CSE 417: Algorithms and Computational Complexity

Winter 2009

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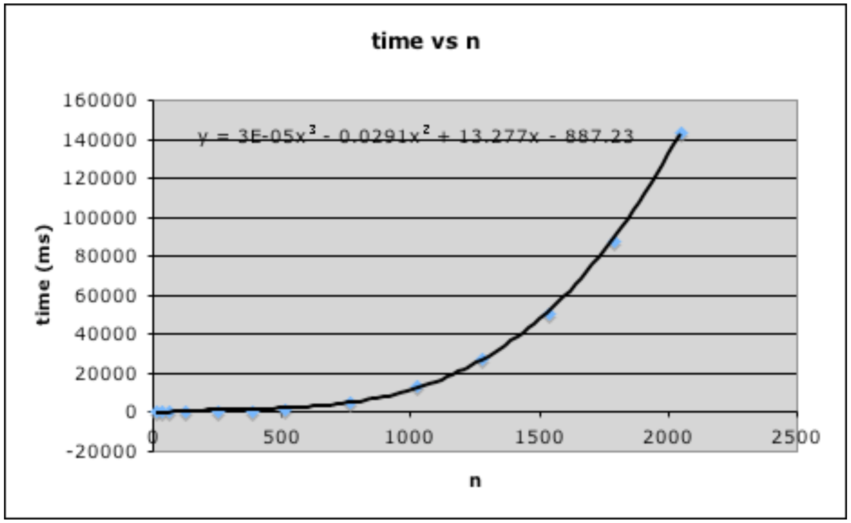
Divide and Conquer Algorithms

HW4 – Empirical Run Times

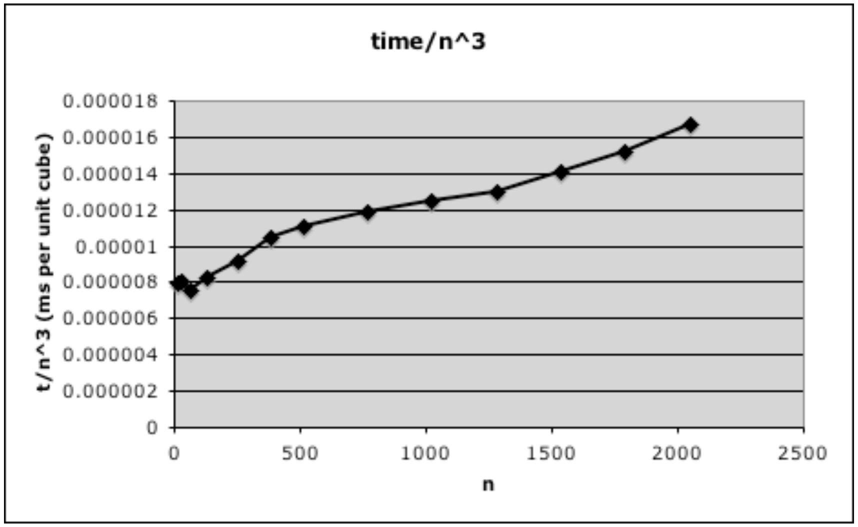


Plot Time vs n

Fit curve to it (e.g., with Excel)

Note: Higher degree polynomials fit better…

Plotting Time/(growth rate) vs n may be more sensitive – should be flat, but small n may be unrepresentative of asymptotics



The Divide and Conquer Paradigm

Outline:

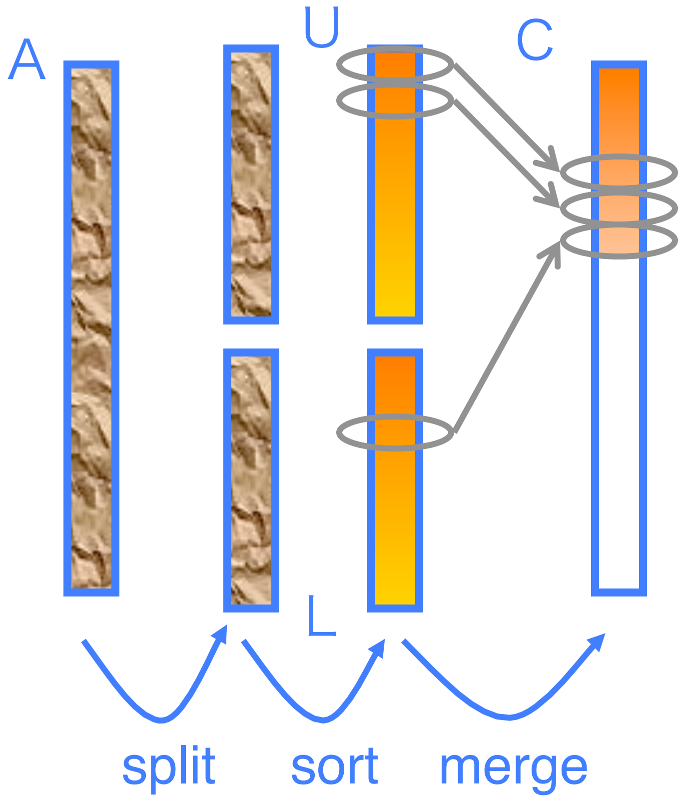
- General Idea
- Review of Merge Sort
- Why does it work?
    - Importance of balance
    - Importance of super-linear growth
- Some interesting applications
    - Closest points
    - Integer Multiplication
- Finding & Solving Recurrences

Algorithm Design Techniques

Divide & Conquer

- Reduce problem to one or more sub-problems of the same type
- Typically, each sub-problem is at most a constant fraction of the size of the original problem
  - e.g. Mergesort, Binary Search, Strassen's Algorithm, Quicksort (kind of)

Merge Sort



```
MS(A: array[1..n]) returns array[1..n] {
    If(n=1) return A[1];
    New U:array[1:n/2] = MS(A[1..n/2]);
    New L:array[1:n/2] = MS(A[n/2+1..n]);
    Return(Merge(U,L));
    }
Merge(U,L: array[1..n]) {
    New C: array[1..2n];
    a=1; b=1;
    For i = 1 to 2n
        C[i] = "smaller of U[a], L[b] and correspondingly a++ or b++";
    Return C;
    }
```

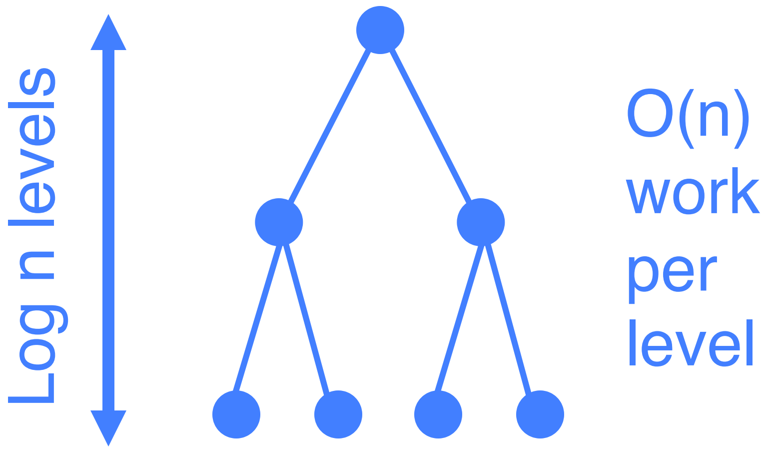
Mergesort (review)

Mergesort: (recursively) sort 2 half-lists, then merge results.

T(n) = 2T(n/2)+cn, n≥2

T(1) = 0

Solution: O(n log n)
(details later)



Why Balanced Subdivision?

Alternative "divide & conquer" algorithm:

Sort n-1

Sort last 1

Merge them

$T(n) = T(n-1)+T(1)+3n$ for $n \geq 2$

$T(1) = 0$

Solution: $3n + 3(n-1) + 3(n-2) \ldots = \Theta(n^2)$

Another D&C Approach

Suppose we've already invented DumbSort, taking time n^2

Try *Just One Level* of divide & conquer:

DumbSort(first n/2 elements)

DumbSort(last n/2 elements)

Merge results

Time: $\boxed{2\,(n/2)^2 + n = n^2/2 + n \ll n^2}$

Almost twice as fast!

D&C in a nutshell

Another D&C Approach, cont.

Moral 1: “two halves are better than a whole”

Two problems of half size are *better* than one full-size problem, even given the O(n) overhead of recombining, since the base algorithm has *super-linear* complexity.

Moral 2: “If a little's good, then more's better”

two levels of D&C would be almost 4 times faster, 3 levels almost 8, etc., even though overhead is growing. Best is usually full recursion down to some small constant size (balancing "work" vs "overhead").

Another D&C Approach, cont.

