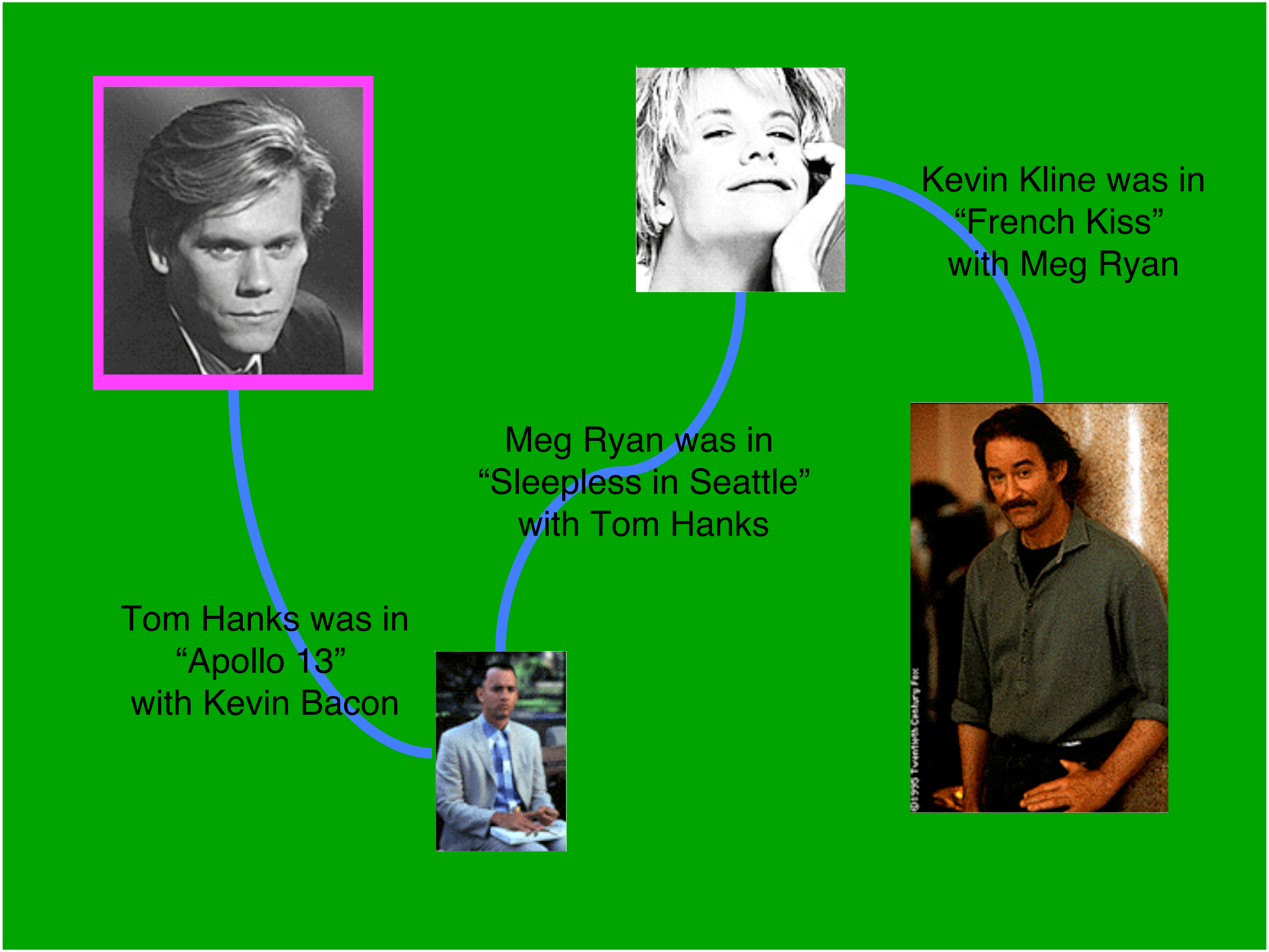
CSE 417: Algorithms and Computational Complexity

Winter 2006

Graphs and Graph Algorithms

Larry Ruzzo


Kevin Kline was in
“French Kiss”
with Meg Ryan
Meg Ryan was in
“Sleepless in Seattle”
with Tom Hanks
Tom Hanks was in
“Apollo 13”
with Kevin Bacon

