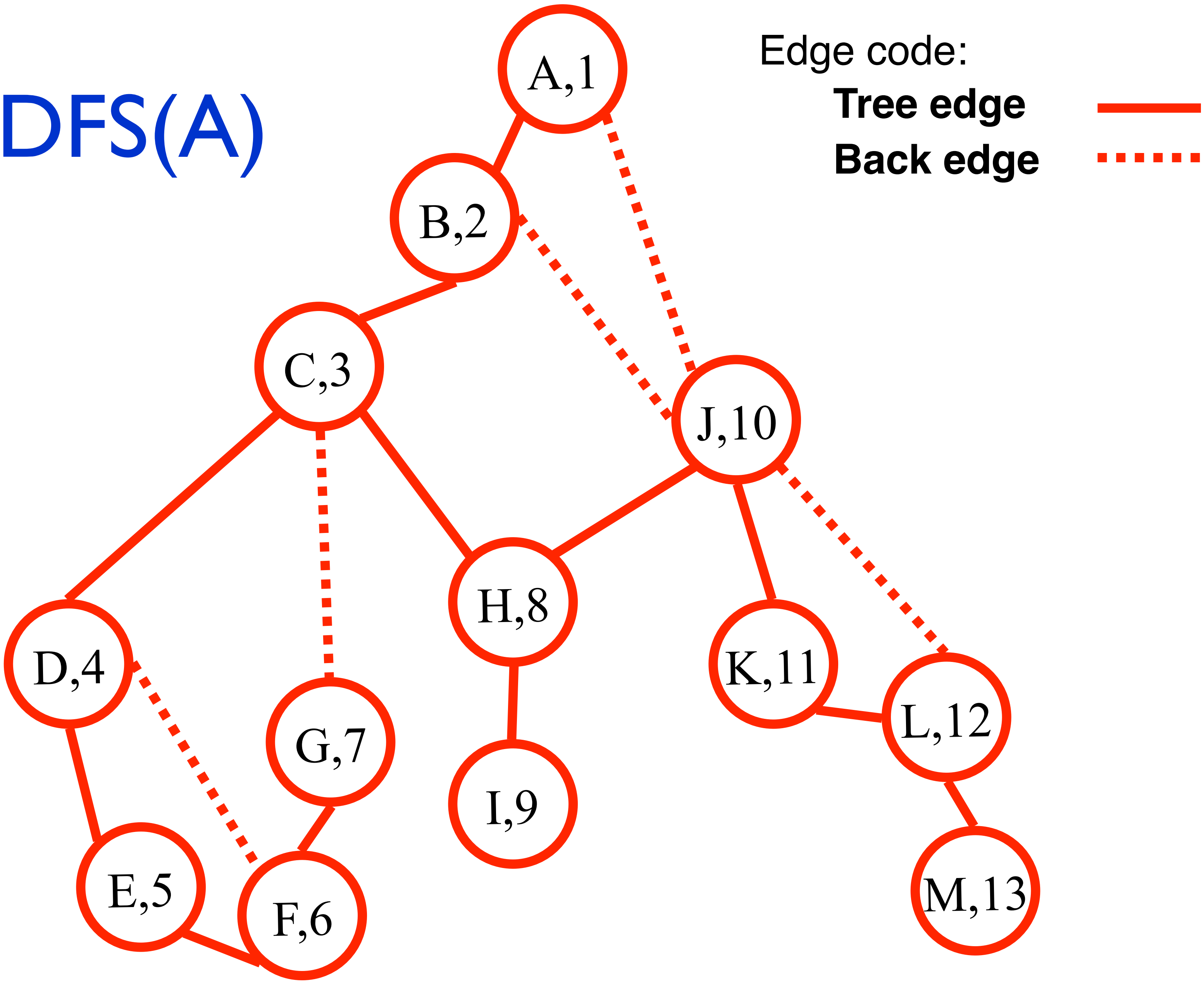
B,2
J,10
C,3
H,8
K,11
L,12
D,4
G,7
I,9
M,13
F,6
E,5

DFS(A)


