CSE 417
Algorithms:
Divide and Conquer

Larry Ruzzo

Thanks to Richard Anderson, Paul Beame, Kevin Wayne for some slides

Outline:

- General Idea
- Review of Merge Sort
- Why does it work?
  - Importance of balance
  - Importance of super-linear growth
- Some interesting applications
  - Closest points
  - Integer Multiplication
- Finding & Solving Recurrences

Divide & Conquer

Reduce problem to one or more sub-problems of the same type

Typically, each sub-problem is at most a constant fraction of the size of the original problem

Subproblems typically disjoint

Often gives significant, usually polynomial, speedup

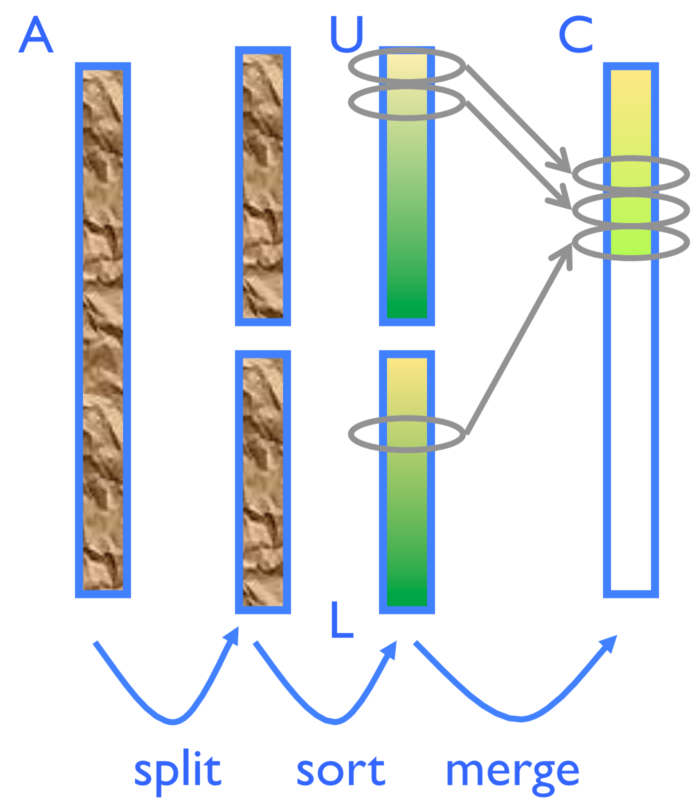
Examples:

Binary Search, Mergesort, Quicksort (roughly), Strassen's Algorithm, integer multiplication, powering, FFT, …

Motivating Example: Mergesort

```
MS(A: array[1..n]) returns array[1..n] {
    If(n=1) return A;
    New U:array[1:n/2] = MS(A[1..n/2]);
    New L:array[1:n/2] = MS(A[n/2+1..n]);
    Return(Merge(U,L));
    }

Merge(U,L: array[1..n]) {
    New C: array[1..2n];
    a=1; b=1;
    For i = 1 to 2n
        "C[i] =  smaller of U[a], L[b] and correspondingly a++ or b++,
                 while being careful about running past end of either";
    Return C;
    }
```



divide & conquer – the key idea

Why does it work? Suppose we've already invented DumbSort, taking time n^2

Try *Just One Level* of divide & conquer:

DumbSort(first n/2 elements)

DumbSort(last n/2 elements)

Merge results

Time: $2\,(n/2)^2 + n = n^2/2 + n \ll n^2$

D&C in a nutshell

Almost twice as fast!

Moral 1: “two halves are better than a whole”

Two problems of half size are *better* than one full-size problem, even given O(n) overhead of recombining, since the base algorithm has *super-linear* complexity.

Moral 2: “If a little's good, then more's better”

Two levels of D&C would be almost 4 times faster, 3 levels almost 8, etc., even though overhead is growing.
Best is usually full recursion down to some small constant size (balancing "work" vs "overhead").

In the limit: you’ve just rediscovered mergesort!

Moral 3: unbalanced division good, but less so:

$$(.1n)^2 + (.9n)^2 + n = .82n^2 + n$$

The 18% savings compounds significantly if you carry recursion to more levels, actually giving O(nlogn), but with a bigger constant. So worth doing if you can't get 50-50 split, but balanced is better if you can.

This is intuitively why Quicksort with random splitter is good – badly unbalanced splits are rare, and not instantly fatal.

Moral 4: but consistent, completely unbalanced division doesn't help much:

$$(1)^2 + (n-1)^2 + n = n^2 - n + 2$$

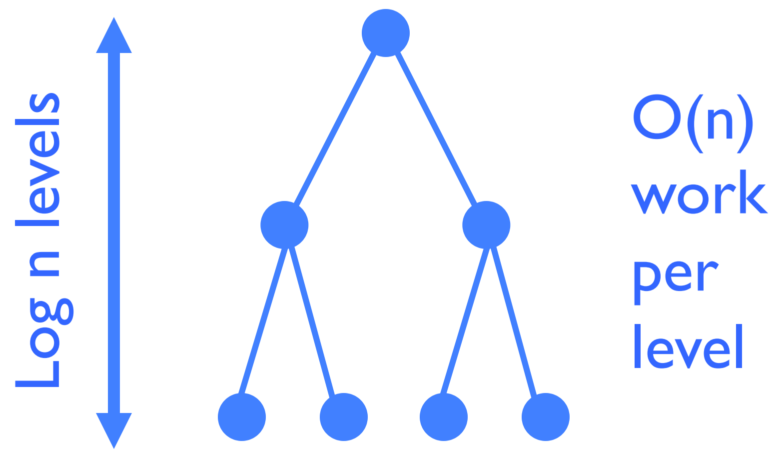
Little improvement here.

Mergesort: (recursively) sort 2 half-lists, then merge results.

$T(n) = 2T(n/2)+cn, \;\; n \geq 2$

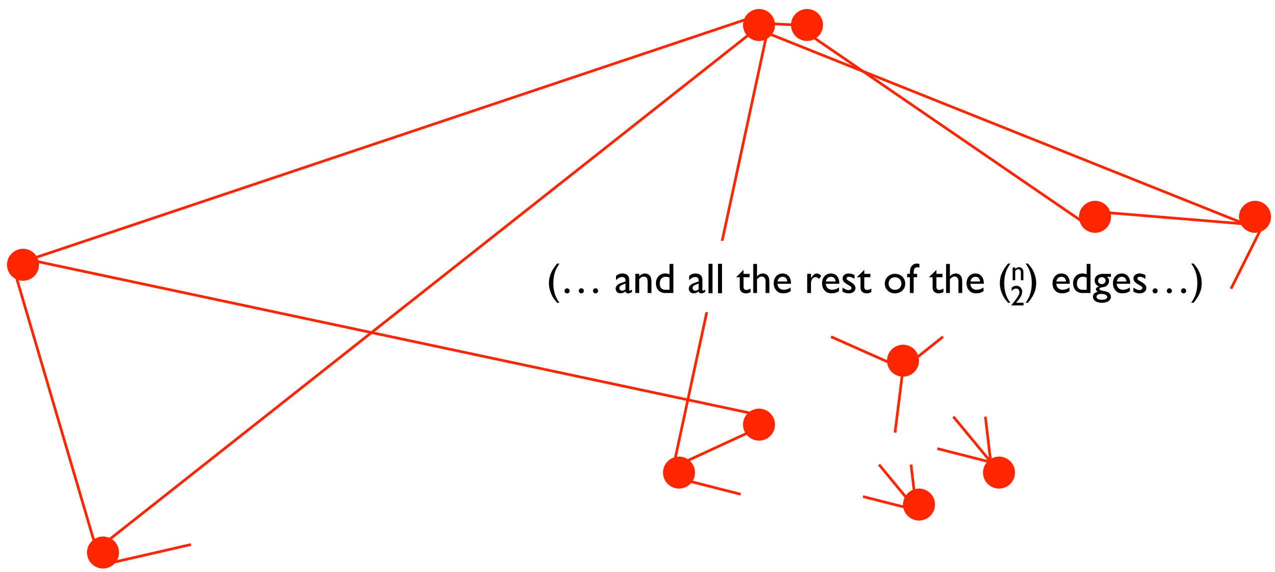
$T(1) = 0$

Solution: $\Theta(n \log n)$
(details later)



A Divide & Conquer Example: Closest Pair of Points

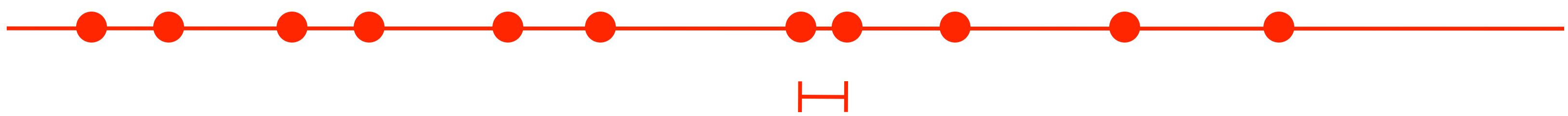
Given n points and *arbitrary* distances between them, find the closest pair. (E.g., think of distance as airfare – definitely *not* Euclidean distance!)