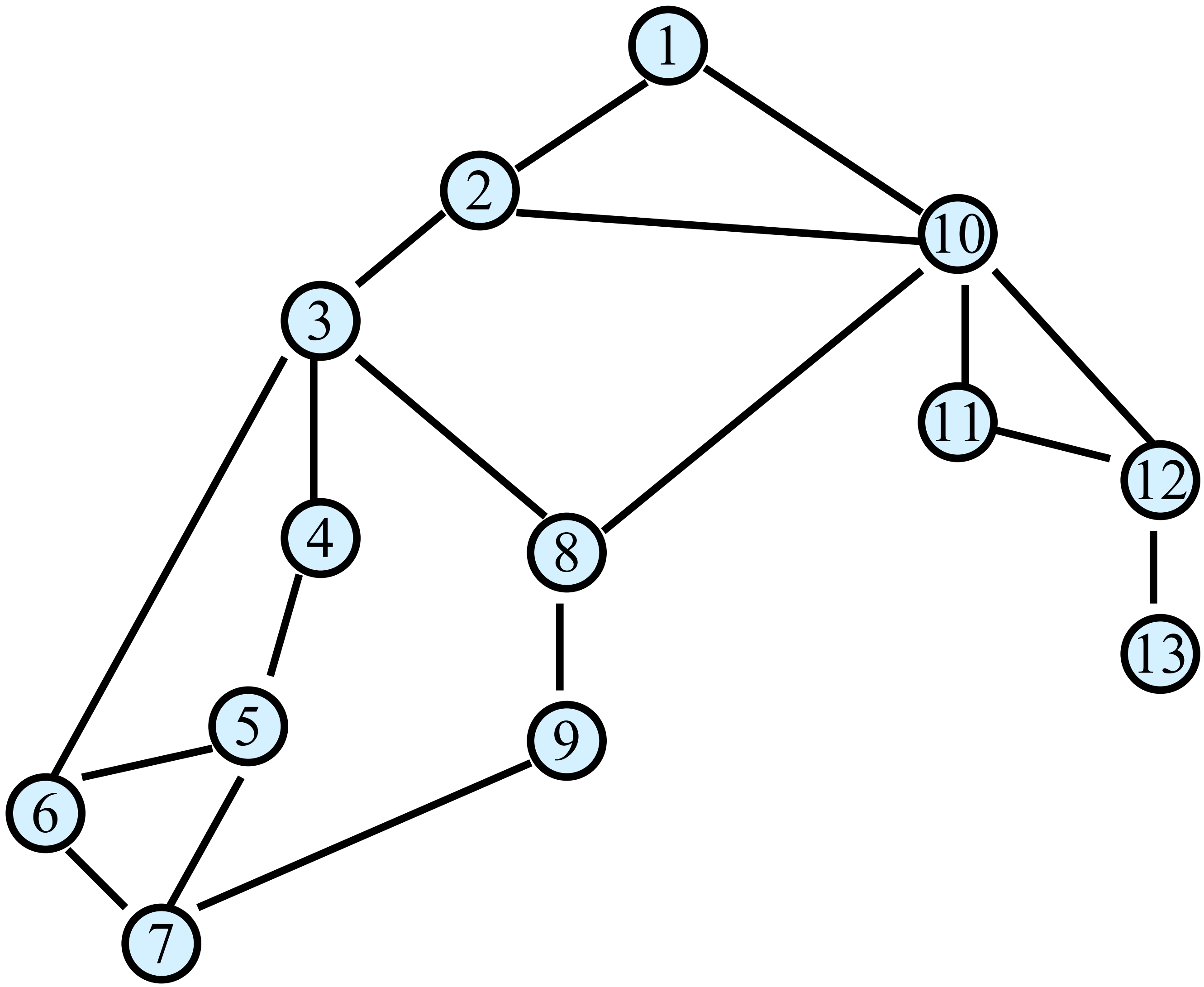
Objects & Relationships

- The Kevin Bacon Game:
  - Actors
  - Two are related if they've been in a movie together
- Exam Scheduling:
  - Classes
  - Two are related if they have students in common
- Traveling Salesperson Problem:
  - Cities
  - Two are related if can travel *directly* between them

