Guessing Game: NP-Complete?

1. LONGEST-PATH: Given a graph G = (V, E), does there exists a simple path of length at least k edges?
   YES

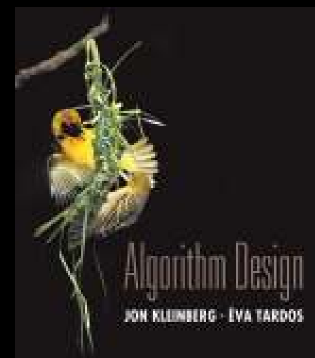
2. SHORTEST-PATH: Given a graph G = (V, E), does there exists a simple path of length at most k edges?
   In P

3. 2-SAT: Give a formula Φ such that each clause has at most 2 literals, is Φ is satisfiable?
   In P

4. 3-COLOR: Given a graph G = (V, E), can we color the nodes of G with 3 colors such that no two nodes joined by an edge have the same coloring
   YES

5. Factoring: Give an integer N. Find the factors of N.
   INAPPLICABLE

1



Chapter 10

Extending the Limits of Tractability

Reading: 10.1-10.2

Coping With NP-Completeness

Q. Suppose I need to solve an NP-complete problem. What should I do?
A. Theory says you're unlikely to find poly-time algorithm.

Must sacrifice one of three desired features.
- Solve problem to optimality.
- Solve problem in polynomial time.
- Solve arbitrary instances of the problem.

This lecture. Solve some special cases of NP-complete problems that arise in practice.

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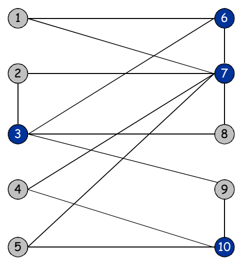
10.1 Finding Small Vertex Covers

Vertex Cover

VERTEX COVER: Given a graph $G = (V, E)$ and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \le k$, and for each edge (u, v) either $u \in S$, or $v \in S$, or both.



$k = 4$
$S = \{ 3, 6, 7, 10 \}$

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Finding Small Vertex Covers

Q. What if k is small?

Brute force. $O(k\, n^{k+1})$.
- Try all $C(n, k) = O(n^k)$ subsets of size k.
- Takes $O(k\, n)$ time to check whether a subset is a vertex cover.

Goal. Limit exponential dependency on k, e.g., to $O(2^k\, k\, n)$.

Ex. $n = 1{,}000$, $k = 10$.
Brute. $k\, n^{k+1} = 10^{34} \Rightarrow$ infeasible.
Better. $2^k\, k\, n = 10^7 \Rightarrow$ feasible.

Remark. If k is a constant, algorithm is poly-time; if k is a small constant, then it's also practical.

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Finding Small Vertex Covers

Claim. Let u-v be an edge of G. G has a vertex cover of size $\le k$ iff at least one of $G - \{ u \}$ and $G - \{ v \}$ has a vertex cover of size $\le k-1$.

delete v and all incident edges

Pf. $\Rightarrow$
- Suppose G has a vertex cover S of size $\le k$.
- S contains either u or v (or both). Assume it contains u.
- $S - \{ u \}$ is a vertex cover of $G - \{ u \}$.

Pf. $\Leftarrow$
- Suppose S is a vertex cover of $G - \{ u \}$ of size $\le k-1$.
- Then $S \cup \{ u \}$ is a vertex cover of G. ▪

Claim. If G has a vertex cover of size k, it has $\le k(n-1)$ edges.
Pf. Each vertex covers at most n-1 edges. ▪

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Finding Small Vertex Covers: Algorithm

Claim. The following algorithm determines if G has a vertex cover of size $\le k$ in $O(2^k\, kn)$ time.

```
boolean Vertex-Cover(G, k) {
   if (G contains no edges)    return true
   if (G contains ≥ kn edges) return false

   let (u, v) be any edge of G
   a = Vertex-Cover(G - {u}, k-1)
   b = Vertex-Cover(G - {v}, k-1)
   return a or b
}
```