A,1
B,2
C,3
D,4
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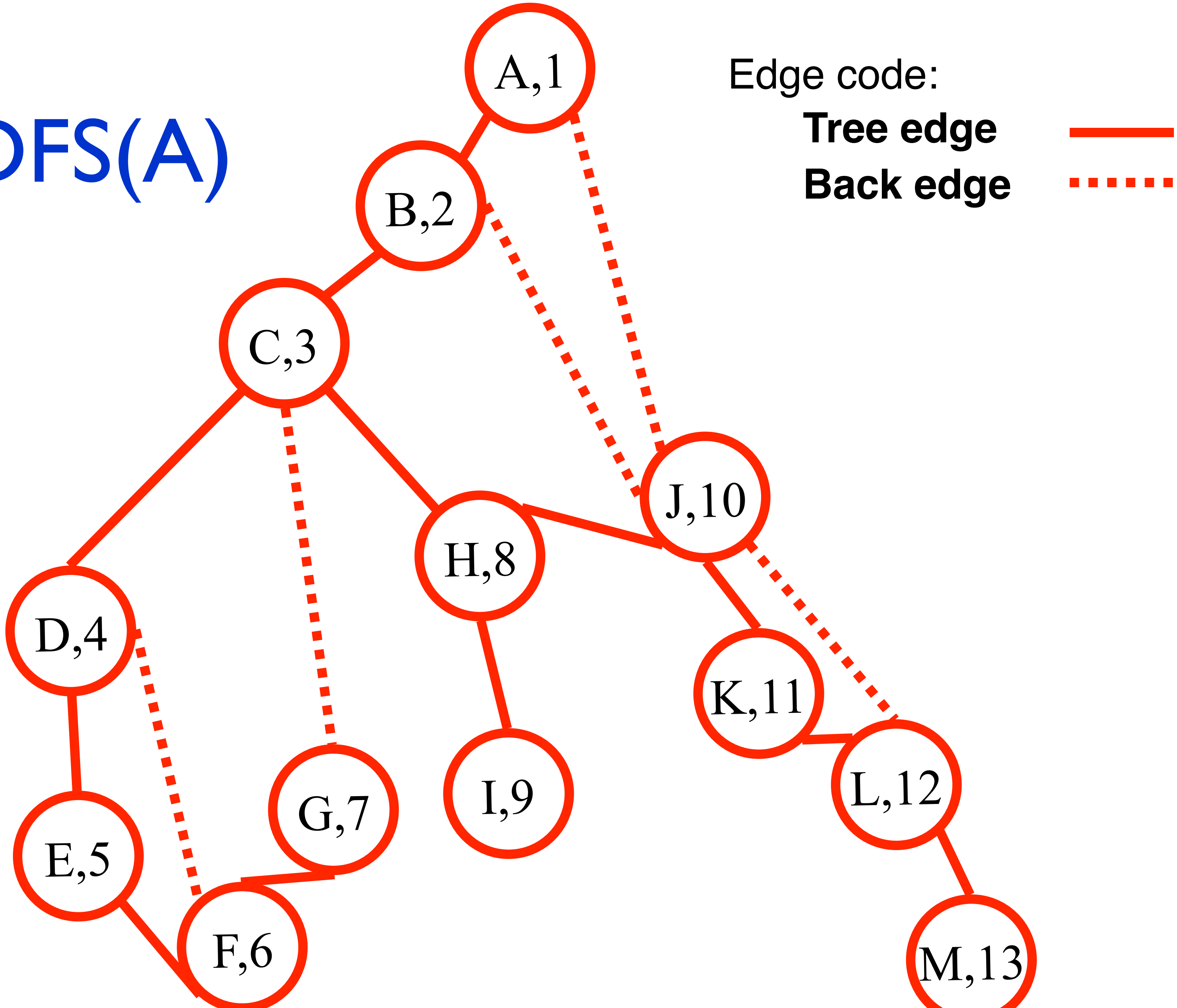
F,6
G,7
H,8
I,9
J,10
K,11
L,12
M,13
Edge code:
Tree edge
Back edge

DFS(A)