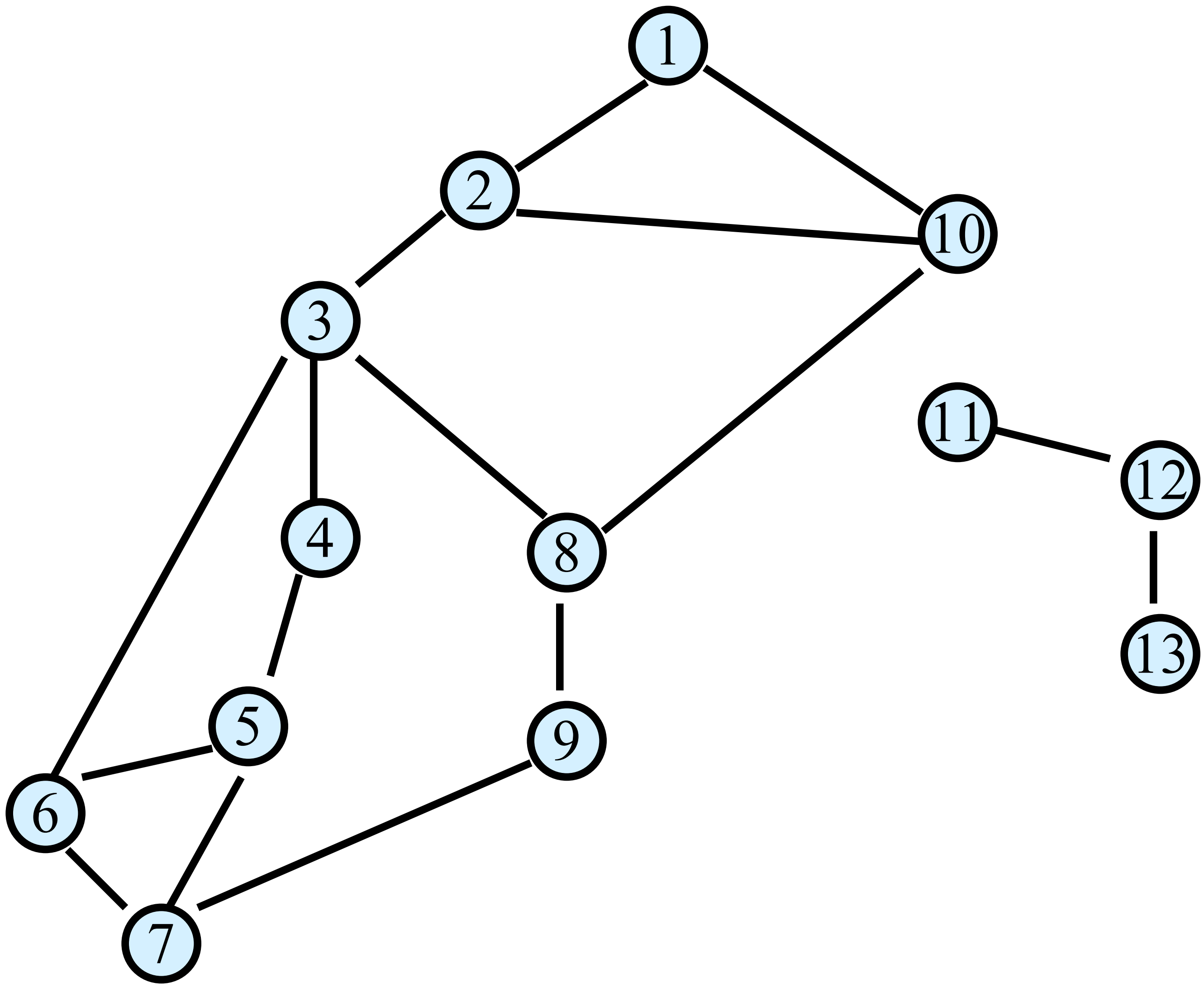
Graphs

- An extremely important formalism for representing (binary) relationships
- Objects: “vertices”, aka “nodes”
- Relationships between pairs: “edges”, aka “arcs”
- Formally, a graph G = (V, E) is a pair of sets, V the vertices and E the edges

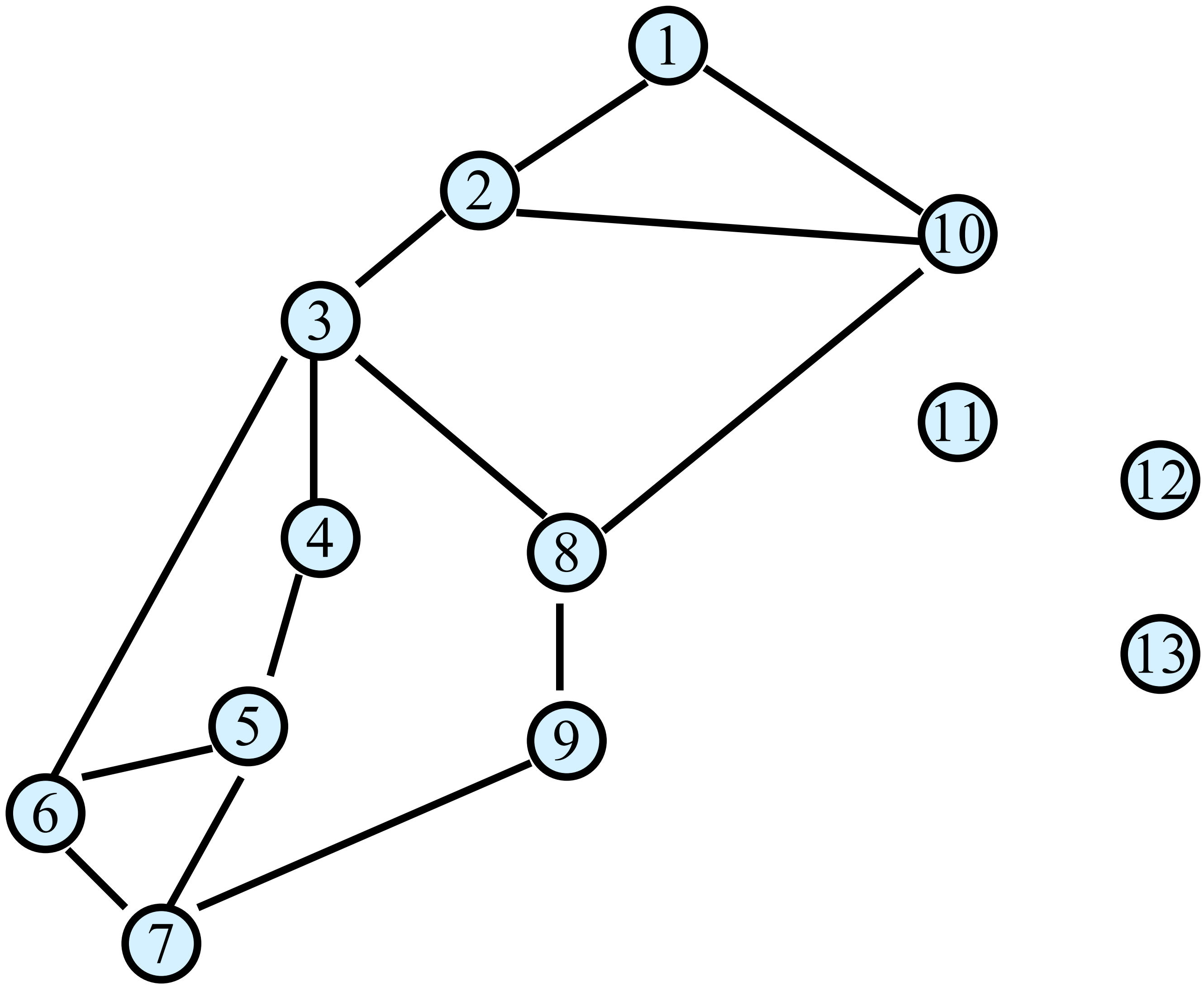
Undirected Graph G = (V,E)