Pf.
- Correctness follows previous two claims.
- There are $\le 2^{k+1}$ nodes in the recursion tree; each invocation takes $O(kn)$ time. ▪

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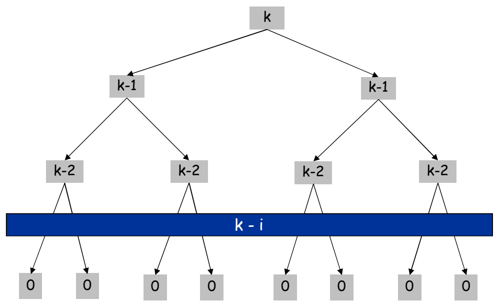
Finding Small Vertex Covers: Recursion Tree

$$T(n,k) \le \begin{cases} cn & \text{if } k = 1 \\ 2T(n,k-1) + ckn & \text{if } k > 1 \end{cases} \quad \Rightarrow \quad T(n,k) \le 2^k c\,k\,n$$



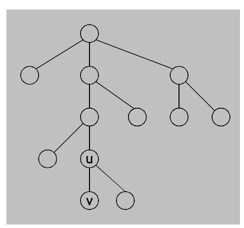
10.2 Solving NP-Hard Problems on Trees

Independent Set on Trees

Independent set on trees. Given a tree, find a maximum cardinality subset of nodes such that no two share an edge.

Fact. A tree on at least two nodes has at least two leaf nodes.

degree = 1



Key observation. If v is a leaf, there exists a maximum size independent set containing v.

Pf. (exchange argument)

- Consider a max cardinality independent set S.
- If v ∈ S, we're done.
- If u ∉ S and v ∉ S, then S ∪ { v } is independent ⇒ S not maximum.
- IF u ∈ S and v ∉ S, then S ∪ { v } – { u } is independent. ▪

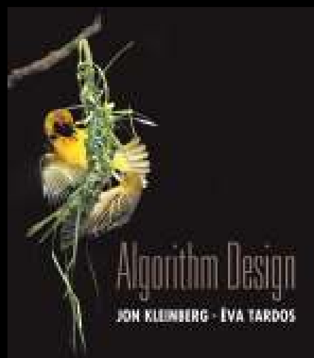
Independent Set on Trees: Greedy Algorithm

Theorem. The following greedy algorithm finds a maximum cardinality independent set in forests (and hence trees).

```
Independent-Set-In-A-Forest(F) {
   S ← φ
   while (F has at least one edge) {
      Let e = (u, v) be an edge such that v is a leaf
      Add v to S
      Delete from F nodes u and v, and all edges
         incident to them.
   }
   return S
}
```

Pf. Correctness follows from the previous key observation. ▪

Remark. Can implement in O(n) time by considering nodes in postorder.



Chapter 11

Approximation Algorithms

Approximation Algorithms

Q. Suppose I need to solve an NP-hard problem. What should I do?
A. Theory says you're unlikely to find a poly-time algorithm.

Must sacrifice one of three desired features.
- Solve problem to optimality.
- Solve problem in poly-time.
- Solve arbitrary instances of the problem.

ρ-approximation algorithm.
- Guaranteed to run in poly-time.
- Guaranteed to solve arbitrary instance of the problem
- Guaranteed to find solution within ratio ρ of true optimum.

Challenge. Need to prove a solution's value is close to optimum, without even knowing what optimum value is!

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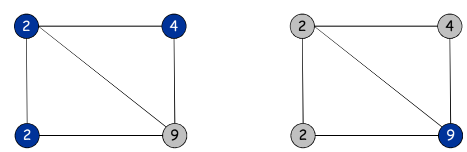
11.4 The Pricing Method: Vertex Cover

Weighted Vertex Cover

Weighted vertex cover. Given a graph G with vertex weights, find a vertex cover of minimum weight.



weight = 2 + 2 + 4 weight = 9

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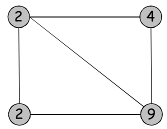
Weighted Vertex Cover

Pricing method. Each edge must be covered by some vertex i. Edge e pays price $p_e \geq 0$ to use vertex i.