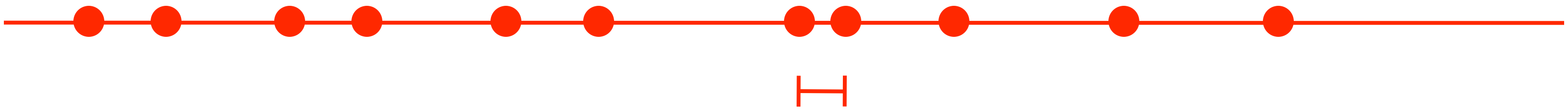
Moral 3: unbalanced division less good:

- $(.1n)^2 + (.9n)^2 + n = .82n^2 + n$
  - The 18% savings compounds significantly if you carry recursion to more levels, actually giving O(nlogn), but with a bigger constant. So worth doing if you can't get 50-50 split, but balanced is better if you can.
  - This is intuitively why Quicksort with random splitter is good – badly unbalanced splits are rare, and not instantly fatal.
- $(1)^2 + (n-1)^2 + n = n^2 - 2n + 2 + n$
  - Little improvement here.

5.4 Closest Pair of Points

Closest pair of points: 1 Dimensional Version

Given n points on the real line, find the closest pair



Closest pair is adjacent in ordered list

Time O(n log n) to sort, if needed

Plus O(n) to scan adjacent pairs

Closest Pair of Points

Closest pair. Given n points in the plane, find a pair with smallest Euclidean distance between them.

Fundamental geometric primitive.

- Graphics, computer vision, geographic information systems, molecular modeling, air traffic control.
- Special case of nearest neighbor, Euclidean MST, Voronoi.

↖ fast closest pair inspired fast algorithms for these problems

Brute force. Check all pairs of points p and q with $\Theta(n^2)$ comparisons.

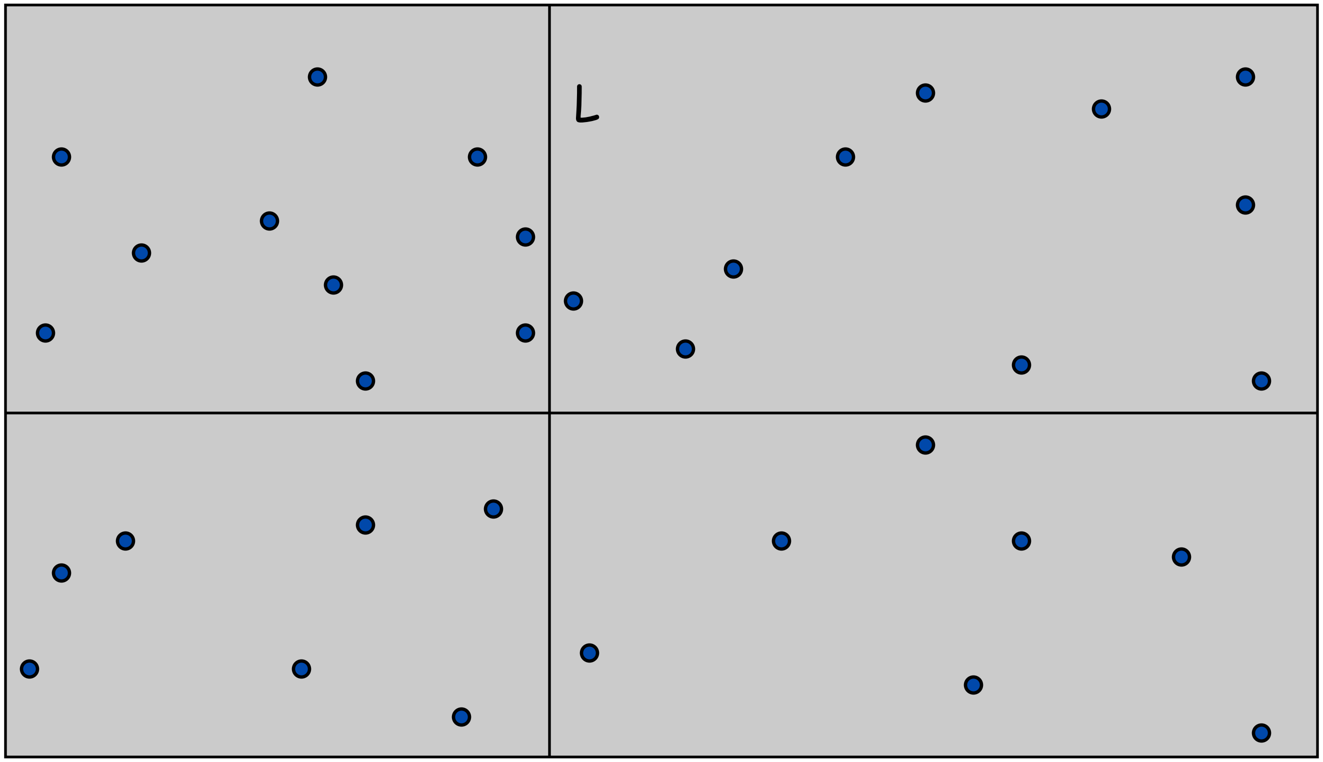
1-D version. O(n log n) easy if points are on a line.

Assumption. No two points have same x coordinate.

↑ to make presentation cleaner

Closest Pair of Points: First Attempt

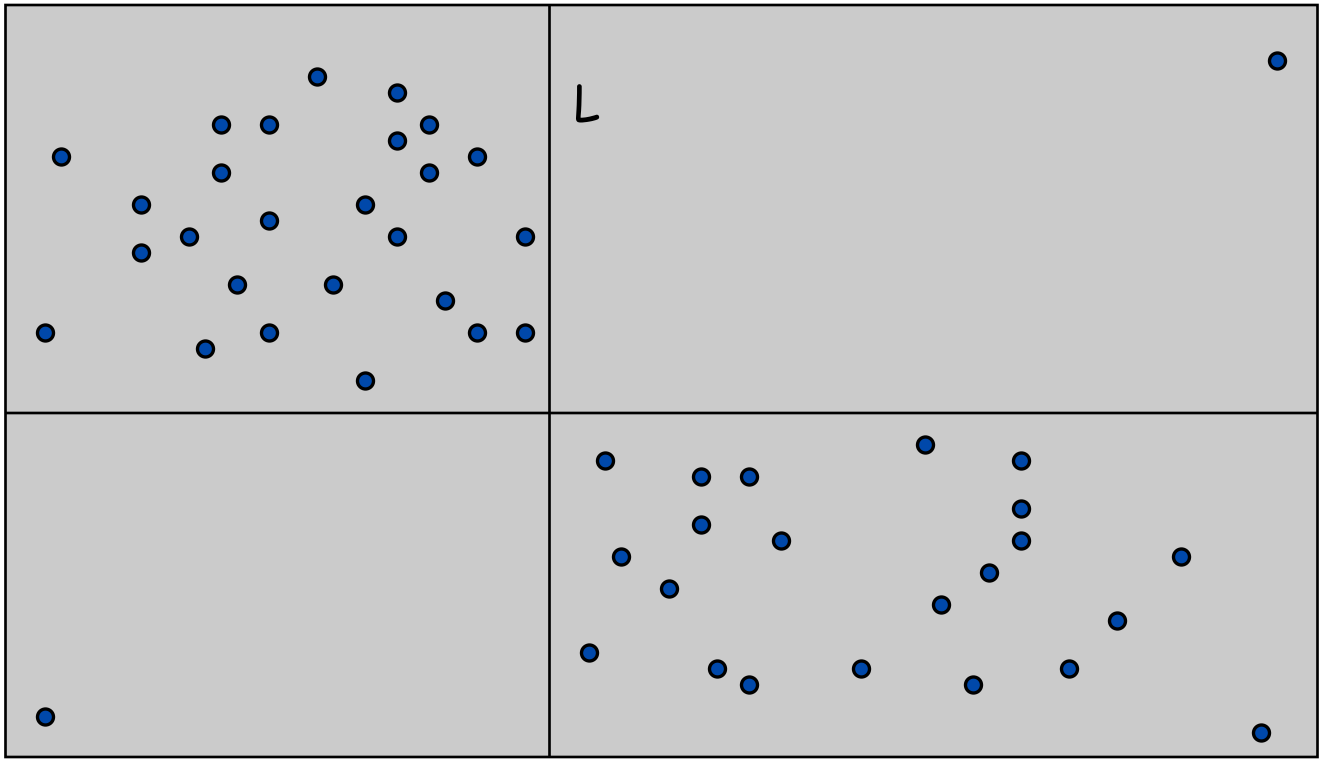
Divide. Sub-divide region into 4 quadrants.



Closest Pair of Points: First Attempt

Divide. Sub-divide region into 4 quadrants.