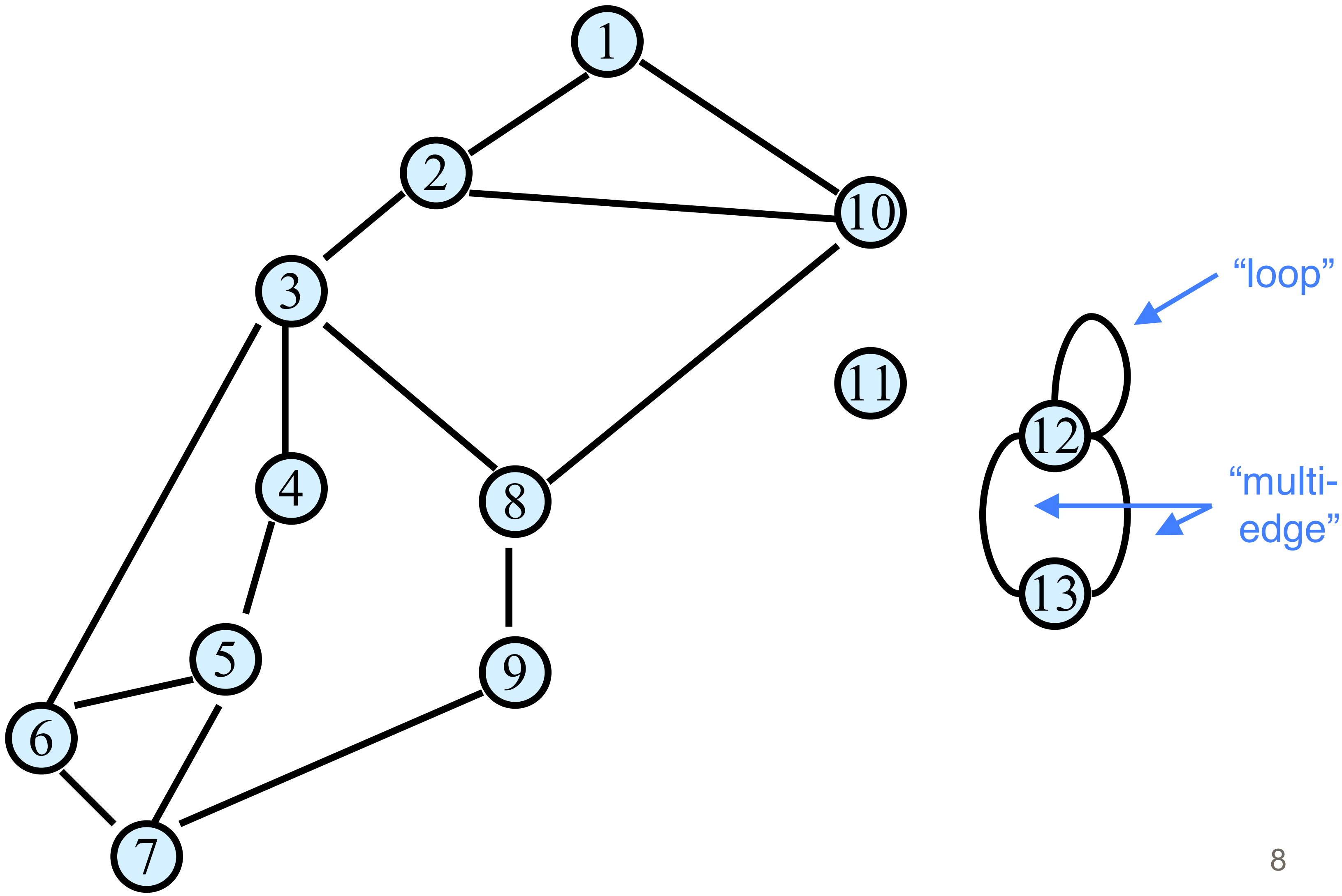
Undirected Graph G = (V,E)