Must look at all n choose 2 pairwise distances, else any one you didn't check might be the shortest.

Also true for Euclidean distance in 1-2 dimensions?

Given n points on the real line, find the closest pair



Closest pair is *adjacent* in ordered list

Time O(n log n) to sort, if needed

Plus O(n) to scan adjacent pairs

Key point: do *not* need to calc distances between all pairs: exploit geometry + ordering

closest pair of points: 2 dimensional version

Closest pair. Given n points in the plane, find a pair with smallest Euclidean distance between them.

Fundamental geometric primitive.

- Graphics, computer vision, geographic information systems, molecular modeling, air traffic control.
- Special case of nearest neighbor, Euclidean MST, Voronoi.

↖ fast closest pair inspired fast algorithms for these problems

Brute force. Check all pairs of points p and q with $\Theta(n^2)$ comparisons.

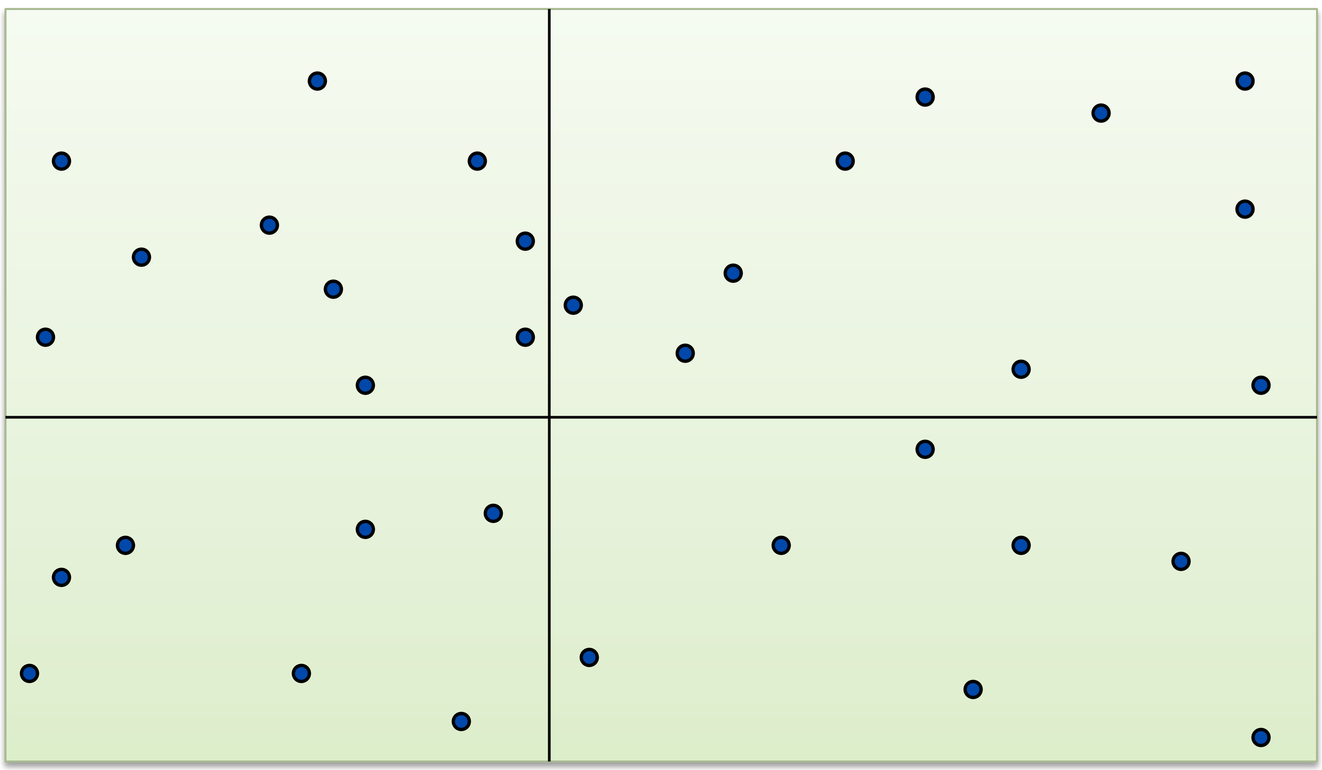
1-D version. O(n log n) easy if points are on a line.

Assumption. No two points have same x coordinate.

↑ Just to simplify presentation

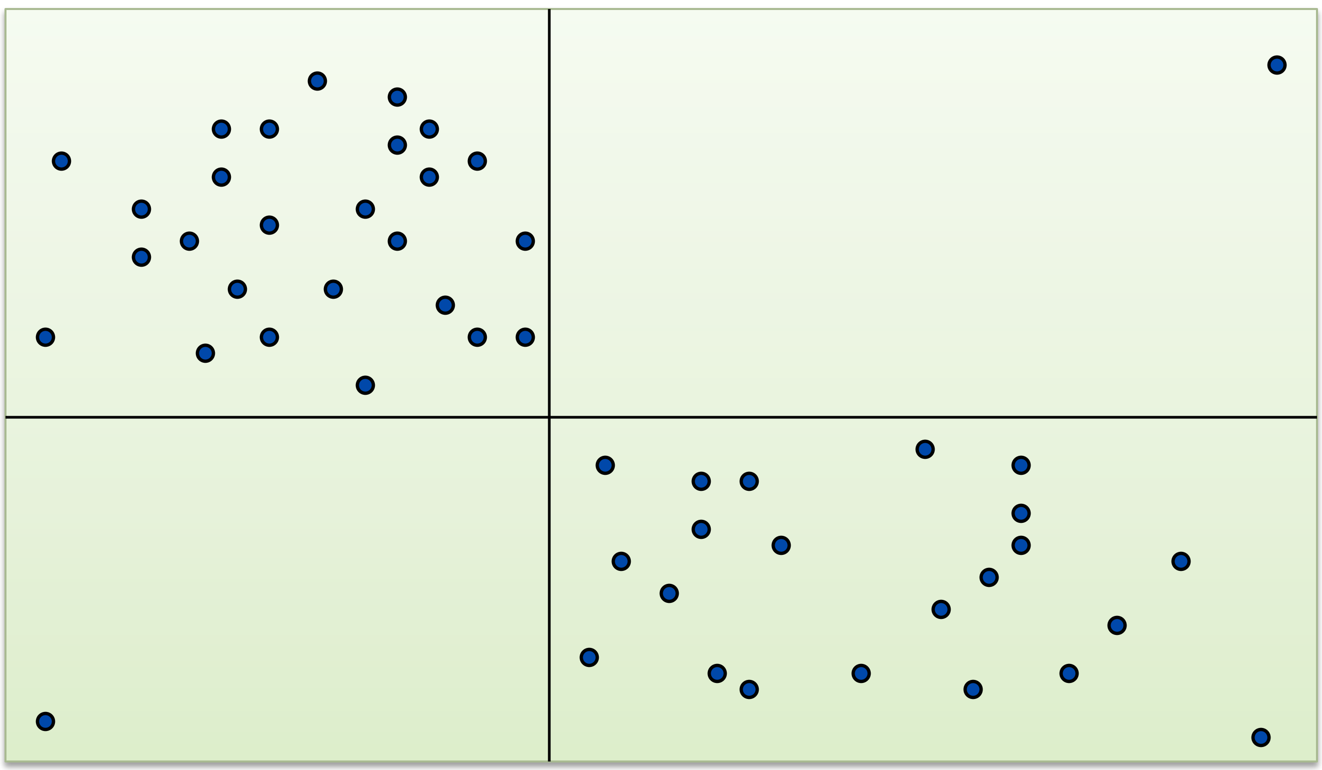
closest pair of points. 2d, Euclidean distance: 1^{st} try

Divide. Sub-divide region into 4 quadrants.