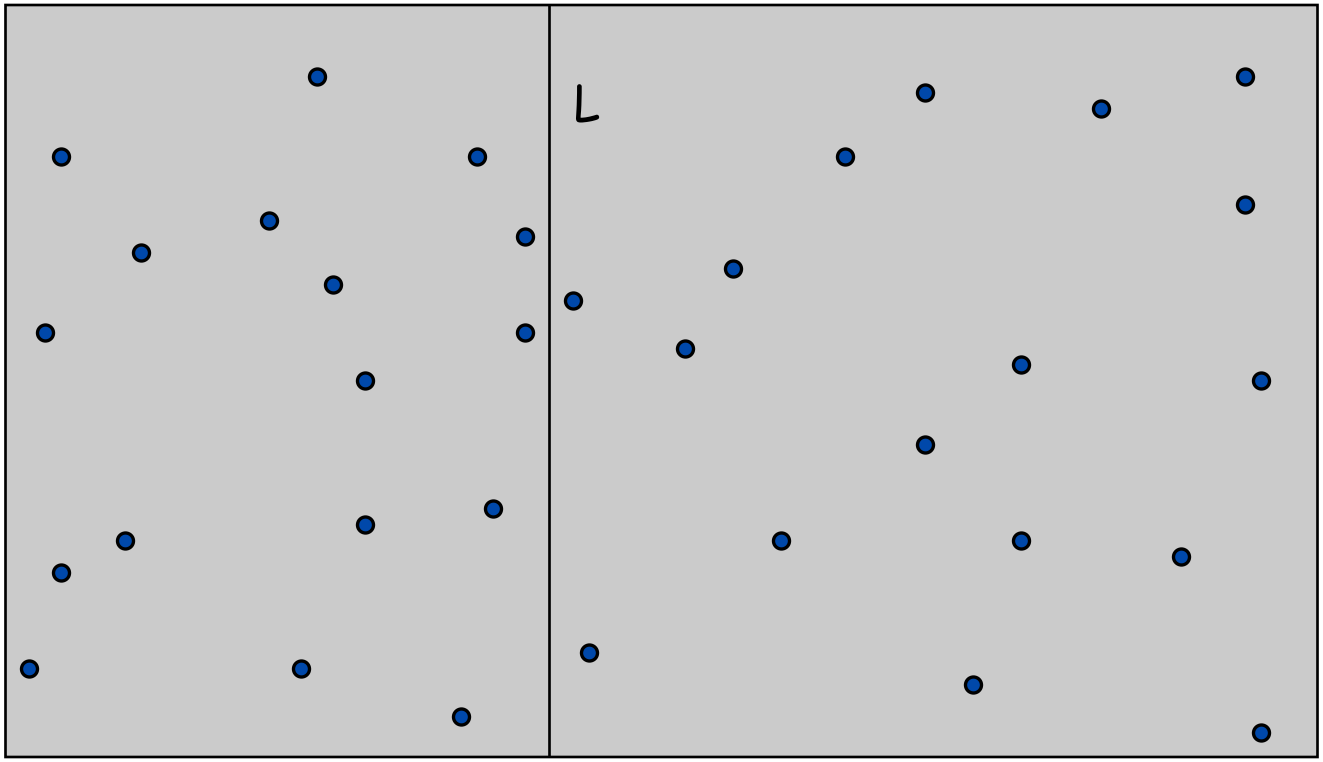
Obstacle. Impossible to ensure n/4 points in each piece.



Closest Pair of Points

Algorithm.

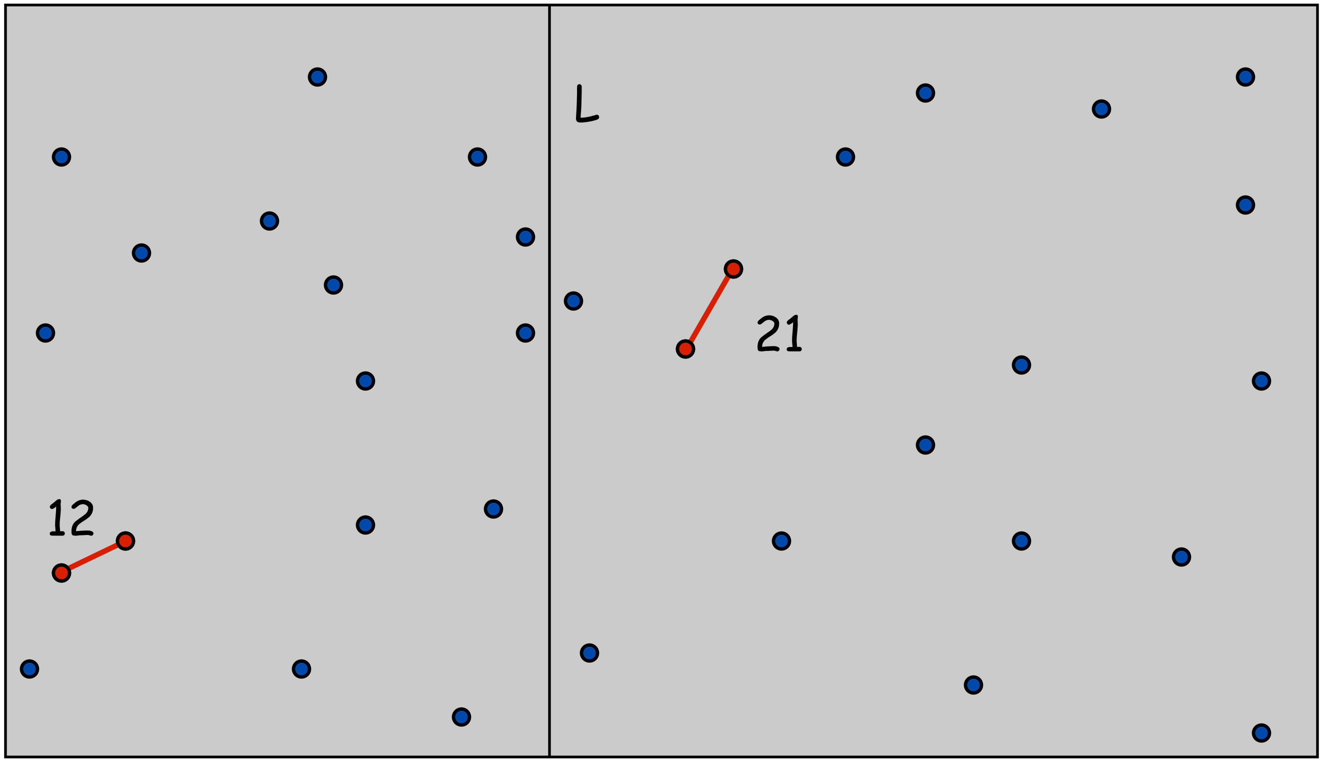
- Divide: draw vertical line L so that roughly $\frac{1}{2}$n points on each side.



Closest Pair of Points

Algorithm.

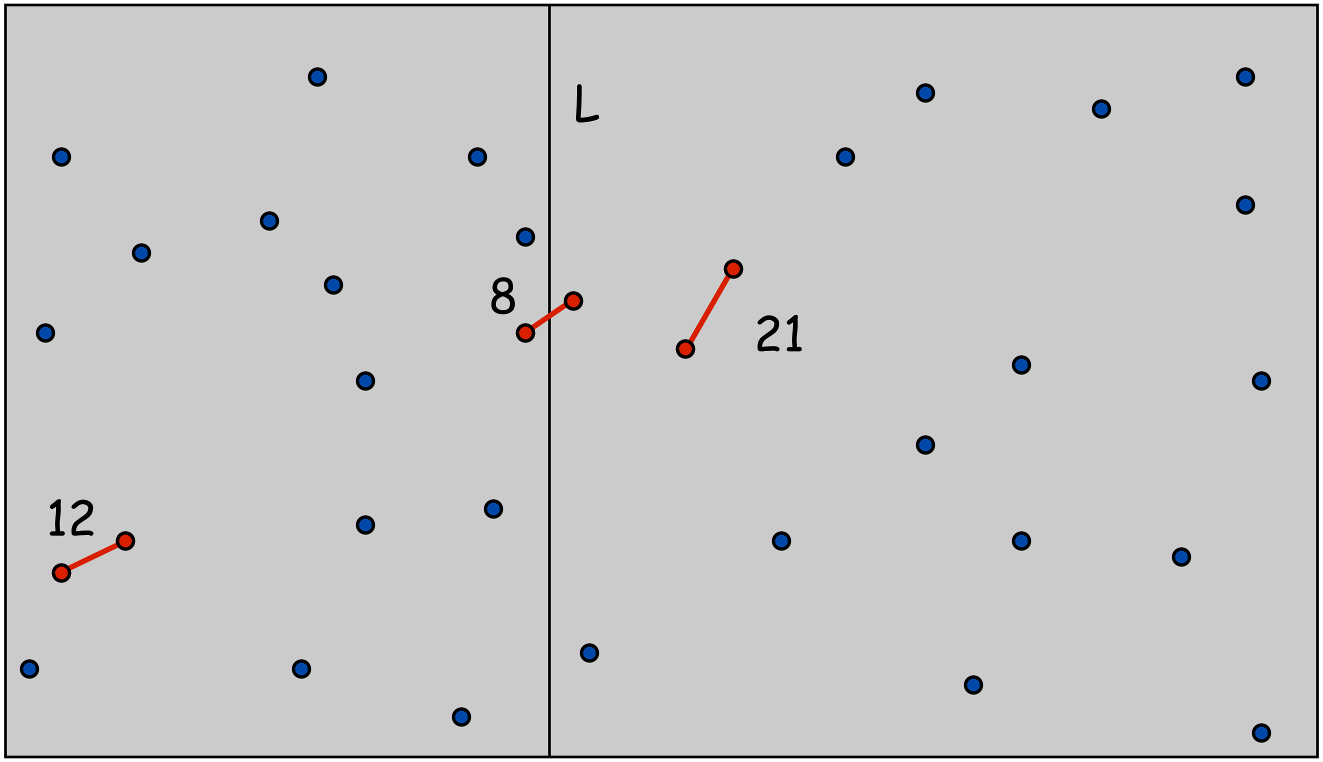
- Divide: draw vertical line L so that roughly $\frac{1}{2}$n points on each side.
- Conquer: find closest pair in each side recursively.