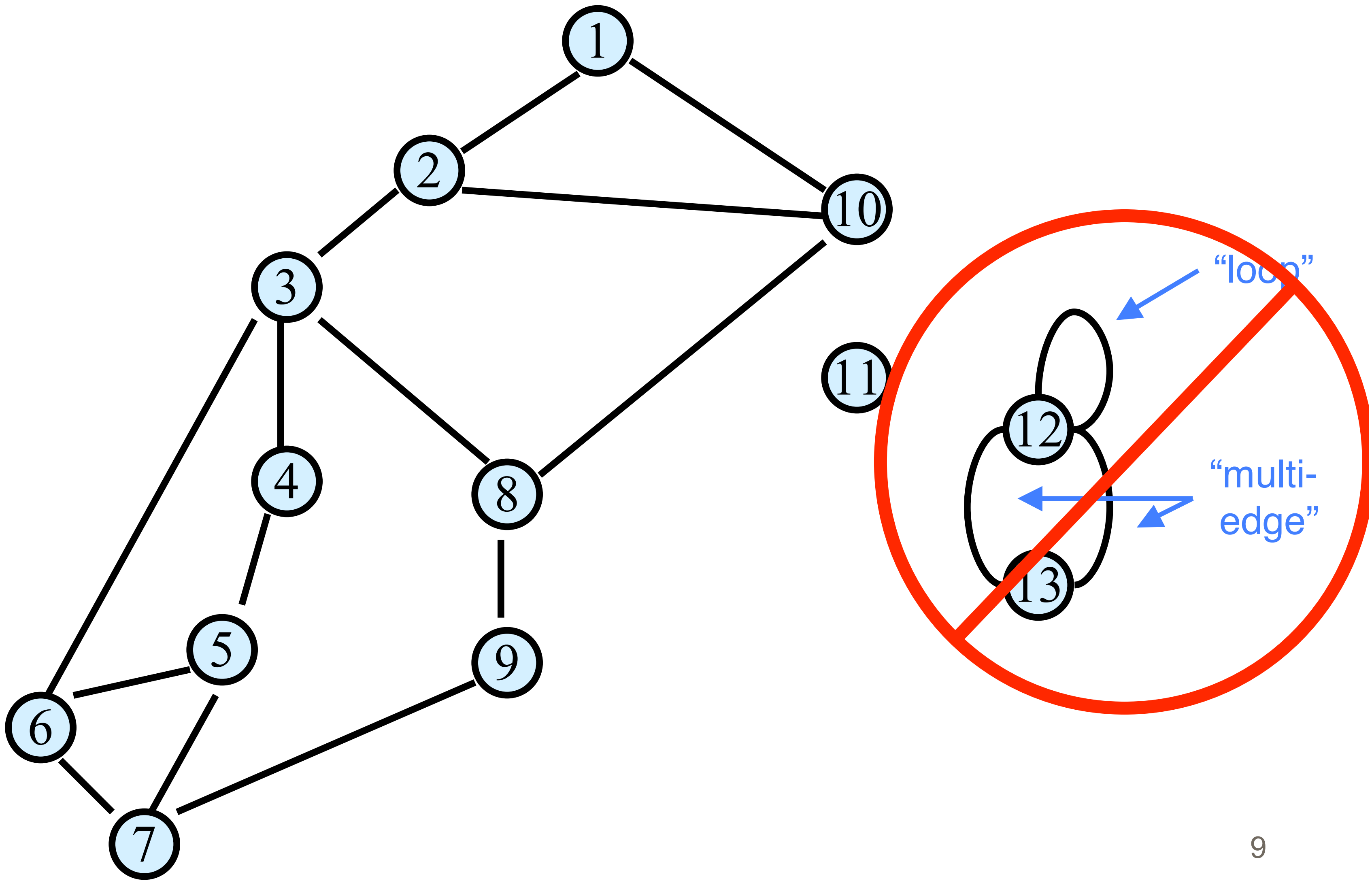
Undirected Graph   G = (V,E)