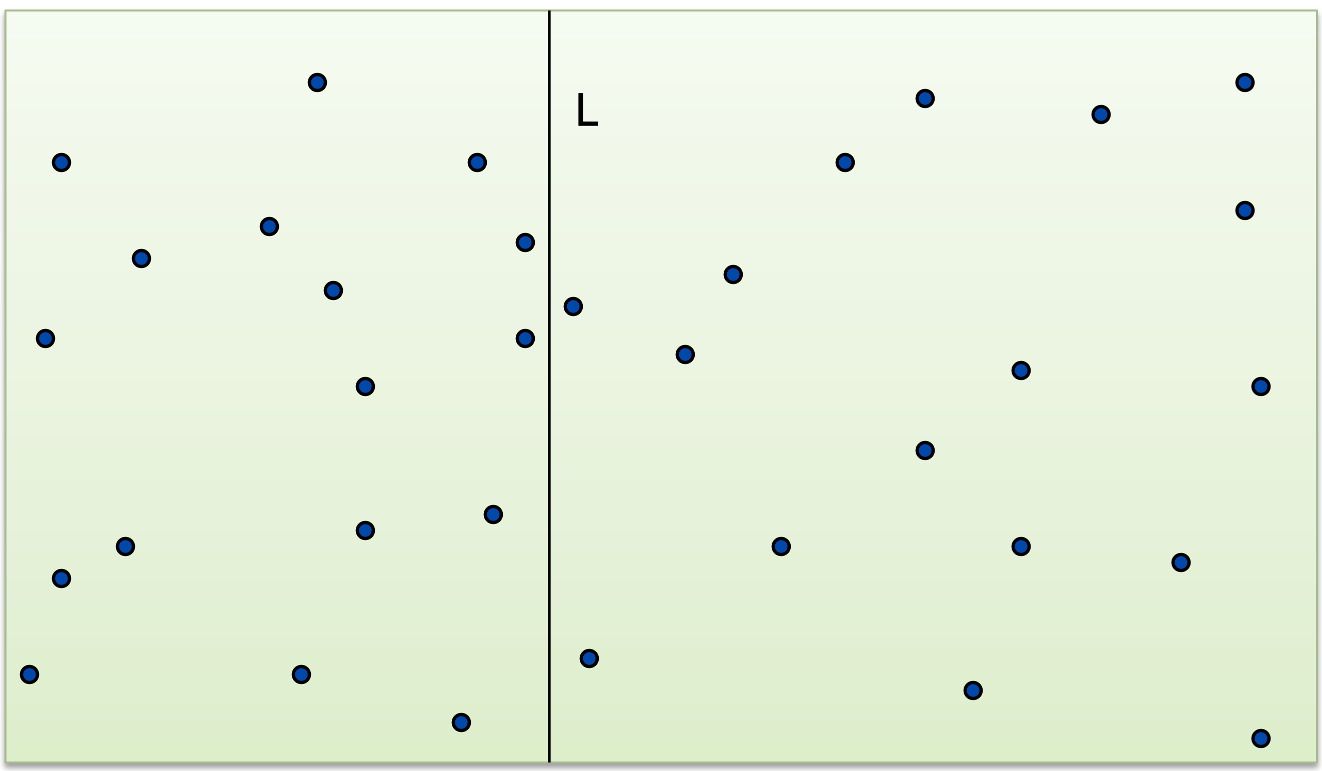
Divide. Sub-divide region into 4 quadrants.

Obstacle. Impossible to ensure n/4 points in each piece, so the “balanced subdivision” goal may be elusive/problematic.



Algorithm.

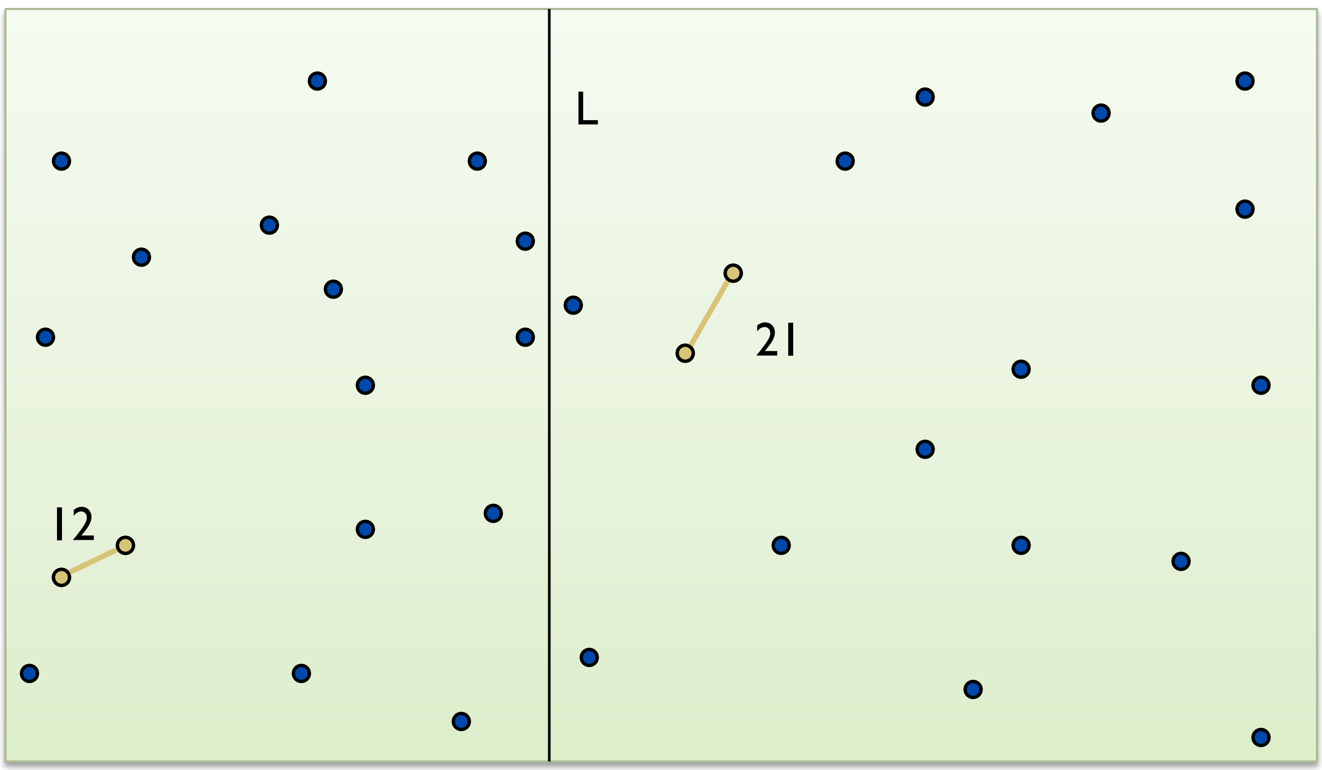
Divide: draw vertical line L with ≈ n/2 points on each side.



Algorithm.

Divide: draw vertical line L with ≈ n/2 points on each side.

Conquer: find closest pair on each side, recursively.



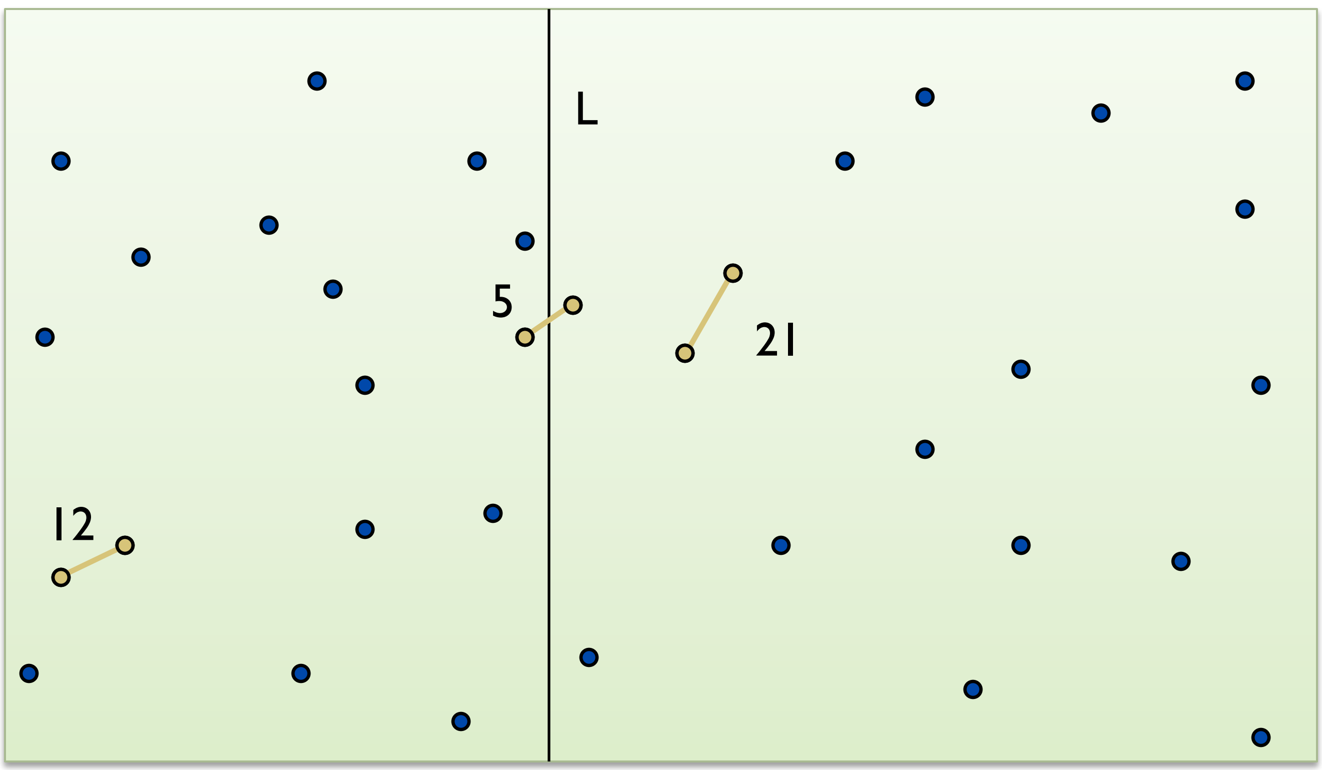
Algorithm.

Divide: draw vertical line L with ≈ n/2 points on each side.