Closest Pair of Points

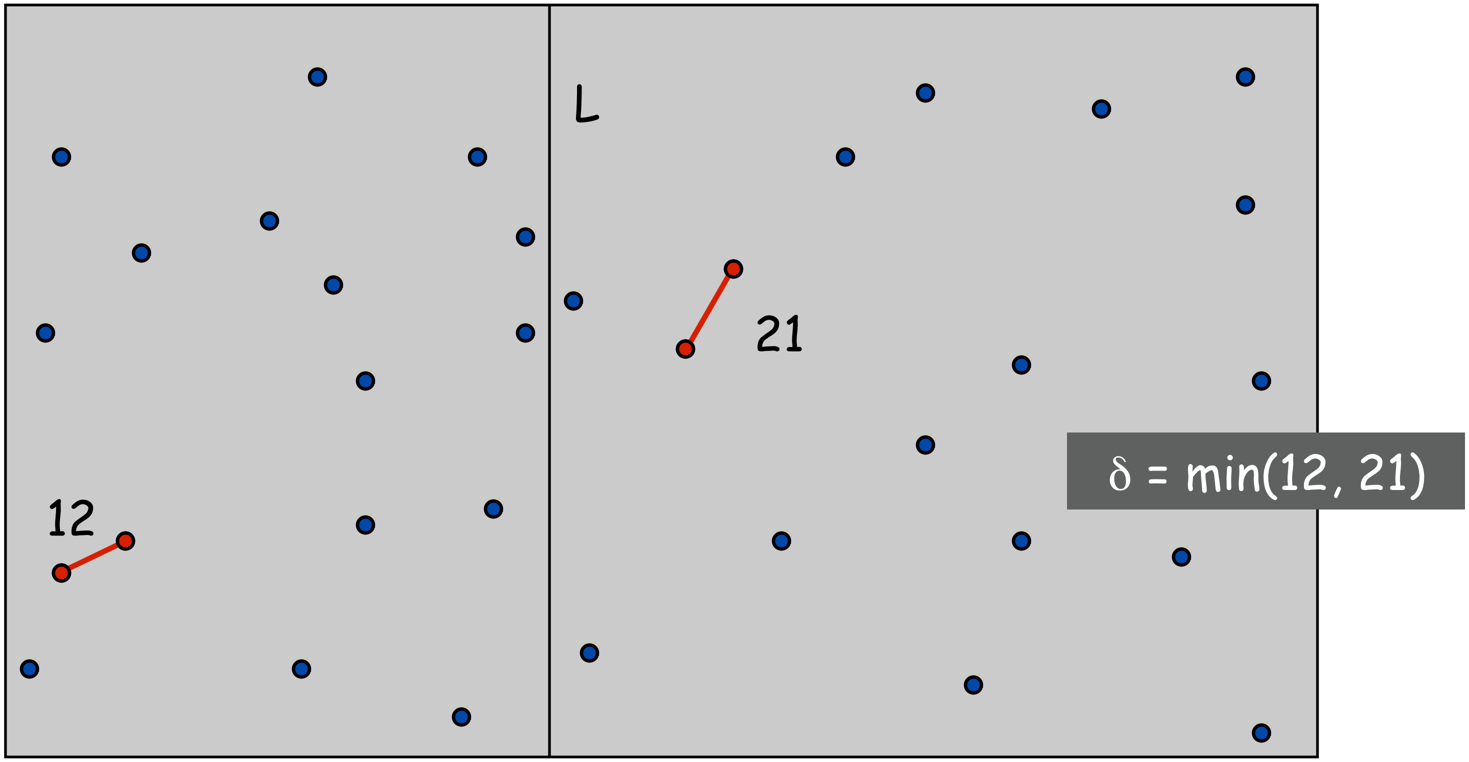
Algorithm.

- Divide: draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.
- Conquer: find closest pair in each side recursively.
- Combine: find closest pair with one point in each side. ← seems like $\Theta(n^2)$
- Return best of 3 solutions.



Closest Pair of Points

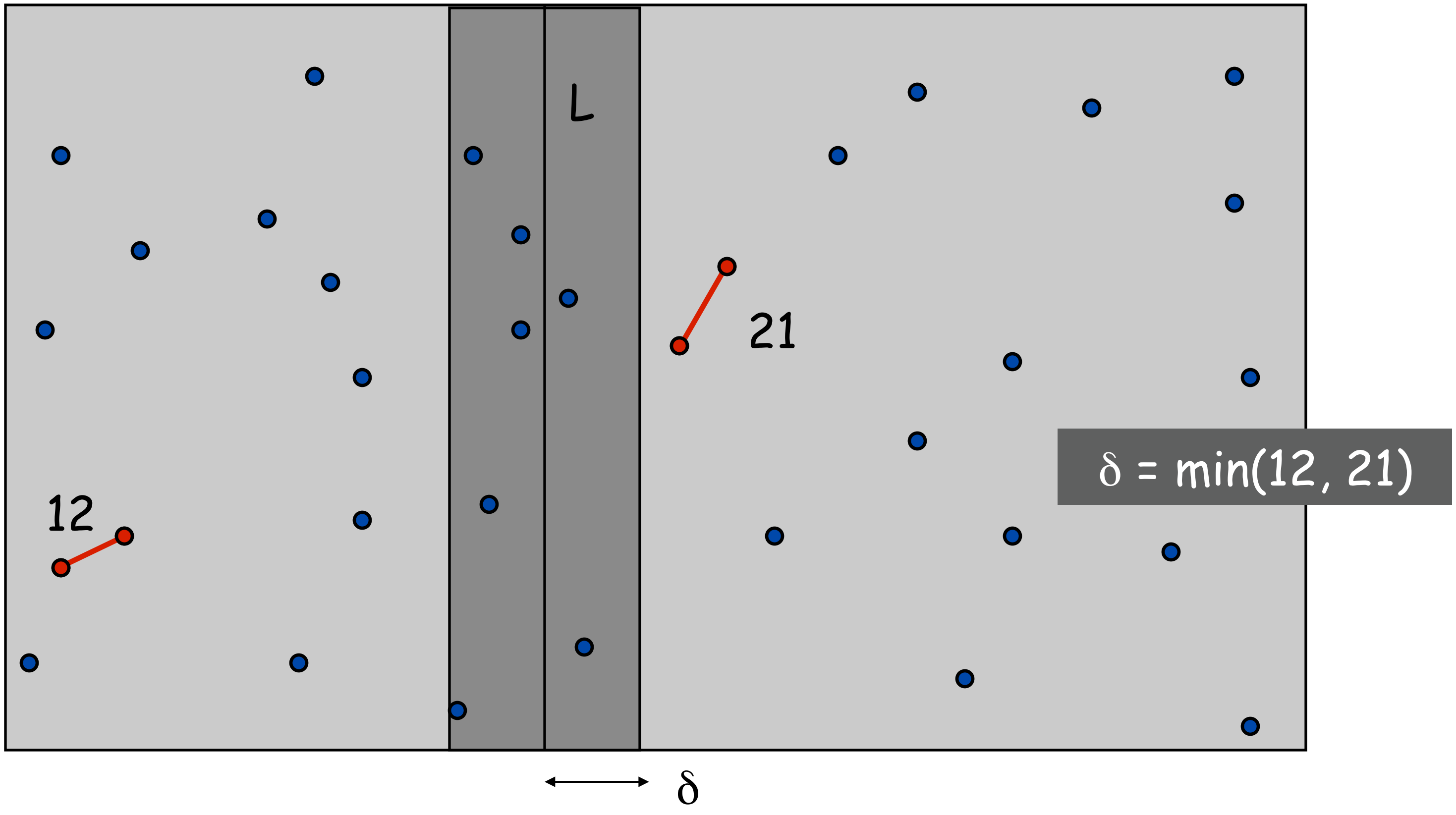
Find closest pair with one point in each side, assuming that distance < δ.



Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

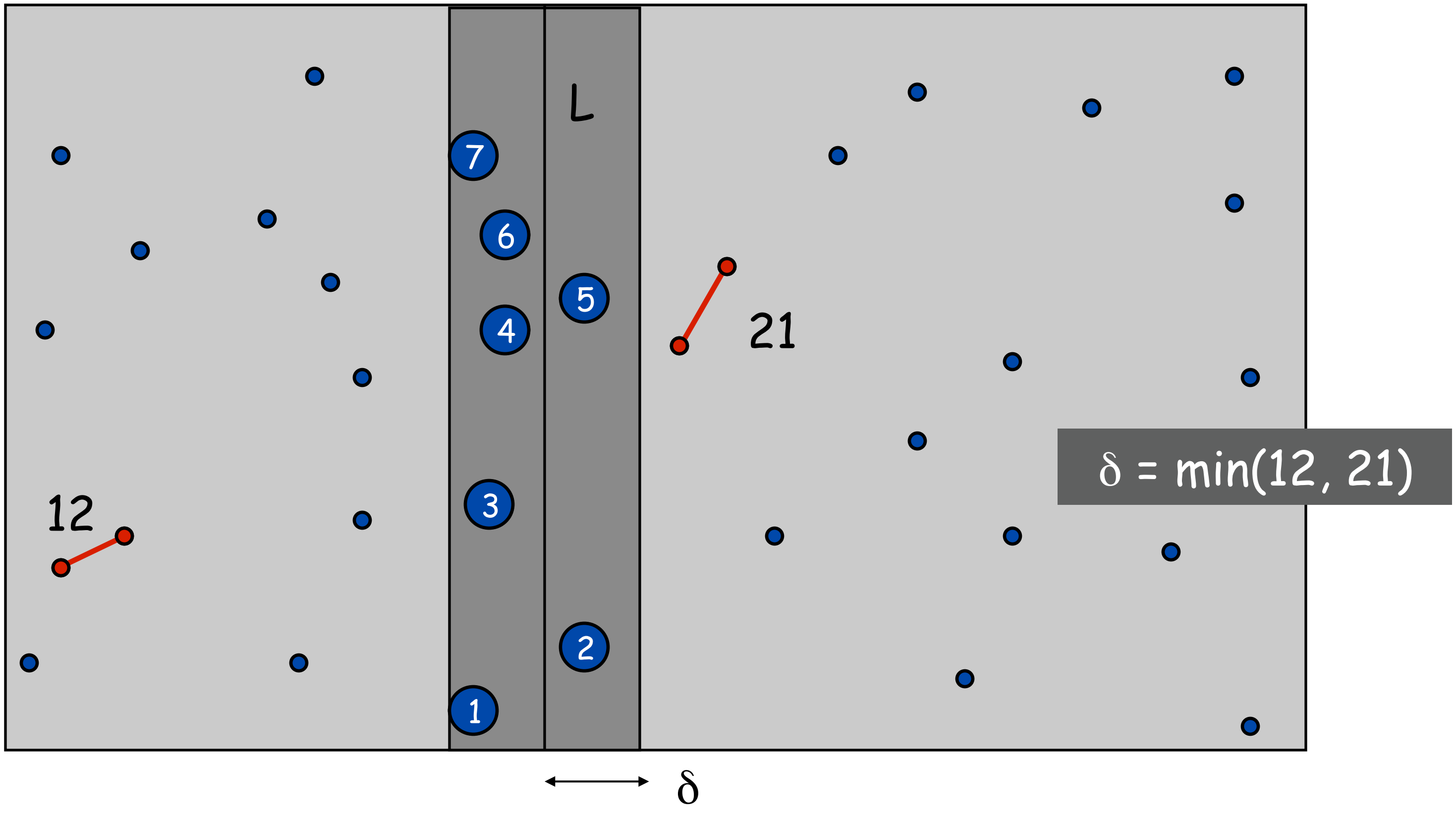
- Observation: only need to consider points within δ of line L.



Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

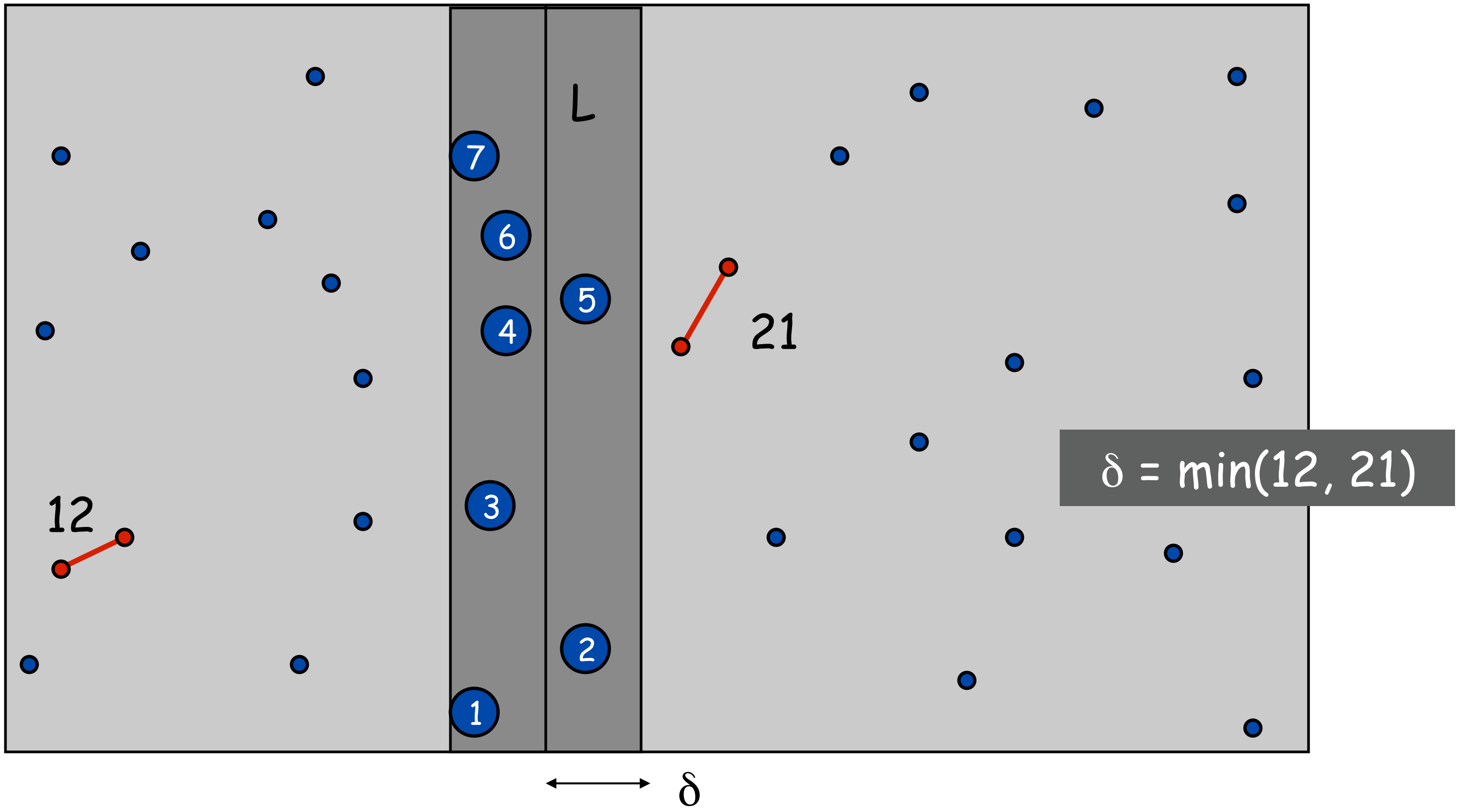
- Observation: only need to consider points within δ of line L.
- Sort points in 2δ-strip by their y coordinate.



Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

- Observation: only need to consider points within δ of line L.
- Sort points in 2δ-strip by their y coordinate.
- Only check distances of those within 8 positions in sorted list!