Fairness. Edges incident to vertex i should pay $\leq w_i$ in total.

$$\text{for each vertex } i: \sum_{e=(i,j)} p_e \leq w_i$$



Claim. For any vertex cover S and any fair prices p_e: $\sum_e p_e \leq w(S)$.

Proof. ▪

$$\sum_{e \in E} p_e \leq \sum_{i \in S} \sum_{e=(i,j)} p_e \leq \sum_{i \in S} w_i = w(S).$$

each edge e covered by at least one node in S

sum fairness inequalities for each node in S

Pricing Method

Pricing method. Set prices and find vertex cover simultaneously.

```
Weighted-Vertex-Cover-Approx(G, w) {
   foreach e in E
      pe = 0

   while (∃ edge i-j such that neither i nor j are tight)
      select such an edge e
      increase pe without violating fairness
   }

   S ← set of all tight nodes
   return S
}
```

$$\sum_{e=(i,j)} p_e = w_i$$

Pricing Method

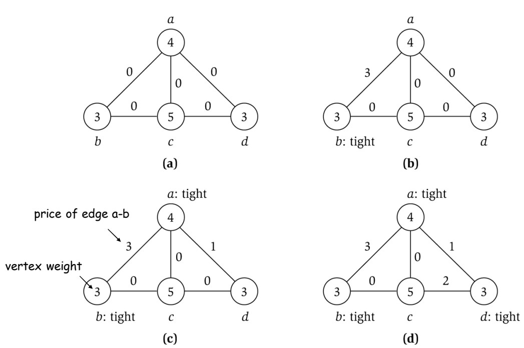


Figure 11.8

Pricing Method: Analysis

Theorem. Pricing method is a 2-approximation.

Pf.

- Algorithm terminates since at least one new node becomes tight after each iteration of while loop.
- Let S = set of all tight nodes upon termination of algorithm. S is a vertex cover: if some edge i-j is uncovered, then neither i nor j is tight. But then while loop would not terminate.
- Let S* be optimal vertex cover. We show $w(S) \leq 2w(S^*)$.

$$w(S) = \sum_{i \in S} w_i = \sum_{i \in S} \sum_{e=(i,j)} p_e \leq \sum_{i \in V} \sum_{e=(i,j)} p_e = 2 \sum_{e \in E} p_e \leq 2w(S^*). \quad ▪$$

all nodes in S are tight

$S \subseteq V$, prices ≥ 0

each edge counted twice

fairness lemma

13.4 MAX 3-SAT

Maximum 3-Satisfiability

exactly 3 distinct literals per clause

MAX-3SAT. Given 3-SAT formula, find a truth assignment that satisfies as many clauses as possible.

$$
\begin{aligned}
C_1 &= x_2 \vee \overline{x_3} \vee \overline{x_4} \\
C_2 &= x_2 \vee x_3 \vee \overline{x_4} \\
C_3 &= \overline{x_1} \vee x_2 \vee x_4 \\
C_4 &= \overline{x_1} \vee \overline{x_2} \vee x_3 \\
C_5 &= x_1 \vee \overline{x_2} \vee \overline{x_4}
\end{aligned}
$$

Remark. NP-hard search problem.

Simple idea. Flip a coin, and set each variable true with probability $\frac{1}{2}$, independently for each variable.

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Maximum 3-Satisfiability: Analysis

Claim. Given a 3-SAT formula with k clauses, the expected number of clauses satisfied by a random assignment is 7k/8.

Pf. Consider random variable $Z_j = \begin{cases} 1 & \text{if clause } C_j \text{ is satisfied} \\ 0 & \text{otherwise.} \end{cases}$

- Let Z = weight of clauses satisfied by assignment Z_j.

linearity of expectation

$$
\begin{aligned}
E[Z] &= \sum_{j=1}^{k} E[Z_j] \\
&= \sum_{j=1}^{k} \Pr[\text{clause } C_j \text{ is satisfied}] \\
&= \tfrac{7}{8} k
\end{aligned}
$$

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The Probabilistic Method

Corollary. For any instance of 3-SAT, there exists a truth assignment that satisfies at least a 7/8 fraction of all clauses.