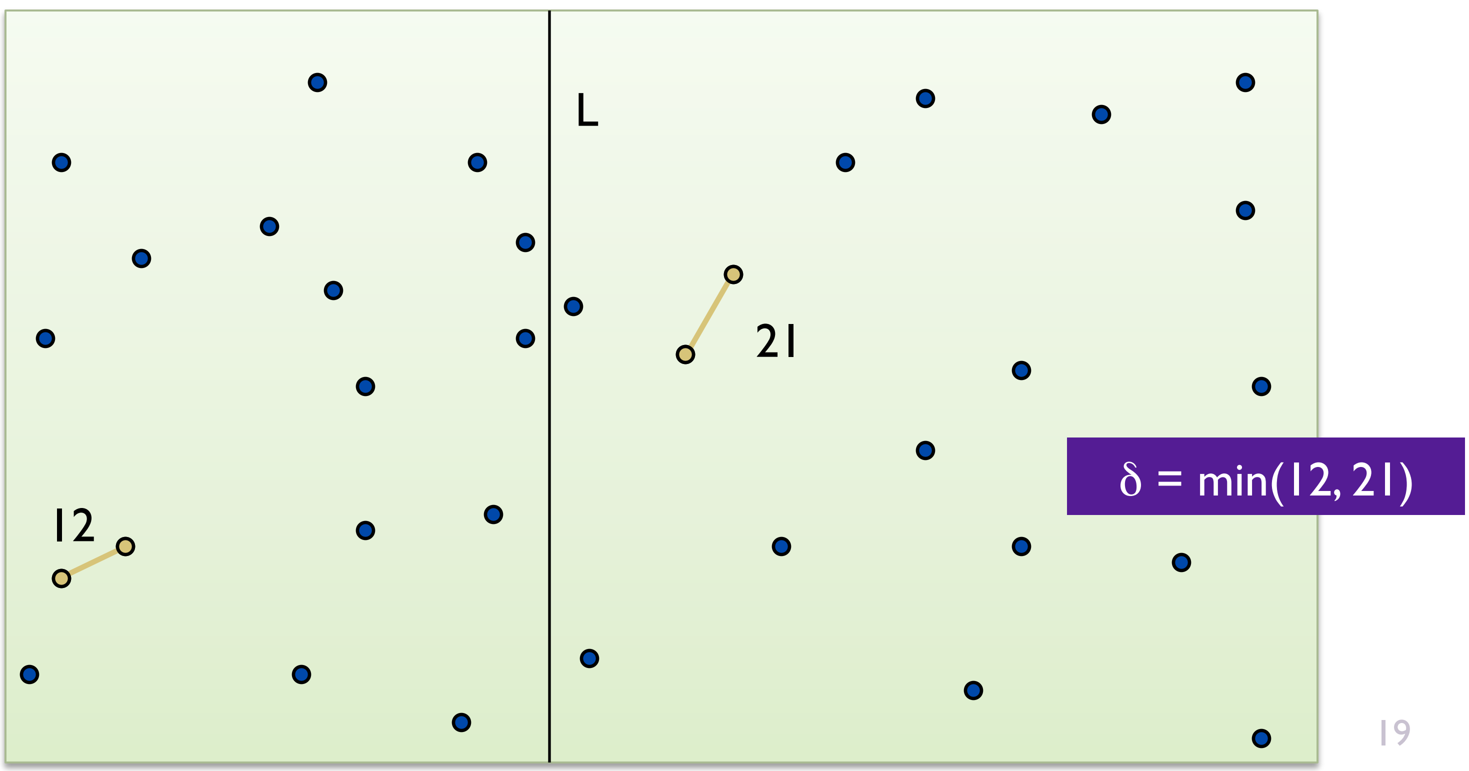
Conquer: find closest pair on each side, recursively.

Combine: find closest pair with one point in each side. ← seems like $\Theta(n^2)$?

Return best of 3 solutions.

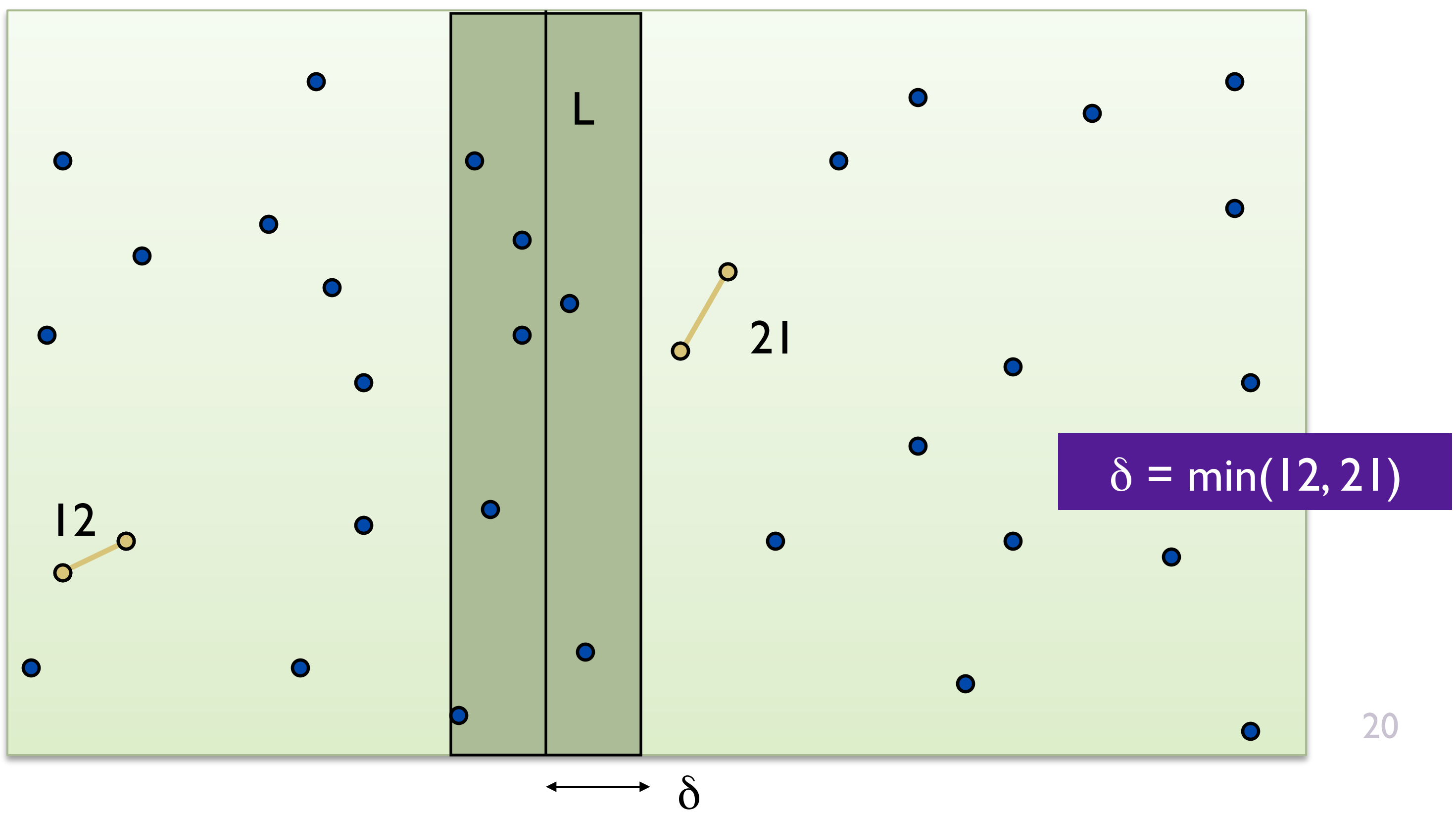


Find closest pair with one point in each side, *assuming* distance < δ.



Find closest pair with one point in each side, *assuming distance < δ*.