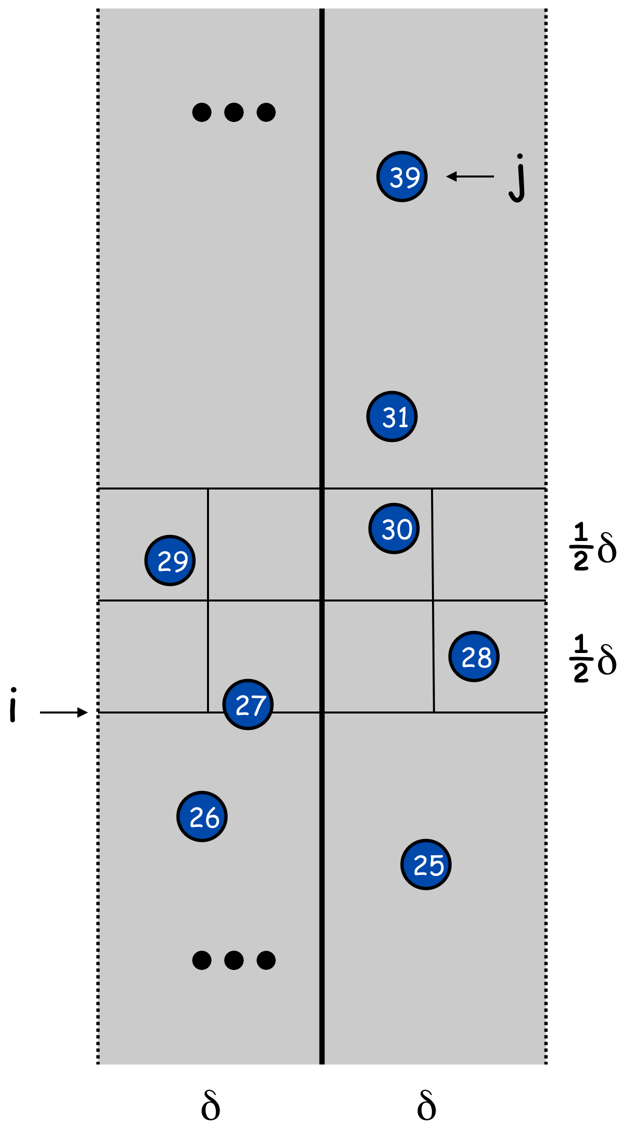
Closest Pair of Points

Def. Let s_i be the point in the 2δ-strip, with the i^{th} smallest y-coordinate.

Claim. If $|i - j| > 8$, then the distance between s_i and s_j is $> \delta$.

Pf.

- No two points lie in same $\frac{1}{2}\delta$-by-$\frac{1}{2}\delta$ box.
- only 8 boxes



Closest Pair Algorithm

```
Closest-Pair($p_1$, …, $p_n$) {
   if(n <= ??) return ??

   Compute separation line L such that half the points
   are on one side and half on the other side.

   $\delta_1$ = Closest-Pair(left half)
   $\delta_2$ = Closest-Pair(right half)
   $\delta$  = min($\delta_1$, $\delta_2$)

   Delete all points further than $\delta$ from separation line L

   Sort remaining points p[1]…p[m] by y-coordinate.

   for i = 1..m
      k = 1
      while i+k <= m && p[i+k].y < p[i].y + $\delta$
        $\delta$ = min($\delta$, distance between p[i] and p[i+k]);
        k++;

   return $\delta$.
}
```

Going From Code to Recurrence

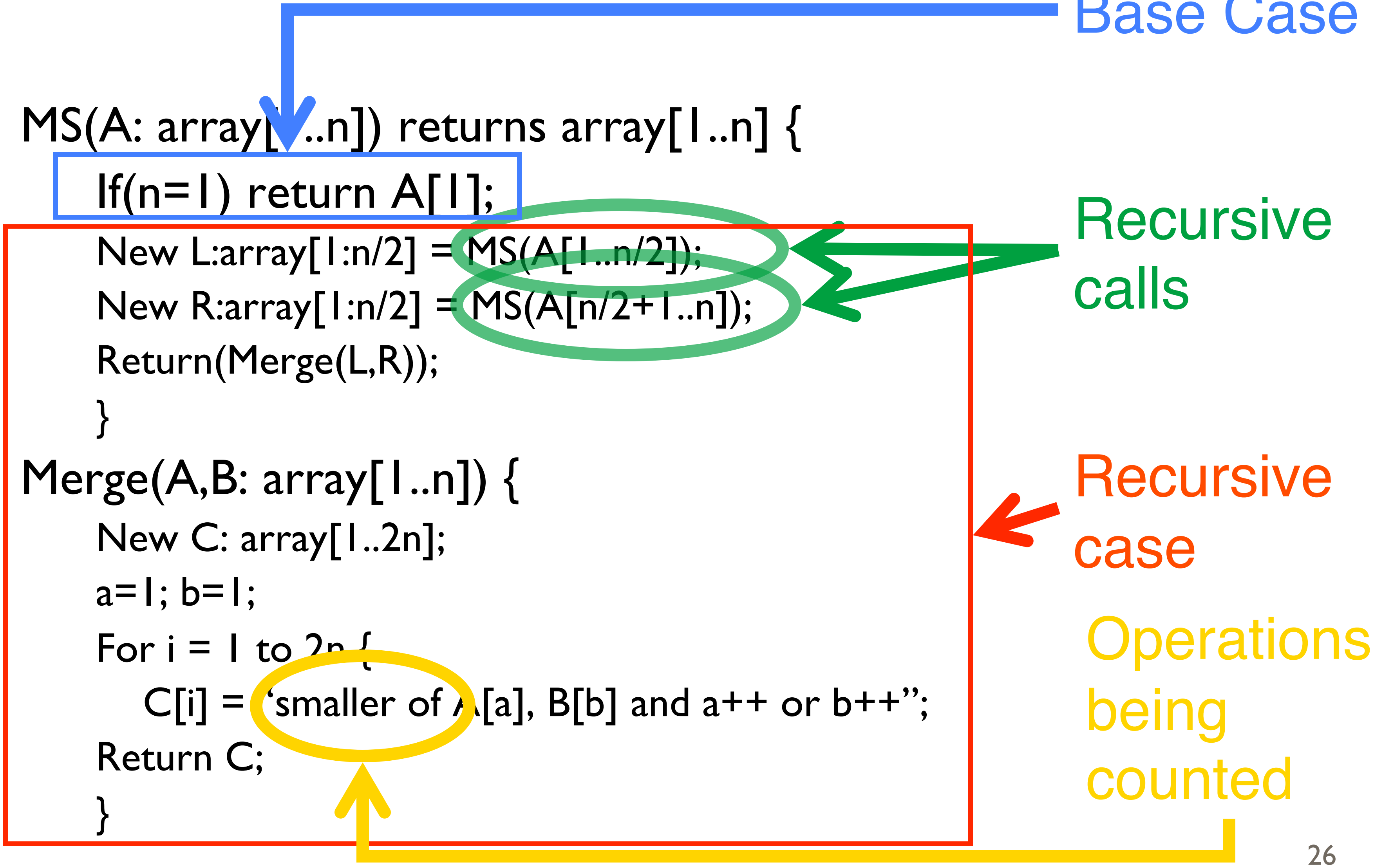
Carefully define what you're counting, and write it down!

> "Let C(n) be the number of comparisons between sort keys used by MergeSort when sorting a list of length n ≥ 1"

In code, clearly separate *base case* from *recursive case,* highlight *recursive calls,* and *operations being counted.*

Write Recurrence(s)

Merge Sort



```
MS(A: array[1..n]) returns array[1..n] {
    If(n=1) return A[1];
    New L:array[1:n/2] = MS(A[1..n/2]);
    New R:array[1:n/2] = MS(A[n/2+1..n]);
    Return(Merge(L,R));
    }
Merge(A,B: array[1..n]) {
    New C: array[1..2n];
    a=1; b=1;
    For i = 1 to 2n {
        C[i] = ‘smaller of A[a], B[b] and a++ or b++”;
    Return C;
    }
```

Base Case

Recursive calls

Recursive case

Operations being counted

The Recurrence

Base case

$$C(n) = \begin{cases} 0 & \text{if } n = 1 \\ 2C(n/2) + (n-1) & \text{if } n > 1 \end{cases}$$

Recursive calls

One compare per element added to merged list, except the last.

Total time: proportional to C(n)

(loops, copying data, parameter passing, etc.)

Going From Code to Recurrence

Carefully define what you're counting, and write it down!

> "Let D(n) be the number of pairwise distance comparisons in the Closest-Pair Algorithm when run on n ≥ 1 points"

In code, clearly separate *base case* from *recursive case,* highlight *recursive calls,* and *operations being counted*.

Write Recurrence(s)

Closest Pair Algorithm

Base Case

Basic operations:
distance calcs

Recursive calls (2)

```
Closest-Pair($p_1$, …, $p_n$) {
   if(n <= 1) return $\infty$

   Compute separation line L such that half the points
   are on one side and half on the other side.

   $\delta_1$ = Closest-Pair(left half)
   $\delta_2$ = Closest-Pair(right half)
   $\delta$  = min($\delta_1$, $\delta_2$)

   Delete all points further than $\delta$ from separation line L

   Sort remaining points p[1]…p[m] by y-coordinate.

   for i = 1..m
      k = 1
      while i+k <= m && p[i+k].y < p[i].y + $\delta$
         $\delta$ = min($\delta$, distance between p[i] and p[i+k]);
         k++;

   return $\delta$.
}
```

Basic operations at
this recursive level

0

2D(n / 2)

O(n)

Closest Pair of Points: *Analysis*

Running time.

$$D(n) \le \begin{cases} 0 & n = 1 \\ 2D(n/2) + 7n & n > 1 \end{cases} \Rightarrow D(n) = O(n \log n)$$

BUT - that's only the number of *distance calculations*

What if we counted comparisons?