Undirected Graph G = (V,E)

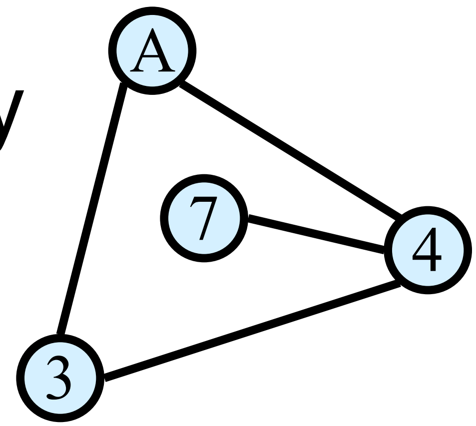


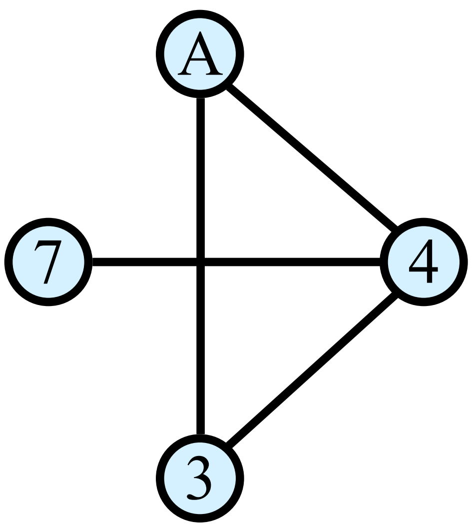
Undirected Graph G = (V,E)

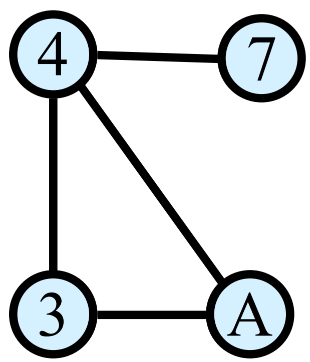


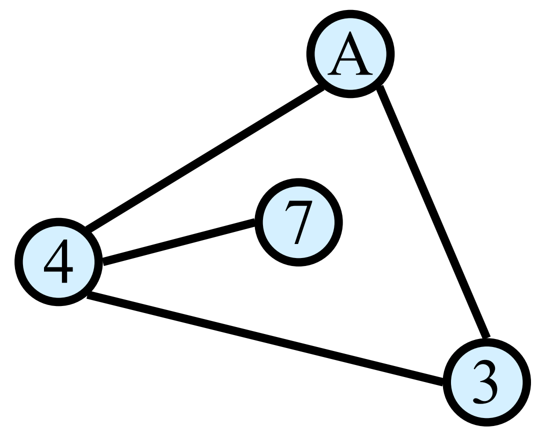
Graphs don’t live in Flatland

- Geometrical drawing is mentally convenient, but mathematically irrelevant: 4 drawings, 1 graph.