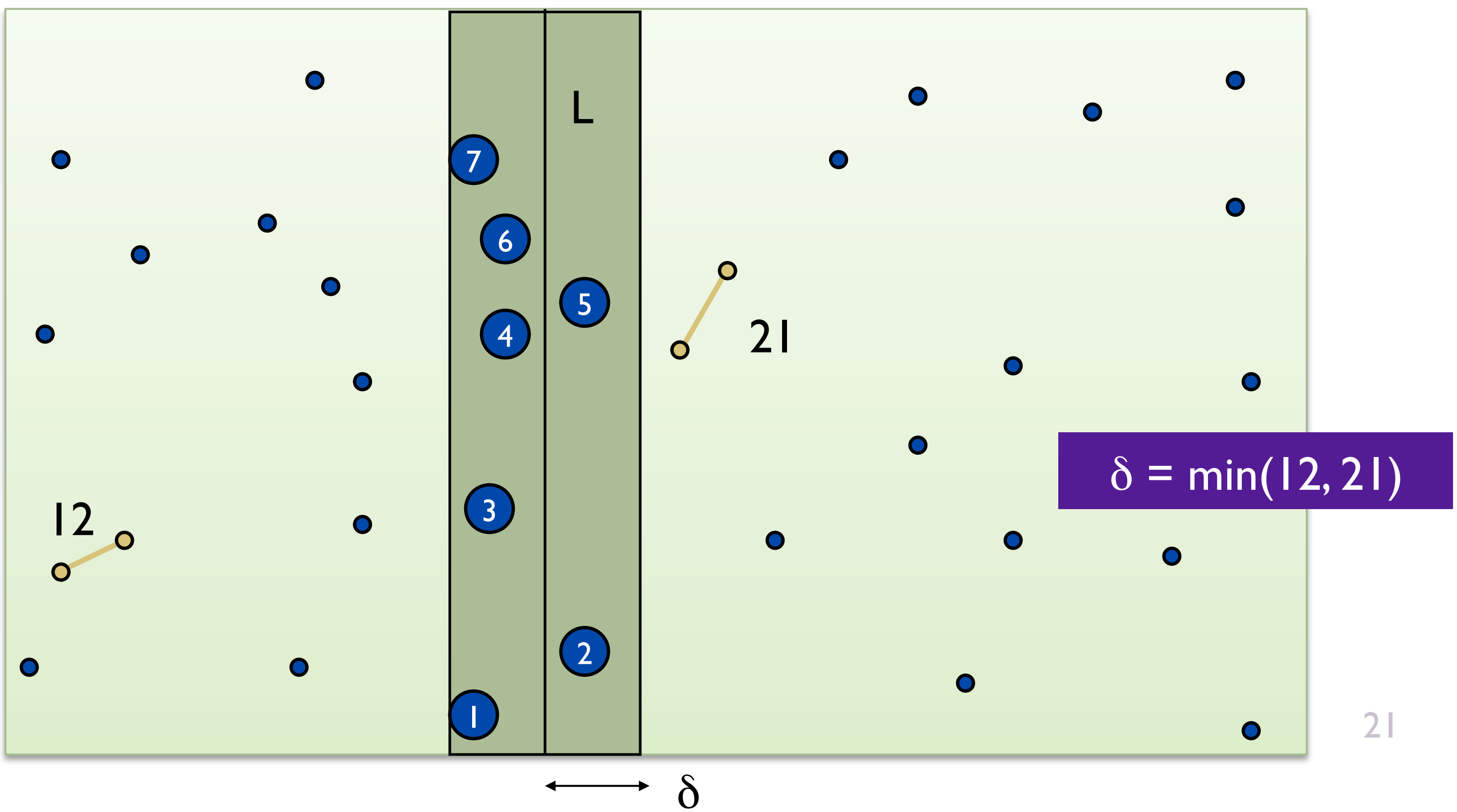
Observation: suffices to consider points within δ of line L.



Find closest pair with one point in each side, *assuming* distance < δ.

Observation: suffices to consider points within δ of line L.

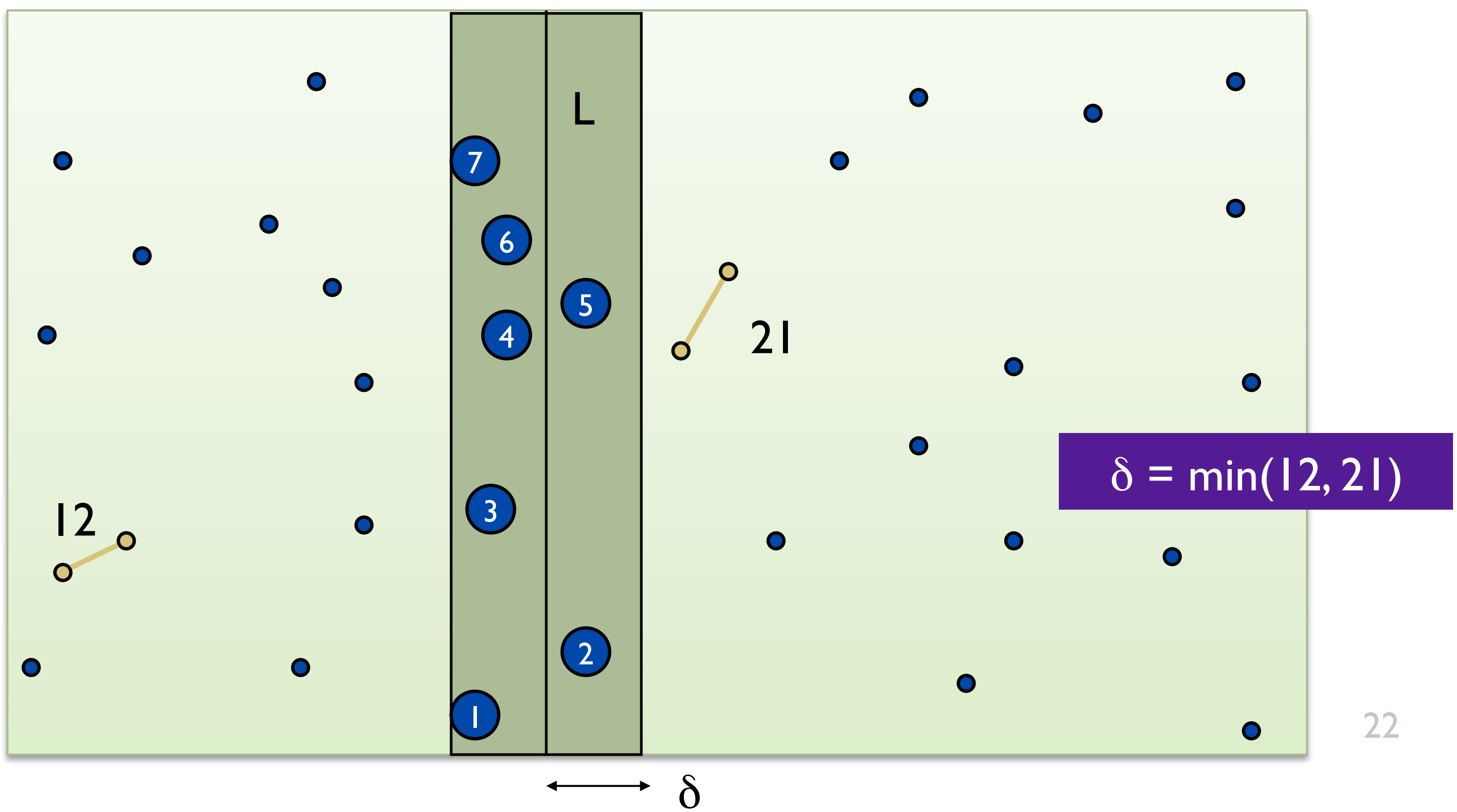
Almost the one-D problem again: Sort points in 2δ-strip by their y coordinate.



Find closest pair with one point in each side, *assuming* distance < δ.

Observation: suffices to consider points within δ of line L.

Almost the one-D problem again: Sort points in 2δ-strip by their y coordinate. Only check pts within 8 in sorted list!



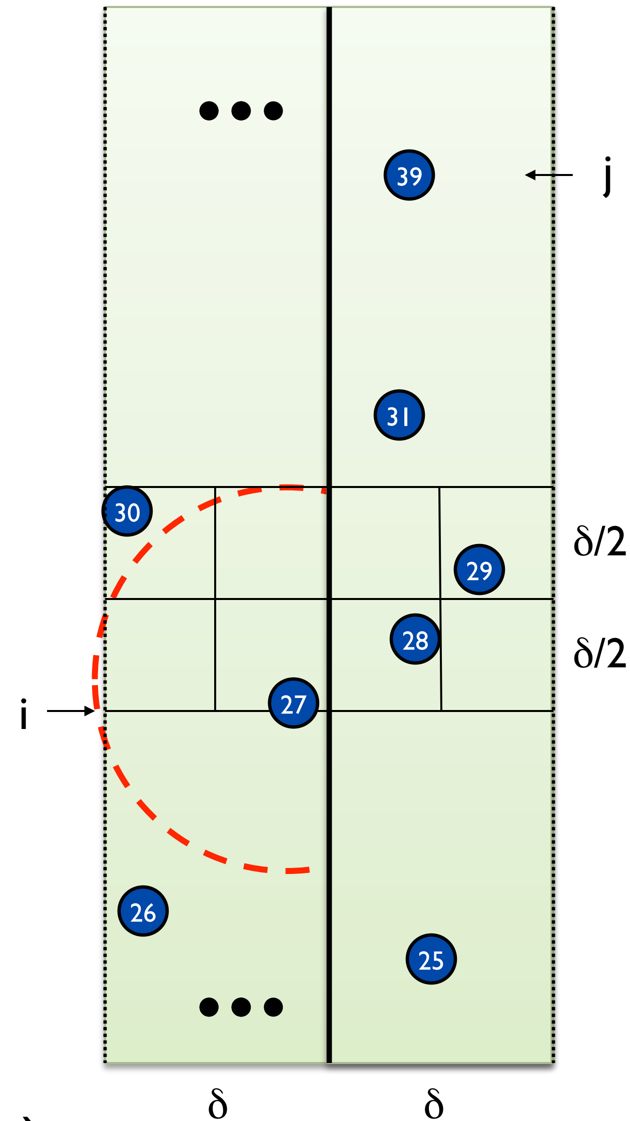
Def. Let s_i have the i^{th} smallest y-coordinate among points in the 2δ-width-strip.