Closest Pair Algorithm

Base Case

Basic operations: *comparisons*

Recursive calls (2)

```
Closest-Pair($p_1$, …, $p_n$) {
   if(n <= 1) return ∞

   Compute separation line L such that half the points
   are on one side and half on the other side.

   $\delta_1$ = Closest-Pair(left half)
   $\delta_2$ = Closest-Pair(right half)
   $\delta$  = min($\delta_1$, $\delta_2$)

   Delete all points further than $\delta$ from separation line L

   Sort remaining points p[1]…p[m]

   for i = 1..m
      k = 1
      while i+k <= m && p[i+k].y < p[i].y + $\delta$
        $\delta$ = min($\delta$, distance between p[i] and p[i+k]);
        k++;

   return $\delta$.
}
```

Basic operations at this recursive level

0

O(n log n)

2C(n / 2)

1

O(n)

O(n log n)

O(n)

Closest Pair of Points: *Analysis*

Running time.

$$C(n) \le \begin{Bmatrix} 0 & n = 1 \\ 2C(n/2) + O(n \log n) & n > 1 \end{Bmatrix} \Rightarrow C(n) = O(n \log^2 n)$$

Q. Can we achieve O(n log n)?

A. Yes. Don't sort points from scratch each time.

- Sort by x at top level only.
- Each recursive call returns δ and list of all points sorted by y
- Sort by merging two pre-sorted lists.

$$T(n) \le 2T(n/2) + O(n) \Rightarrow \mathrm{T}(n) = O(n \log n)$$

5.5 Integer Multiplication

Integer Arithmetic

Add. Given two n-digit integers a and b, compute a + b.

- O(n) bit operations.

Multiply. Given two n-digit integers a and b, compute a × b.

- The "grade school" method: $\Theta(n^2)$ bit operations.

1	1	1	1	1	1	0	1	
	1	1	0	1	0	1	0	1
+	0	1	1	1	1	1	0	1
1	0	1	0	1	0	0	1	0

Add

								1	1	0	1	0	1	0	1
							*	0	1	1	1	1	1	0	1
								1	1	0	1	0	1	0	1
							0	0	0	0	0	0	0	0	0
						1	1	0	1	0	1	0	1	0	
					1	1	0	1	0	1	0	1	0		
				1	1	0	1	0	1	0	1	0			
			1	1	0	1	0	1	0	1	0				
		1	1	0	1	0	1	0	1	0					
	0	0	0	0	0	0	0	0	0						
0	1	1	0	1	0	0	0	0	0	0	0	0	0	0	1

Multiply

Divide-and-Conquer Multiplication: Warmup

To multiply two n-digit integers:

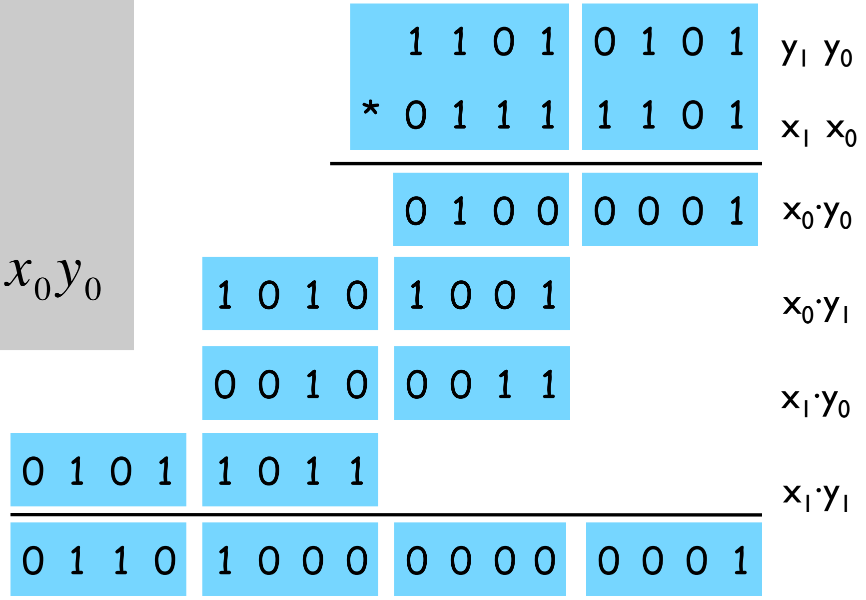
- Multiply four ½n-digit integers.
- Add two ½n-digit integers, and shift to obtain result.

$$
\begin{aligned}
x &= 2^{n/2} \cdot x_1 + x_0 \\
y &= 2^{n/2} \cdot y_1 + y_0 \\
xy &= \left(2^{n/2} \cdot x_1 + x_0\right)\left(2^{n/2} \cdot y_1 + y_0\right) \\
&= 2^n \cdot x_1 y_1 + 2^{n/2} \cdot \left(x_1 y_0 + x_0 y_1\right) + x_0 y_0
\end{aligned}
$$

$$
\mathrm{T}(n) = \underbrace{4T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, shift}} \Rightarrow \mathrm{T}(n) = \Theta(n^2)
$$

↑

assumes n is a power of 2



		1 1 0 1	0 1 0 1	y_1 y_0
		* 0 1 1 1	1 1 0 1	x_1 x_0
		0 1 0 0	0 0 0 1	$x_0 \cdot y_0$
	1 0 1 0	1 0 0 1		$x_0 \cdot y_1$
	0 0 1 0	0 0 1 1		$x_1 \cdot y_0$
0 1 0 1	1 0 1 1			$x_1 \cdot y_1$
0 1 1 0	1 0 0 0	0 0 0 0	0 0 0 1	

Key trick: 2 multiplies for the price of 1:

$$
\begin{aligned}
x &= 2^{n/2} \cdot x_1 + x_0 \\
y &= 2^{n/2} \cdot y_1 + y_0 \\
xy &= \left(2^{n/2} \cdot x_1 + x_0\right)\left(2^{n/2} \cdot y_1 + y_0\right) \\
&= 2^{n} \cdot x_1 y_1 + 2^{n/2} \cdot \left(x_1 y_0 + x_0 y_1\right) + x_0 y_0
\end{aligned}
$$

Well, ok, 4 for 3 is more accurate…

$$
\begin{aligned}
\alpha &= x_1 + x_0 \\
\beta &= y_1 + y_0 \\
\alpha\beta &= \left(x_1 + x_0\right)\left(y_1 + y_0\right) \\
&= x_1 y_1 + \left(x_1 y_0 + x_0 y_1\right) + x_0 y_0 \\
\left(x_1 y_0 + x_0 y_1\right) &= \alpha\beta - x_1 y_1 - x_0 y_0
\end{aligned}
$$

Karatsuba Multiplication

To multiply two n-digit integers:

- Add two $\frac{1}{2}$n digit integers.
- Multiply three $\frac{1}{2}$n-digit integers.
- Add, subtract, and shift $\frac{1}{2}$n-digit integers to obtain result.

$$\begin{aligned}
x &= 2^{n/2} \cdot x_1 + x_0 \\
y &= 2^{n/2} \cdot y_1 + y_0 \\
xy &= 2^{n} \cdot x_1 y_1 + 2^{n/2} \cdot \left(x_1 y_0 + x_0 y_1\right) + x_0 y_0 \\
&= 2^{n} \cdot \underset{A}{x_1 y_1} + 2^{n/2} \cdot \left(\underset{B}{(x_1 + x_0)(y_1 + y_0)} - \underset{A}{x_1 y_1} - \underset{C}{x_0 y_0} \right) + \underset{C}{x_0 y_0}
\end{aligned}$$

Theorem. [Karatsuba-Ofman, 1962] Can multiply two n-digit integers in $O(n^{1.585})$ bit operations.

$$\mathrm{T}(n) \le \underbrace{T\left(\lfloor n/2 \rfloor\right) + T\left(\lceil n/2 \rceil\right) + T\left(1 + \lceil n/2 \rceil\right)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, subtract, shift}}$$

$$\textit{Sloppy version}: \ T(n) \le 3T(n/2) + O(n)$$

$$\Rightarrow \mathrm{T}(n) = O(n^{\log_2 3}) = O(n^{1.585})$$

Multiplication – The Bottom Line

Naïve: $\Theta(n^2)$

Karatsuba: $\Theta(n^{1.59\ldots})$

Amusing exercise: generalize Karatsuba to do 5 size n/3 subproblems => $\Theta(n^{1.46\ldots})$

Best known: $\Theta(n \log n \log\log n)$

"Fast Fourier Transform"

but mostly unused in practice (unless you need really big numbers - a billion digits of π, say)

High precision arithmetic *IS* important for crypto

Recurrences

Where they come from,
how to find them (above)

Next: how to solve them

Mergesort (review)

Mergesort: (recursively) sort 2 half-lists, then merge results.