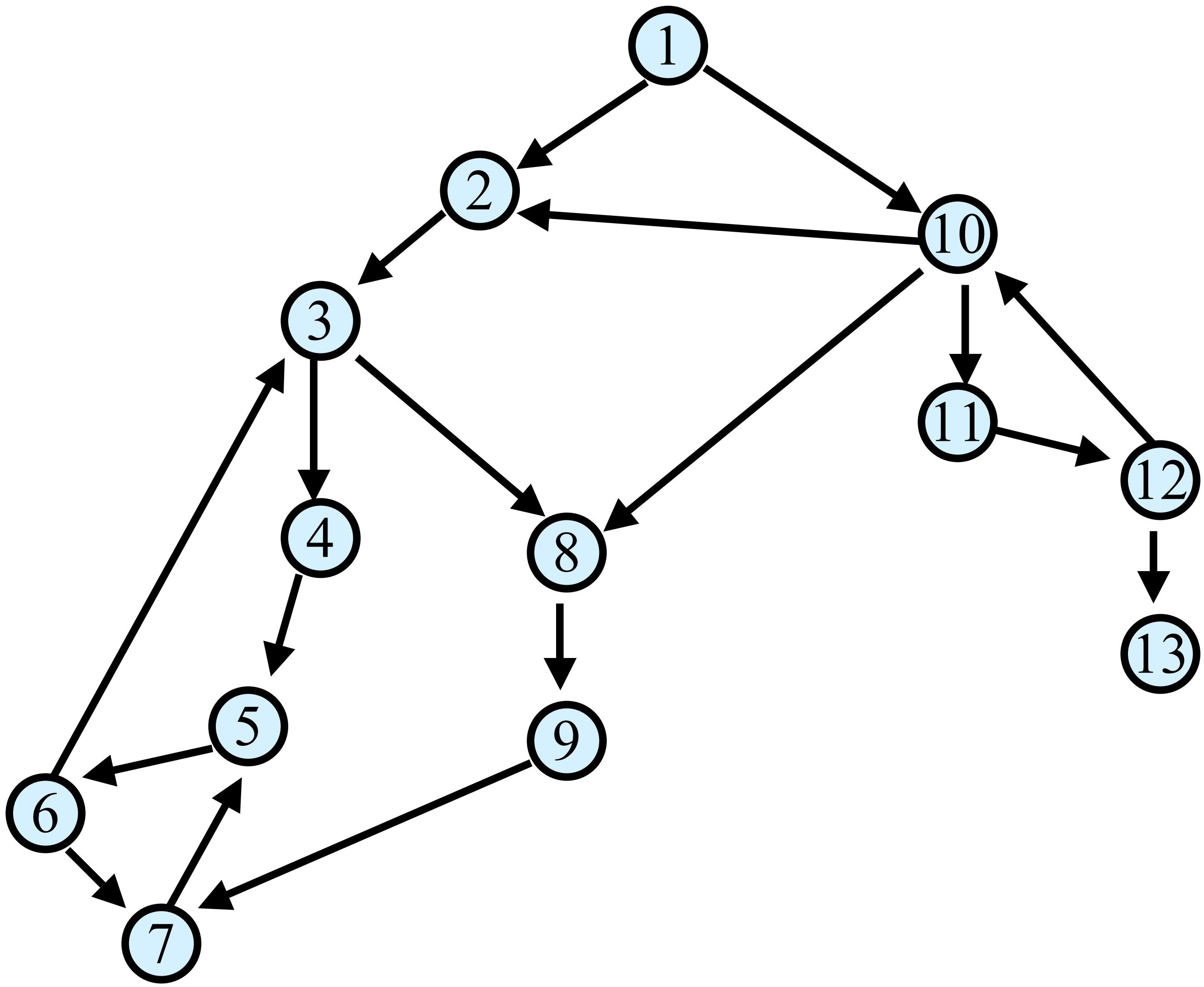




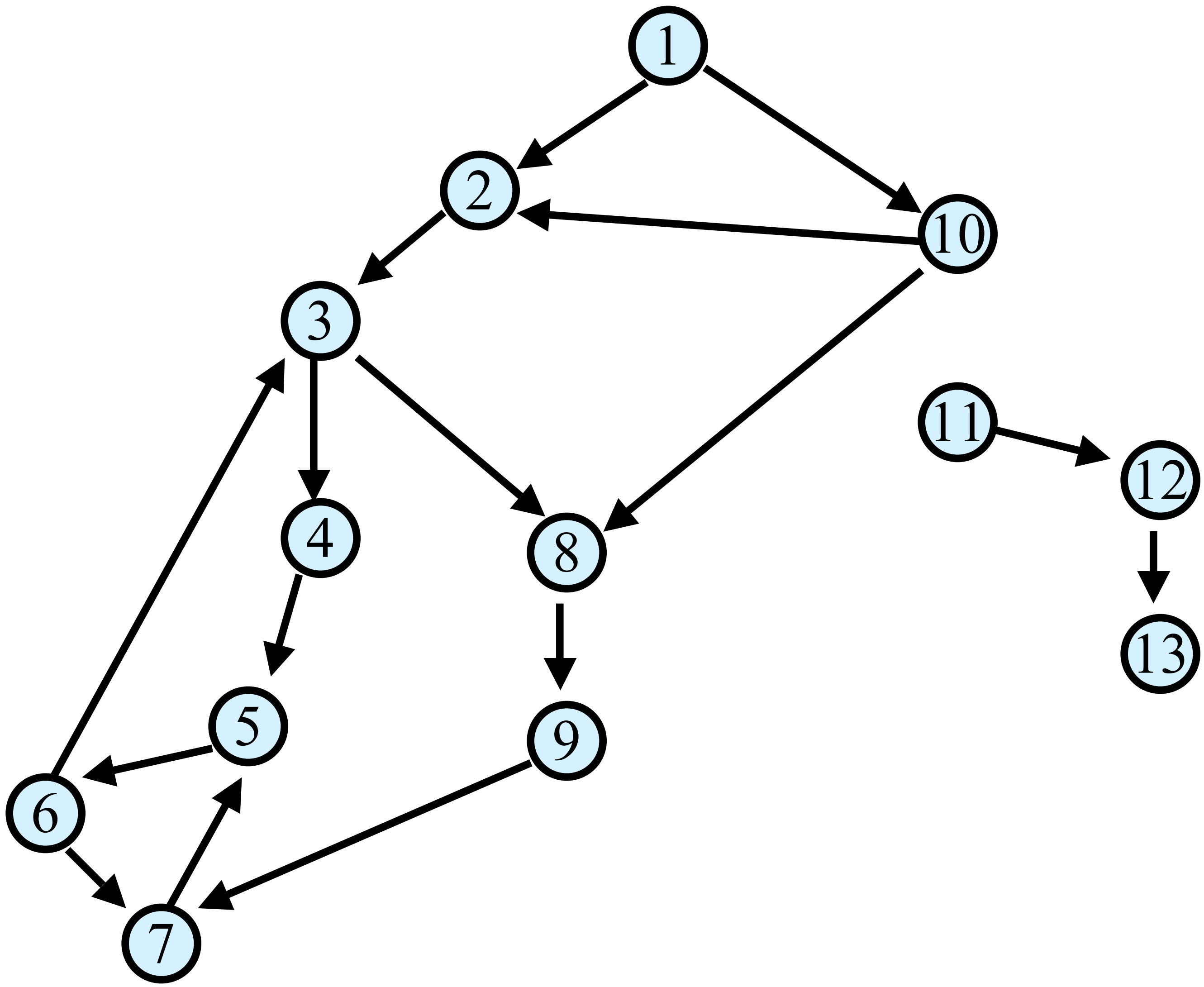