Claim. If $|i - j| > 8$, then the distance between s_i and s_j is $> \delta$.

Pf: No two points lie in the same δ/2-by-δ/2 square:

$$\sqrt{\left(\frac{\delta}{2}\right)^2 + \left(\frac{\delta}{2}\right)^2} = \frac{\sqrt{2}}{2}\delta \approx 0.7\delta < \delta$$

so ≤ 8 points within +δ of $y(s_i)$.



```
Closest-Pair(p₁, …, pₙ) {
   if(n <= ??) return ??

   Compute separation line L such that half the points
   are on one side and half on the other side.

   δ₁ = Closest-Pair(left half)
   δ₂ = Closest-Pair(right half)
   δ  = min(δ₁, δ₂)

   Delete all points further than δ from separation line L

   Sort remaining points p[1]…p[m] by y-coordinate.

   for i = 1..m
      k = 1
      while i+k <= m && p[i+k].y < p[i].y + δ
        δ = min(δ, distance between p[i] and p[i+k]);
        k++;

   return δ.
}
```

Analysis, I: Let D(n) be the number of pairwise distance calculations in the Closest-Pair Algorithm when run on n ≥ 1 points

$$D(n) \le \begin{Bmatrix} 0 & n = 1 \\ 2D(n/2) + 7n & n > 1 \end{Bmatrix} \Rightarrow D(n) = O(n \log n)$$

BUT – that's only the number of *distance calculations*

What if we counted comparisons?

Analysis, II: Let C(n) be the number of comparisons between coordinates/distances in the Closest-Pair Algorithm when run on n ≥ 1 points

$$C(n) \le \begin{cases} 0 & n = 1 \\ 2C(n/2) + kn\log n & n > 1 \end{cases} \Rightarrow C(n) = O(n\log^2 n)$$

for some constant k

Q. Can we achieve O(n log n)?

A. Yes. Don't sort points from scratch each time.

Sort by x at top level only.

Each recursive call returns δ *and* list of all points sorted by y

Sort by merging two pre-sorted lists.

$$T(n) \le 2T(n/2) + O(n) \Rightarrow T(n) = O(n\log n)$$

Code is longer & more complex

O(n log n) vs O(n^2) may hide 10x in constant?

How many points?

n	Speedup: n^2 / (10 n $\log_2$ n)
10	0.3
100	1.5
1,000	10
10,000	75
100,000	602
1,000,000	5,017
10,000,000	43,004

Going From Code to Recurrence

Carefully define what you're counting, and *write it down*!

"Let C(n) be the number of comparisons between sort keys used by MergeSort when sorting a list of length $n \geq 1$"

In code, clearly separate *base case* from *recursive case,* highlight *recursive calls,* and *operations being counted.*

Write Recurrence(s)

```
MS(A: array[1..n]) returns array[1..n] {
    If(n=1) return A;
    New L:array[1:n/2] = MS(A[1..n/2]);
    New R:array[1:n/2] = MS(A[n/2+1..n]);
    Return(Merge(L,R));
    }
Merge(A,B: array[1..n]) {
    New C: array[1..2n];
    a=1; b=1;
    For i = 1 to 2n {
        C[i] = "smaller of A[a], B[b] and a++ or b++";
    Return C;
    }
```

Base Case

Recursive calls

One Recursive Level

Operations being counted

Base case

$$C(n) = \begin{cases} 0 & \text{if } n = 1 \\ 2C(n/2) + (n-1) & \text{if } n > 1 \end{cases}$$

Recursive calls

One compare per element added to merged list, except the last.

Total time: proportional to C(n)

(loops, copying data, parameter passing, etc.)

Carefully define what you're counting, and *write it down*!

"Let D(n) be the number of pairwise distance calculations in the Closest-Pair Algorithm when run on n ≥ 1 points"

In code, clearly separate *base case* from *recursive case,* highlight *recursive calls,* and *operations being counted*.

Write Recurrence(s)

closest pair algorithm

Basic operations: distance calcs

```
Closest-Pair(p1, …, pn) {
   if(n <= 1) return ∞

   Compute separation line L such that half the points
   are on one side and half on the other side.

   δ1 = Closest-Pair(left half)
   δ2 = Closest-Pair(right half)
   δ  = min(δ1, δ2)

   Delete all points further than δ from separation line L

   Sort remaining points p[1]…p[m] by y-coordinate.

   for i = 1..m
      k = 1
      while i+k <= m && p[i+k].y < p[i].y + δ
        δ = min(δ, distance between p[i] and p[i+k]);
        k++;

   return δ.
}
```

Base Case — 0

Recursive calls (2) — 2D(n / 2)

One recursive level

Basic operations at this recursive level — 7n

Analysis, I: Let D(n) be the number of pairwise distance calculations in the Closest-Pair Algorithm when run on n ≥ 1 points

$$D(n) \le \begin{cases} 0 & n = 1 \\ 2D(n/2) + 7n & n > 1 \end{cases} \Rightarrow D(n) = O(n \log n)$$

BUT – that's only the number of *distance calculations*

What if we counted comparisons?

Carefully define what you're counting, and *write it down*!

> "Let D(n) be the number of comparisons between coordinates/distances in the Closest-Pair Algorithm when run on n ≥ 1 points"

In code, clearly separate *base case* from *recursive case,* highlight *recursive calls,* and *operations being counted*.

Write Recurrence(s)

closest pair algorithm

Basic operations: *comparisons*

Base Case

Recursive calls (2)

```
Closest-Pair(p1, …, pn) {
  if(n <= 1) return ∞

  Compute separation line L such that half the points
  are on one side and half on the other side.

  δ1 = Closest-Pair(left half)
  δ2 = Closest-Pair(right half)
  δ  = min(δ1, δ2)

  Delete all points further than δ from separation line L

  Sort remaining points p[1]…p[m] by y-coordinate.

  for i = 1..m
     k = 1
     while i+k <= m && p[i+k].y < p[i].y + δ
       δ = min(δ, distance between p[i] and p[i+k]);
       k++;

  return δ.
}
```

Basic operations at this recursive level

Base case	0
Compute separation line	$k_1 n \log n$
Recursive calls	$2C(n / 2)$
min	1
Delete points	$k_2 n$
Sort	$k_3 n \log n$
for loop	8n 7n

One recursive level

Analysis, II: Let C(n) be the number of comparisons of coordinates/distances in the Closest-Pair Algorithm when run on n ≥ 1 points

$$C(n) \le \begin{cases} 0 & n = 1 \\ 2C(n/2) + k_4 n \log n + 1 & n > 1 \end{cases} \Rightarrow C(n) = O(n \log^2 n)$$

for some $k_4 \le k_1 + k_2 + k_3 + 15$

Q. Can we achieve time O(n log n)?

A. Yes. Don't sort points from scratch each time.

Sort by x at top level only.

Each recursive call returns δ *and* list of all points sorted by y

Sort by merging two pre-sorted lists.

$$T(n) \le 2T(n/2) + O(n) \Rightarrow \mathrm{T}(n) = O(n \log n)$$

Integer Multiplication

Add. Given two n-bit integers a and b, compute a + b.

Add

1	1	1	1	1	1	0	1	
	1	1	0	1	0	1	0	1
+	0	1	1	1	1	1	0	1
1	0	1	0	1	0	0	1	0

O(n) bit operations.

Add. Given two n-bit integers a and b, compute a + b.

Add

1	1	1	1	1	1	0	1	Carries
	1	1	0	1	0	1	0	1
+	0	1	1	1	1	1	0	1
1	0	1	0	1	0	0	1	0

O(n) bit operations.

Multiply. Given two n-bit integers a and b, compute a × b.
The "grade school" method:

Multiply

								1	1	0	1	0	1	0	1
							*	0	1	1	1	1	1	0	1
								1	1	0	1	0	1	0	1
							0	0	0	0	0	0	0	0	
						1	1	0	1	0	1	0	1		
					1	1	0	1	0	1	0	1			
				1	1	0	1	0	1	0	1				
			1	1	0	1	0	1	0	1					
		1	1	0	1	0	1	0	1						
	0	0	0	0	0	0	0	0							
0	1	1	0	1	0	0	0	0	0	0	0	0	0	0	1

$\Theta(n^2)$ bit operations.

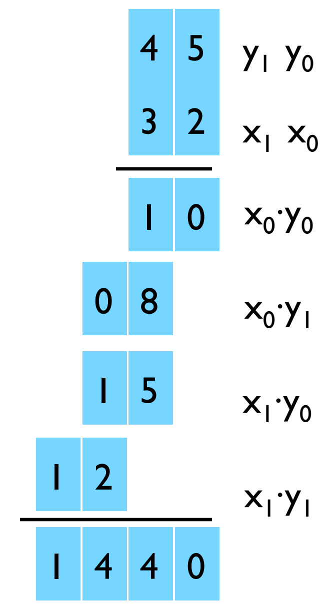
divide & conquer multiplication: warmup

To multiply two 2-digit integers:

Multiply four 1-digit integers.