A,1
B,2
C,3
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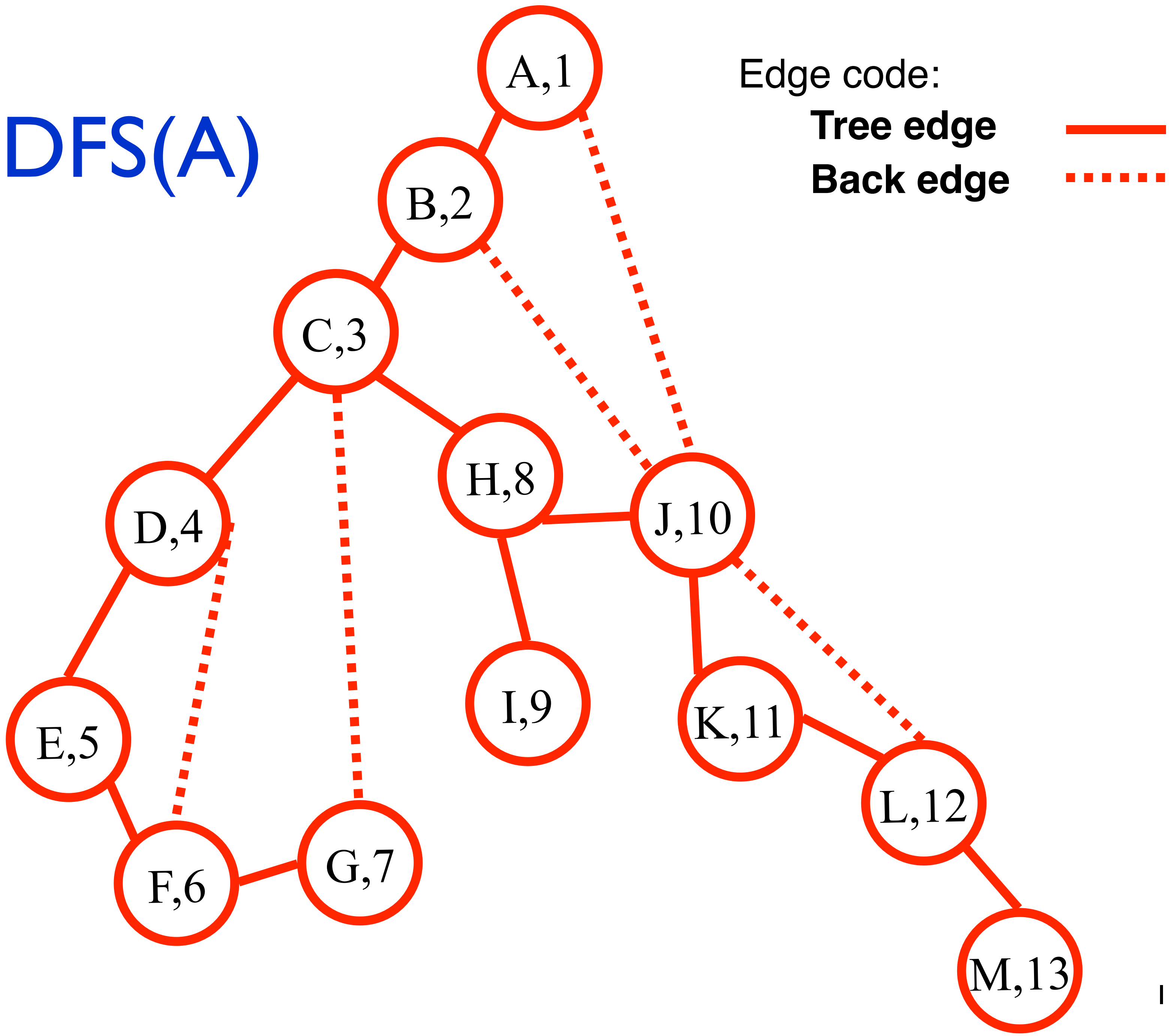
J,10
K,11
L,12
M,13
Edge code:
Tree edge
Back edge

DFS(A)