Pf. Random variable is at least its expectation some of the time. ▪

Probabilistic method. We showed the existence of a non-obvious property of 3-SAT by showing that a random construction produces it with positive probability!

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Maximum 3-Satisfiability: Analysis

Q. Can we turn this idea into a 7/8-approximation algorithm? In general, a random variable can almost always be below its mean.

Lemma. The probability that a random assignment satisfies $\geq 7k/8$ clauses is at least $1/(8k)$.

Pf. Let p_j be probability that exactly j clauses are satisfied; let p be probability that $\geq 7k/8$ clauses are satisfied.

$$\begin{aligned} \tfrac{7}{8}k \;=\; E[Z] &= \sum_{j \geq 0} j\,p_j \\ &= \sum_{j < 7k/8} j\,p_j \;+\; \sum_{j \geq 7k/8} j\,p_j \\ &\leq \left(\tfrac{7k}{8} - \tfrac{1}{8}\right) \sum_{j < 7k/8} p_j \;+\; k \sum_{j \geq 7k/8} p_j \\ &\leq \left(\tfrac{7}{8}k - \tfrac{1}{8}\right) \cdot 1 \;+\; k\,p \end{aligned}$$

Rearranging terms yields $p \geq 1/(8k)$. ▪

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Maximum 3-Satisfiability: Analysis

Johnson's algorithm. Repeatedly generate random truth assignments until one of them satisfies $\geq 7k/8$ clauses.

Theorem. Johnson's algorithm is a 7/8-approximation algorithm.

Pf. By previous lemma, each iteration succeeds with probability at least $1/(8k)$. By the waiting-time bound, the expected number of trials to find the satisfying assignment is at most 8k. ▪

Waiting for a first success. Coin is heads with probability p and tails with probability 1-p. How many independent flips X until first heads?

$$E[X] = \sum_{j=0}^{\infty} j \cdot \Pr[X=j] = \sum_{j=0}^{\infty} j\,\underbrace{(1-p)^{j-1}}_{\text{j-1 tails}}\,\underbrace{p}_{\text{1 head}} = \frac{p}{1-p}\sum_{j=0}^{\infty} j\,(1-p)^{j} = \frac{p}{1-p}\cdot\frac{1-p}{p^2} = \frac{1}{p}$$

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Maximum Satisfiability

Extensions.

- Allow one, two, or more literals per clause.
- Find max weighted set of satisfied clauses.

Theorem. [Asano-Williamson 2000] There exists a 0.784-approximation algorithm for MAX-SAT.

Theorem. [Karloff-Zwick 1997, Zwick+computer 2002] There exists a 7/8-approximation algorithm for version of MAX-3SAT where each clause has at most 3 literals.

Theorem. [Håstad 1997] Unless P = NP, no ρ-approximation algorithm for MAX-3SAT (and hence MAX-SAT) for any $\rho > 7/8$.

very unlikely to improve over simple randomized algorithm for MAX-3SAT

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What to do if the problem you want to solve is NP-hard

- More on approximation algorithms
  - Recent research has classified problems based on what kinds of approximations are possible if **P≠NP**
    - Best: **(1+ε)** factor for any **ε>0**.
      - packing and some scheduling problems, TSP in plane
    - Some fixed constant factor **> 1**, e.g. **2, 3/2, 100**
      - Vertex Cover, TSP in space, other scheduling problems
    - **Θ(log n)** factor
      - Set Cover, Graph Partitioning problems
    - Worst: **Ω(n^{1-ε})** factor for any **ε>0**
      - Clique, Independent-Set, Coloring

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What to do if the problem you want to solve is NP-hard