Add, shift some 2-digit integers to obtain result.

$$
\begin{aligned}
x &= 10 \cdot x_1 + x_0 \\
y &= 10 \cdot y_1 + y_0 \\
xy &= \left(10 \cdot x_1 + x_0\right)\left(10 \cdot y_1 + y_0\right) \\
&= 100 \cdot x_1 y_1 + 10 \cdot \left(x_1 y_0 + x_0 y_1\right) + x_0 y_0
\end{aligned}
$$



Same idea works for *long* integers – can split them into 4 half-sized ints ("10" becomes "10^k", k = length/2)

divide & conquer multiplication: warmup

To multiply two n-bit integers:

Multiply four ½n-bit integers.

Shift/add four n-bit integers to obtain result.

$$
\begin{aligned}
x &= 2^{n/2} \cdot x_1 + x_0 \\
y &= 2^{n/2} \cdot y_1 + y_0 \\
xy &= \left(2^{n/2} \cdot x_1 + x_0\right)\left(2^{n/2} \cdot y_1 + y_0\right) \\
&= 2^{n} \cdot x_1 y_1 + 2^{n/2} \cdot \left(x_1 y_0 + x_0 y_1\right) + x_0 y_0
\end{aligned}
$$

		1 1 0 1	0 1 0 1	y_1 y_0
		* 0 1 1 1	1 1 0 1	x_1 x_0
		0 1 0 0	0 0 0 1	$x_0 \cdot y_0$
	1 0 1 0	1 0 0 1		$x_0 \cdot y_1$
	0 0 1 0	0 0 1 1		$x_1 \cdot y_0$
0 1 0 1	1 0 1 1			$x_1 \cdot y_1$
0 1 1 0	1 0 0 0	0 0 0 0	0 0 0 1	

$$
T(n) = \underbrace{4T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, shift}} \Rightarrow T(n) = \Theta(n^2)
$$

↑

assumes n is a power of 2

key trick: 2 multiplies for the price of 1:

$$
\begin{aligned}
x &= 2^{n/2} \cdot x_1 + x_0 \\
y &= 2^{n/2} \cdot y_1 + y_0 \\
xy &= \left(2^{n/2} \cdot x_1 + x_0\right)\left(2^{n/2} \cdot y_1 + y_0\right) \\
&= 2^{n} \cdot x_1 y_1 + 2^{n/2} \cdot \left(x_1 y_0 + x_0 y_1\right) + x_0 y_0
\end{aligned}
$$

Well, ok, 4 for 3 is more accurate…

$$
\begin{aligned}
\alpha &= x_1 + x_0 \\
\beta &= y_1 + y_0 \\
\alpha\beta &= \left(x_1 + x_0\right)\left(y_1 + y_0\right) \\
&= x_1 y_1 + \left(x_1 y_0 + x_0 y_1\right) + x_0 y_0 \\
\left(x_1 y_0 + x_0 y_1\right) &= \alpha\beta - x_1 y_1 - x_0 y_0
\end{aligned}
$$

To multiply two n-bit integers:

- Add two pairs of ½n bit integers.
- Multiply three pairs of ½n-bit integers.
- Add, subtract, and shift n-bit integers to obtain result.

$$
\begin{aligned}
x &= 2^{n/2} \cdot x_1 + x_0 \\
y &= 2^{n/2} \cdot y_1 + y_0 \\
xy &= 2^{n} \cdot x_1 y_1 + 2^{n/2} \cdot (x_1 y_0 + x_0 y_1) + x_0 y_0 \\
&= 2^{n} \cdot \underset{A}{x_1 y_1} + 2^{n/2} \cdot \left(\underset{B}{(x_1 + x_0)(y_1 + y_0)} - \underset{A}{x_1 y_1} - \underset{C}{x_0 y_0} \right) + \underset{C}{x_0 y_0}
\end{aligned}
$$

Theorem. [Karatsuba-Ofman, 1962] Can multiply two n-digit integers in $O(n^{1.585})$ bit operations.

$$
T(n) \le \underbrace{T\left(\lfloor n/2 \rfloor\right) + T\left(\lceil n/2 \rceil\right) + T\left(1 + \lceil n/2 \rceil\right)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, subtract, shift}}
$$

$$
\textit{Sloppy version}: \ T(n) \le 3T(n/2) + O(n)
$$

$$
\Rightarrow T(n) = O(n^{\log_2 3}) = O(n^{1.585})
$$

Theorem. [Karatsuba-Ofman, 1962] Can multiply two n-digit integers in $O(n^{1.585})$ bit operations.

$$T(n) \le \underbrace{T\big(\lfloor n/2 \rfloor\big) + T\big(\lceil n/2 \rceil\big) + T\big(1+\lceil n/2 \rceil\big)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, subtract, shift}}$$

Best to solve it directly (but messy). Instead, it nearly always suffices to solve a simpler recurrence:

$$\textit{Sloppy version}: \; T(n) \le 3T(n/2) + O(n)$$

Intuition: If $T(n) = n^k$, then $T(n+1) = n^k + kn^{k-1} + \ldots = O(n^k)$

$$\Rightarrow \; T(n) = O(n^{\log_2 3}) = O(n^{1.585})$$

(Proof later.)

Naïve: $\Theta(n^2)$

Karatsuba: $\Theta(n^{1.59\ldots})$

Amusing exercise: generalize Karatsuba to do 5 size n/3 subproblems $\rightarrow \Theta(n^{1.46\ldots})$

Best known: $\Theta(n \log n \log\log n)$

"Fast Fourier Transform"

but mostly unused in practice (unless you need really big numbers - a billion digits of π, say)

High precision arithmetic *IS* important for crypto

Recurrences

Above: Where they come from, how to find them

Next: how to solve them

Mergesort: (recursively) sort 2 half-lists, then merge results.

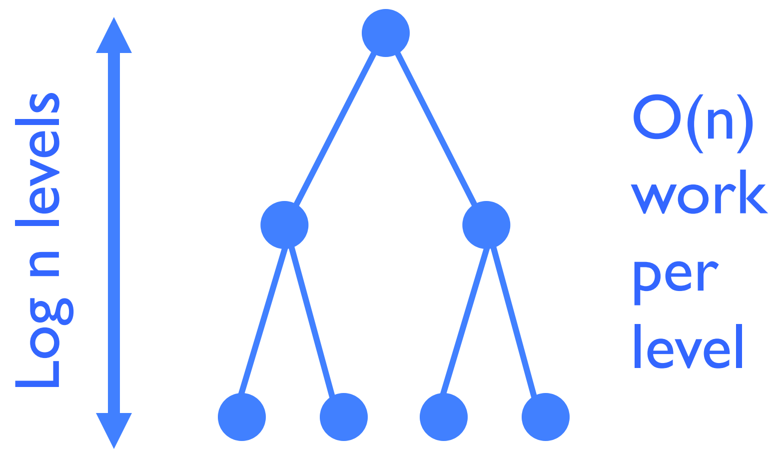
$T(n) = 2T(n/2)+cn, \;\; n \geq 2$

$T(1) = 0$

Solution: $\Theta(n \log n)$

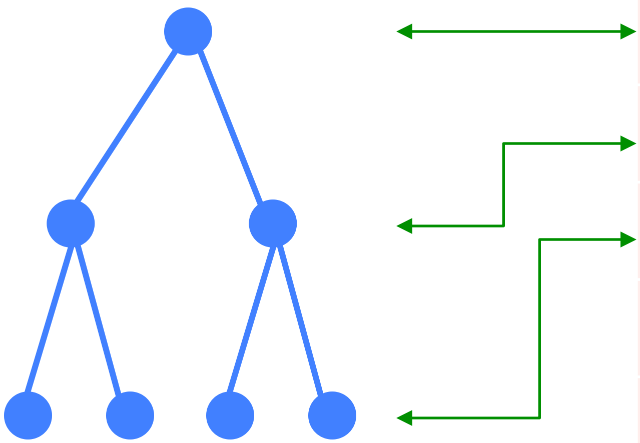
(~~details~~ later)

now!