A,1
B,2
C,3
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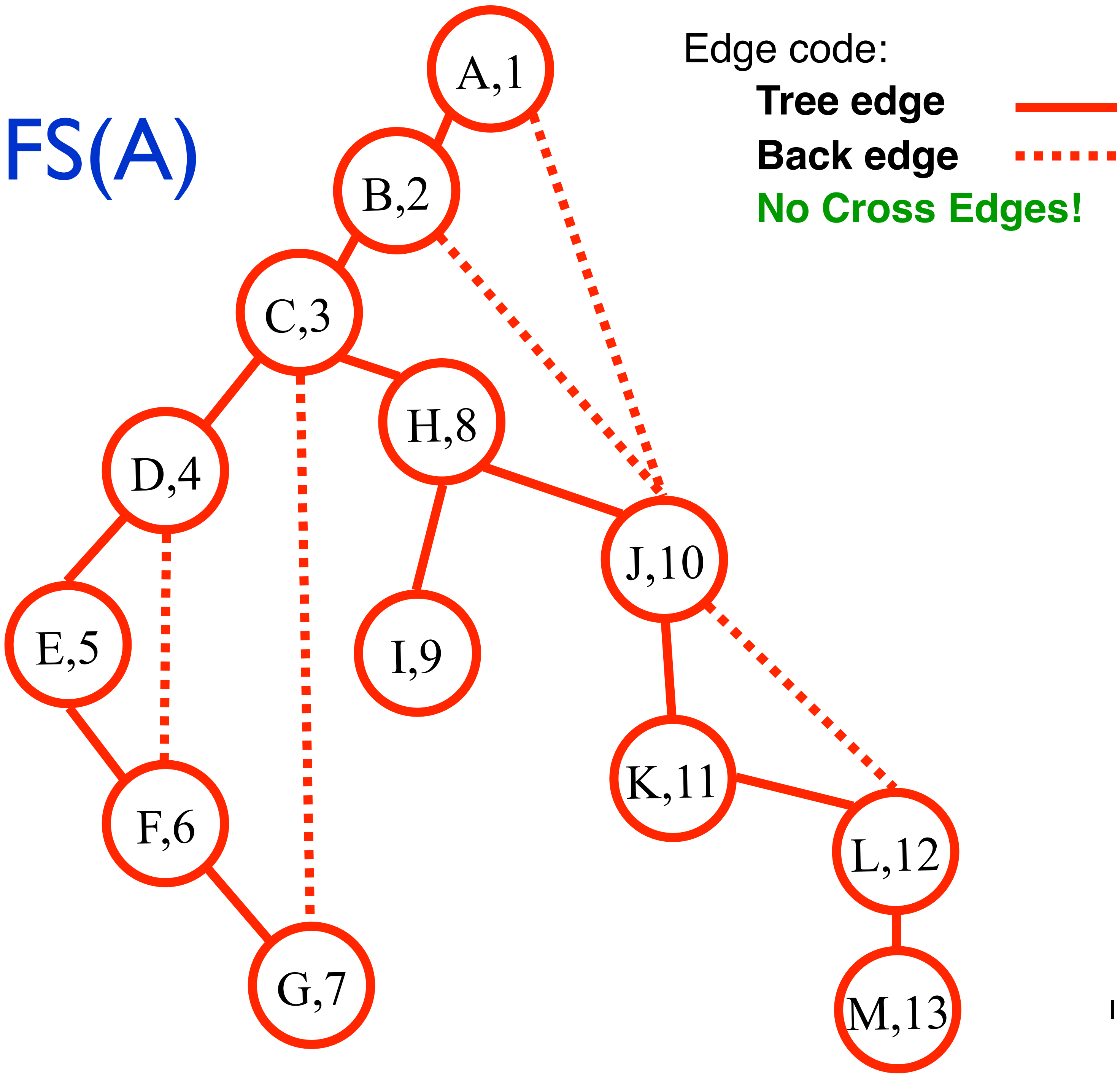
M,13
Edge code:
Tree edge
Back edge

DFS(A)



Edge code:

Tree edge

Back edge

No Cross Edges!

Properties of (Undirected) DFS(v)

Like BFS(v):

- DFS(v) visits x if and only if there is a path in G from v to x (through previously unvisited vertices)
- Edges into then-undiscovered vertices define a ***tree*** – the "depth first spanning tree" of G

Unlike the BFS tree:

- the DF spanning tree isn't minimum depth
- its levels don't reflect min distance from the root
- non-tree edges never join vertices on the same or adjacent levels

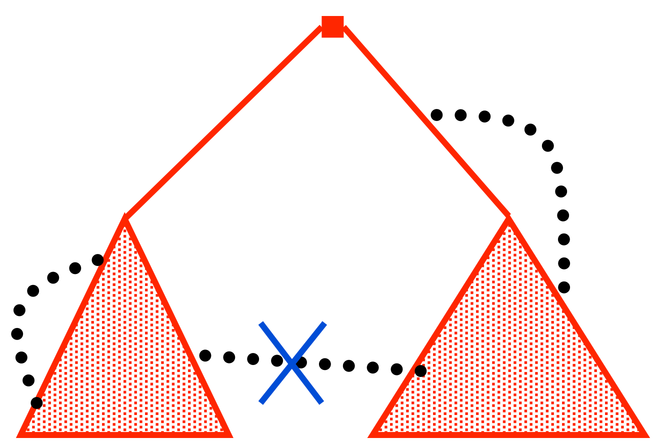
BUT…

Non-tree edges

All non-tree edges join a vertex and one of its descendents/ancestors in the DFS tree

(undirected graphs)

No cross edges!



Why fuss about trees (again)?

As with BFS, DFS has found a tree in the graph s.t. non-tree edges are "simple"--only descendant/ancestor

DFS(v) – Recursive version

Global Initialization:

    for all nodes v, v.dfs# = -1   // mark v "**undiscovered**"
    dfscounter = 0
    for v = 1 to n do
        if state(v) != **fully-explored** then
        DFS(v):

DFS(v)

    v.dfs# = dfscounter++   // v **"discovered"**, number it
    Mark v **``discovered".**
    for each edge (v,x)
        if (x.dfs# = -1)   // (x previously undiscovered)
            DFS(x)
        else …
    Mark v **"fully-explored"**

Kinds of edges – DFS on directed graphs

Edge (u,v)

Tree

[u [v v] u]

Forward

[u [v v] u]

Cross

[v v] [u u]

Back

[v [u u] v]