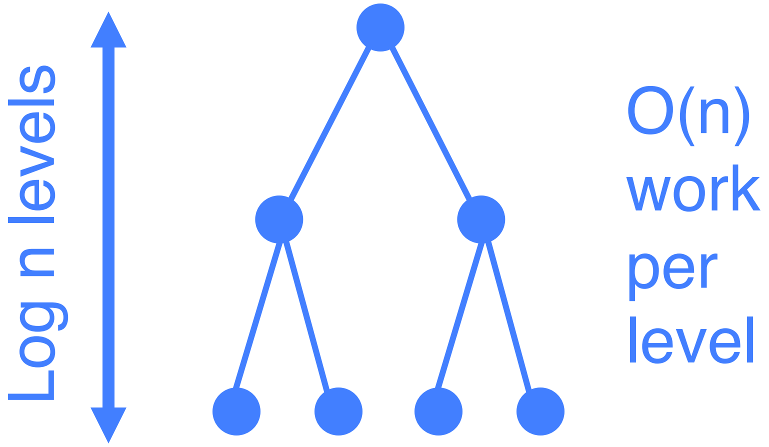
$T(n) = 2T(n/2)+cn, \quad n \geq 2$

$T(1) = 0$

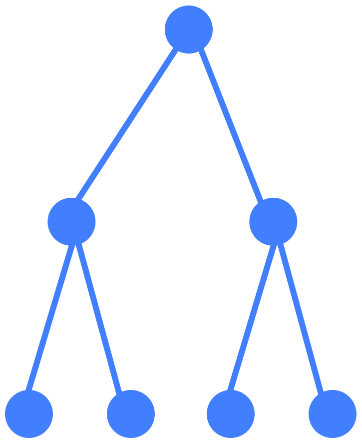
Solution: $\Theta(n \log n)$

(details ~~later~~)

now



Solve: $T(1) = c$
$T(n) = 2\,T(n/2) + cn$

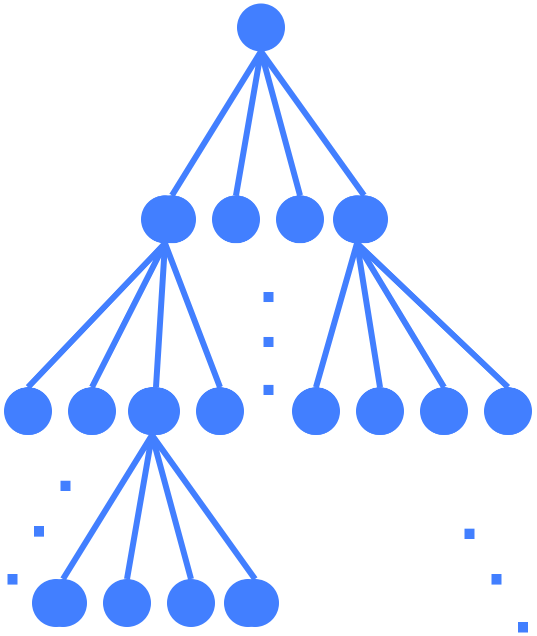


Level	Num	Size	Work
0	$1 = 2^0$	n	cn
1	$2 = 2^1$	$n/2$	$2cn/2$
2	$4 = 2^2$	$n/4$	$4cn/4$
…	…	…	…
i	2^i	$n/2^i$	$2^i\ c\ n/2^i$
…	…	…	…
$k-1$	2^{k-1}	$n/2^{k-1}$	$2^{k-1}\ c\ n/2^{k-1}$
k	2^k	$n/2^k = 1$	$2^k\ T(1)$

$n = 2^k$; $k = \log_2 n$

Total Work: $c\ n \log_2 n$ (add last col)

Solve: $T(1) = c$
$T(n) = 4\ T(n/2) + cn$

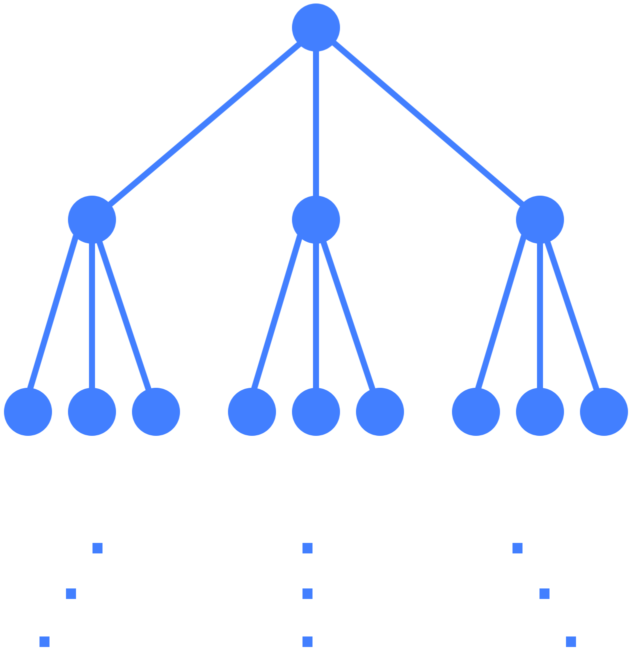


Level	Num	Size	Work
0	$1 = 4^0$	n	cn
1	$4 = 4^1$	n/2	4cn/2
2	$16 = 4^2$	n/4	16cn/4
…	…	…	…
i	4^i	$n/2^i$	$4^i\ c\ n/2^i$
…	…	…	…
k-1	4^{k-1}	$n/2^{k-1}$	$4^{k-1}\ c\ n/2^{k-1}$
k	4^k	$n/2^k = 1$	$4^k\ T(1)$

$n = 2^k$; $k = \log_2 n$

Total Work: $T(n) = \sum_{i=0}^{k} 4^i cn / 2^i = O(n^2)$

Solve: $T(1) = c$
$T(n) = 3\,T(n/2) + cn$



Level	Num	Size	Work
0	$1 = 3^0$	n	cn
1	$3 = 3^1$	$n/2$	$3cn/2$
2	$9 = 3^2$	$n/4$	$9cn/4$
…	…	…	…
i	3^i	$n/2^i$	$3^i\ c\ n/2^i$
…	…	…	…
$k-1$	3^{k-1}	$n/2^{k-1}$	$3^{k-1}\ c\ n/2^{k-1}$
k	3^k	$n/2^k = 1$	$3^k\ T(1)$

$n = 2^k$; $k = \log_2 n$

Total Work: $T(n) = \sum_{i=0}^{k} 3^i cn / 2^i$

Solve: T(1) = c
T(n) = 3 T(n/2) + cn (cont.)

$$T(n) = \sum_{i=0}^{k} 3^i cn/2^i$$

$$= cn \sum_{i=0}^{k} 3^i/2^i$$

$$= cn \sum_{i=0}^{k} \left(\tfrac{3}{2}\right)^i$$

$$= cn \frac{\left(\tfrac{3}{2}\right)^{k+1} - 1}{\left(\tfrac{3}{2}\right) - 1}$$

$$\sum_{i=0}^{k} x^i = \frac{x^{k+1} - 1}{x - 1}$$

$$(x \neq 1)$$

Solve: T(1) = c
T(n) = 3 T(n/2) + cn (cont.)

$$= 2cn\left(\left(\tfrac{3}{2}\right)^{k+1} - 1\right)$$

$$< 2cn\left(\tfrac{3}{2}\right)^{k+1}$$

$$= 3cn\left(\tfrac{3}{2}\right)^{k}$$

$$= 3cn\,\frac{3^k}{2^k}$$

Solve: T(1) = c
T(n) = 3 T(n/2) + cn (cont.)

$$= 3cn \frac{3^{\log_2 n}}{2^{\log_2 n}}$$

$$= 3cn \frac{3^{\log_2 n}}{n}$$

$$= 3c\,3^{\log_2 n}$$

$$= 3c\left(n^{\log_2 3}\right)$$

$$\boxed{= O\left(n^{1.59\ldots}\right)}$$

$$a^{\log_b n}$$

$$= \left(b^{\log_b a}\right)^{\log_b n}$$

$$= \left(b^{\log_b n}\right)^{\log_b a}$$

$$= n^{\log_b a}$$

Divide and Conquer Master Recurrence

If $T(n) = aT(n/b)+cn^k$ for $n > b$ then

if $a > b^k$ then $T(n)$ is $\Theta(n^{\log_b a})$

[many subproblems => leaves dominate]

if $a < b^k$ then $T(n)$ is $\Theta(n^k)$

[few subproblems => top level dominates]

if $a = b^k$ then $T(n)$ is $\Theta(n^k \log n)$

[balanced => all log n levels contribute]

True even if it is $\lceil n/b \rceil$ instead of n/b.

D & C Summary

Idea:

- "Two halves are better than a whole"
  - if the base algorithm has super-linear complexity.
- "If a little's good, then more's better"
  - repeat above, recursively

Analysis: recursion tree or Master Recurrence

Applications: Many.

- Binary Search, Merge Sort, (Quicksort), Closest points, Integer multiply,…