Solve: $T(1) = c$

$T(n) = 2\,T(n/2) + cn$

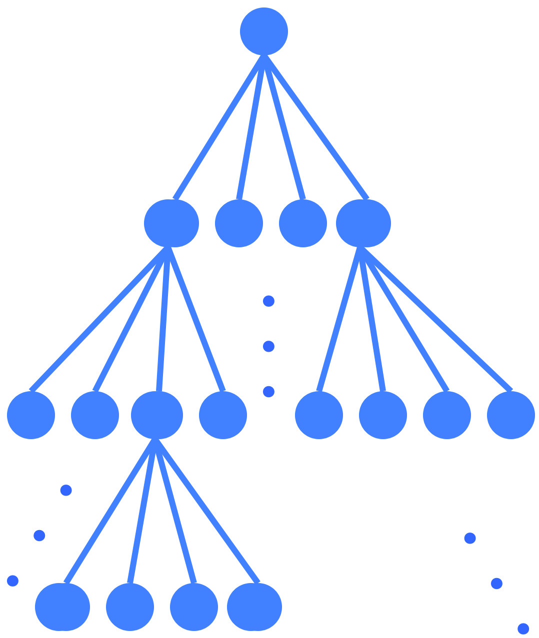


Level	Num	Size	Work
0	$1 = 2^0$	n	cn
1	$2 = 2^1$	$n/2$	$2cn/2$
2	$4 = 2^2$	$n/4$	$4cn/4$
…	…	…	…
i	2^i	$n/2^i$	$2^i\ c\ n/2^i$
…	…	…	…
$k-1$	2^{k-1}	$n/2^{k-1}$	$2^{k-1}\ c\ n/2^{k-1}$
k	2^k	$n/2^k = 1$	$2^k\,T(1)$

$n = 2^k$; $k = \log_2 n$

Total Work: $c\ n\ (1+\log_2 n)$ (add last col)

Solve: $T(1) = c$
$T(n) = 4\,T(n/2) + cn$



Level	Num	Size	Work
0	$1 = 4^0$	n	cn
1	$4 = 4^1$	n/2	4cn/2
2	$16 = 4^2$	n/4	16cn/4
…	…	…	…
i	4^i	$n/2^i$	$4^i\ c\ n/2^i$
…	…	…	…
k-1	4^{k-1}	$n/2^{k-1}$	$4^{k-1}\ c\ n/2^{k-1}$
k	4^k	$n/2^k = 1$	$4^k\, T(1)$

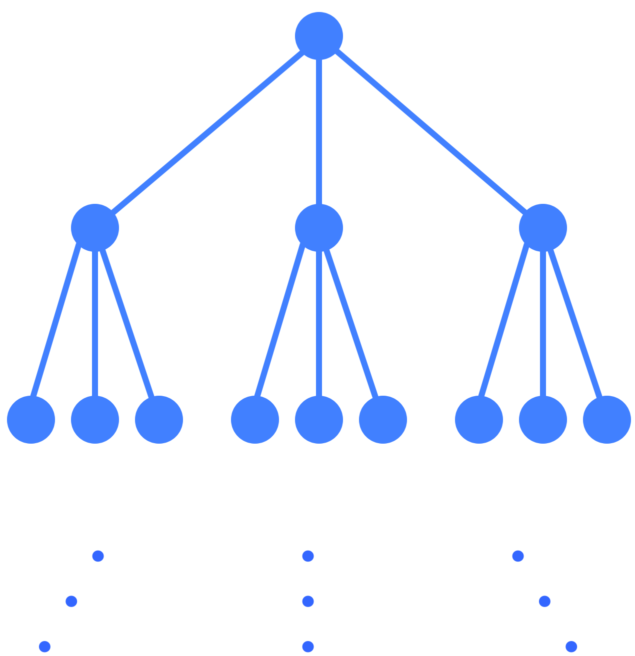
$n = 2^k$; $k = \log_2 n$

$4^k = (2^2)^k = (2^k)^2 = n^2$

Total Work: $T(n) = \sum_{i=0}^{k} 4^i cn/2^i = O(n^2)$

Solve: $T(1) = c$

$T(n) = 3\,T(n/2) + cn$



Level	Num	Size	Work
0	$1 = 3^0$	n	cn
1	$3 = 3^1$	$n/2$	$3cn/2$
2	$9 = 3^2$	$n/4$	$9cn/4$
…	…	…	…
i	3^i	$n/2^i$	$3^i\ c\ n/2^i$
…	…	…	…
$k-1$	3^{k-1}	$n/2^{k-1}$	$3^{k-1}\ c\ n/2^{k-1}$
k	3^k	$n/2^k = 1$	$3^k\ T(1)$

$n = 2^k$; $k = \log_2 n$

Total Work: $T(n) = \sum_{i=0}^{k} 3^i cn / 2^i$

Theorem: for $x \neq 1$,

$$1 + x + x^2 + x^3 + \ldots + x^k = (x^{k+1}-1)/(x-1)$$

proof:

$$\begin{aligned}
y &= 1 + x + x^2 + x^3 + \ldots + x^k \\
xy &= \phantom{1 + {}} x + x^2 + x^3 + \ldots + x^k + x^{k+1} \\
xy-y &= x^{k+1} - 1 \\
y(x-1) &= x^{k+1} - 1 \\
y &= (x^{k+1}-1)/(x-1)
\end{aligned}$$

Solve: $T(1) = c$

$T(n) = 3\,T(n/2) + cn$ (cont.)

$$T(n) = \sum_{i=0}^{k} 3^i cn / 2^i$$

$$= cn \sum_{i=0}^{k} 3^i / 2^i$$

$$= cn \sum_{i=0}^{k} \left(\tfrac{3}{2}\right)^i$$

$$= cn \frac{\left(\tfrac{3}{2}\right)^{k+1} - 1}{\left(\tfrac{3}{2}\right) - 1}$$

$$\sum_{i=0}^{k} x^i = \frac{x^{k+1} - 1}{x - 1}$$

$$(x \neq 1)$$

Solve: T(1) = c
T(n) = 3 T(n/2) + cn (cont.)

$$cn\frac{\left(\frac{3}{2}\right)^{k+1}-1}{\left(\frac{3}{2}\right)-1} = 2cn\left(\left(\frac{3}{2}\right)^{k+1}-1\right)$$

$$< 2cn\left(\frac{3}{2}\right)^{k+1}$$

$$= 3cn\left(\frac{3}{2}\right)^{k}$$

$$= 3cn\frac{3^k}{2^k}$$

Solve: $T(1) = c$
$T(n) = 3\,T(n/2) + cn$ (cont.)

$$3cn\frac{3^k}{2^k} = 3cn\frac{3^{\log_2 n}}{2^{\log_2 n}}$$

$$= 3cn\frac{3^{\log_2 n}}{n}$$

$$= 3c3^{\log_2 n}$$

$$= 3c\left(n^{\log_2 3}\right)$$

$$= O\left(n^{1.585\ldots}\right)$$

$$a^{\log_b n}$$

$$= \left(b^{\log_b a}\right)^{\log_b n}$$

$$= \left(b^{\log_b n}\right)^{\log_b a}$$

$$= n^{\log_b a}$$

$T(n) = aT(n/b)+cn^k$ for $n > b$ then

$a > b^k \Rightarrow T(n) = \Theta(n^{\log_b a})$ [many subprobs → leaves dominate]

$a < b^k \Rightarrow T(n) = \Theta(n^k)$ [few subprobs → top level dominates]

$a = b^k \Rightarrow T(n) = \Theta(n^k \log n)$ [balanced → all log n levels contribute]

Fine print:

$T(1) = d$; $a \geq 1$; $b > 1$; $c, d, k \geq 0$; $n = b^t$ for some $t > 0$; a, b, k, t integers. True even if it is $\lceil n/b \rceil$ instead of n/b.

Expand recurrence as in earlier examples, to get

$$T(n) = n^h \, (d + c\,S)$$

where $h = \log_b(a)$ (and n^h = number of tree leaves) and $S = \sum_{j=1}^{\log_b n} x^j$, where $x = b^k/a$.

If $c = 0$ the sum S is irrelevant, and $T(n) = O(n^h)$: all work happens in the base cases, of which there are n^h, one for each leaf in the recursion tree.

If $c > 0$, then the sum matters, and splits into 3 cases (like previous slide):

- if $x < 1$, then $S < x/(1-x) = O(1)$. [S is the first log n terms of the infinite series with that sum.]
- if $x = 1$, then $S = \log_b(n) = O(\log n)$. [All terms in the sum are 1 and there are that many terms.]
- if $x > 1$, then $S = x \bullet (x^{1+\log_b(n)}-1)/(x-1)$. [And after some algebra, $n^h * S = O(n^k)$.]

Another Example: Exponentiation

Power(a,n)

Input: integer n and number a

Output: a^n

Obvious algorithm

$n-1$ multiplications

Observation:

if n is even, $n = 2m$, then $a^n = a^m \cdot a^m$

```
Power(a,n)
   if n = 0 then return(1)
   if n = 1 then return(a)
   x ← Power(a,⌊n/2⌋)
   x ← x•x
   if n is odd then
        x ← a•x
   return(x)
```

Let M(n) be number of multiplies

Worst-case recurrence:

$$M(n) = \begin{cases} 0 & n \le 1 \\ M(\lfloor n/2 \rfloor) + 2 & n > 1 \end{cases}$$

By master theorem

$M(n) = O(\log n)$ (a=1, b=2, k=0)

More precise analysis:

$M(n) = \lfloor \log_2 n \rfloor$ + (# of 1's in n's binary representation) - 1

Time is O(M(n)) if numbers < word size, else also depends on length, multiply algorithm

Instead of a^n want a^n mod N

a^{i+j} mod N = ((a^i mod N) • (a^j mod N)) mod N

same algorithm applies with each $x \cdot y$ replaced by

((x mod N) • (y mod N)) mod N

In RSA cryptosystem (widely used for security)

need a^n mod N where a, n, N each typically have 1024 bits

Power: at most 2048 multiplies of 1024 bit numbers

relatively easy for modern machines

Naive algorithm: 2^{1024} multiplies

Idea:

"Two halves are better than a whole"

if the base algorithm has super-linear complexity.

"If a little's good, then more's better"

repeat above, recursively

Analysis: recursion tree or Master Recurrence

Applications: Many.

Binary Search, Merge Sort, (Quicksort), Closest Points, Integer Multiply, Exponentiation,…