- Try an algorithm that is provably fast "on average".
  - To even try this one needs a model of what a typical instance is.
  - Typically, people consider "random graphs"
    - e.g. all graphs with a given # of edges are equally likely
  - Problems:
    - real data doesn't look like the random graphs
    - distributions of real data aren't analyzable

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What to do if the problem you want to solve is NP-hard

- Try to search the space of possible hints/certificates in a more efficient way and hope it is quick enough
  - **Backtracking search**
    - E.g. For SAT there are 2^n possible truth assignments
    - If we set the truth values one-by-one we might be able to figure out whole parts of the space to avoid,
      - e.g. After setting $x_1 \leftarrow 1$, $x_2 \leftarrow 0$ we don't even need to set x_3 or x_4 to know that it won't satisfy

$$(\neg x_1 \vee x_2) \wedge (\neg x_2 \vee x_3) \wedge (x_4 \vee \neg x_3) \wedge (x_1 \vee \neg x_4)$$

    - Related technique: **branch-and-bound**
  - Backtracking search can be very effective even with exponential worst-case time
    - For example, the best SAT algorithms used in practice are all variants on backtracking search and can solve surprisingly large problems

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What to do if the problem you want to solve is NP-hard

- Use heuristic algorithms and hope they give good answers
  - No guarantees of quality
  - Many different types of heuristic algorithms
  - Many different options, especially for optimization problems, such as TSP, where we want the best solution.
    - We'll mention several on following slides

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Heuristic algorithms for NP-hard problems

- **local search** for optimization problems
  - need a notion of two solutions being neighbors
  - Start at an arbitrary solution **S**
  - While there is a neighbor **T** of **S** that is better than **S**
    - **S←T**
- Usually fast but often gets stuck in a local optimum and misses the global optimum
  - With some notions of neighbor can take a long time in the worst case

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e.g., Neighboring solutions for TSP

Solution S   Solution T

Two solutions are neighbors iff there is a pair of edges you can swap to transform one to the other

Heuristic algorithms for NP-hard problems

- randomized local search
  - start local search several times from random starting points and take the best answer found from each point
    - **more expensive than plain local search but usually much better answers**
- simulated annealing
  - like local search but at each step sometimes move to a worse neighbor with some probability
    - probability of going to a worse neighbor is set to decrease with time as, presumably, solution is closer to optimal
    - helps avoid getting stuck in a local optimum but often **slow to converge** (much more expensive than randomized local search)
    - analogy with slow cooling to get to lowest energy state in a crystal (or in forging a metal)

Heuristic algorithms

- **artificial neural networks**
  - based on very elementary model of human neurons
  - **Set up a circuit of artificial neurons**
    - each artificial neuron is an analog circuit gate whose computation depends on a set of **connection strengths**
  - **Train the circuit**
    - Adjust the connection strengths of the neurons by giving many positive & negative training examples and seeing if it behaves correctly
  - **The network is now ready to use**
  - **useful for ill-defined classification problems such as optical character recognition but not typical cut & dried problems**

Other directions

- Quantum computing
  - **Use physical processes at the quantum level to implement "weird" kinds of circuit gates**
    - unitary transformations
  - **Quantum objects can be in a superposition of many pure states at once**
    - can have **n** objects together in a superposition of 2^n states
  - **Each quantum circuit gate operates on the whole superposition of states at once**
    - inherent **parallelism** but classical randomized algorithms have a similar parallelism: **not enough on its own**
    - **Advantage over classical: parallel copies interfere with each other.**
  - **Need totally new kinds of algorithms to work well. Theoretically able to factor efficiently but huge practical problems: errors, decoherence.**

Loose Ends

Space Complexity:
- Amount of memory used by an algorithm
- If an algorithm runs in time T, then it uses at most T units of memory
- Every poly-time algorithm uses poly-space
- If an algorithm uses S units of memory, it run in time $O(2^S)$

PSPACE: class of algorithms solvable by algorithms that use a polynomial amount of space.

P ⊆ PSPACE

Another big question in complexity is whether P = PSPACE.

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