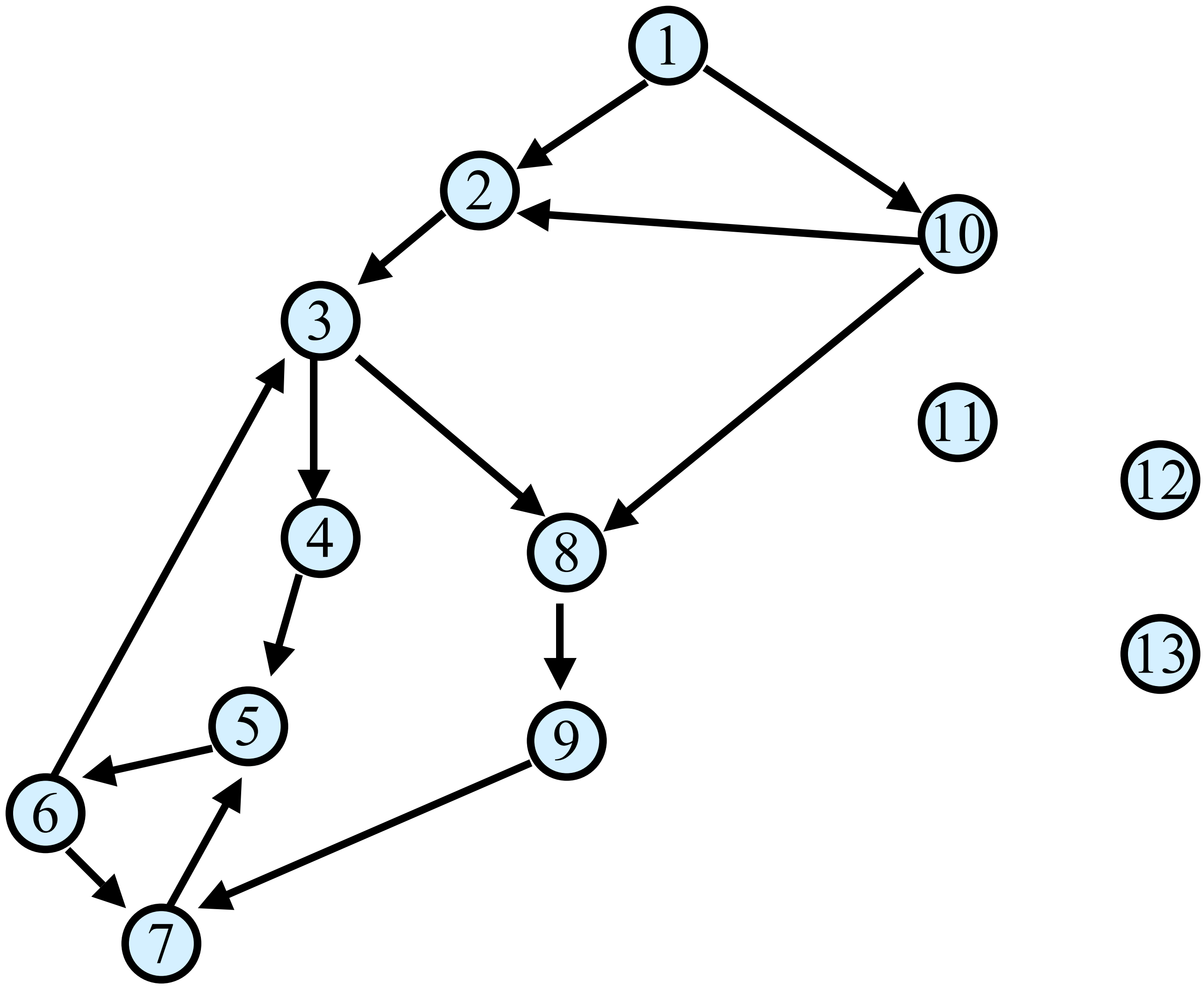
Directed Graph G = (V,E)



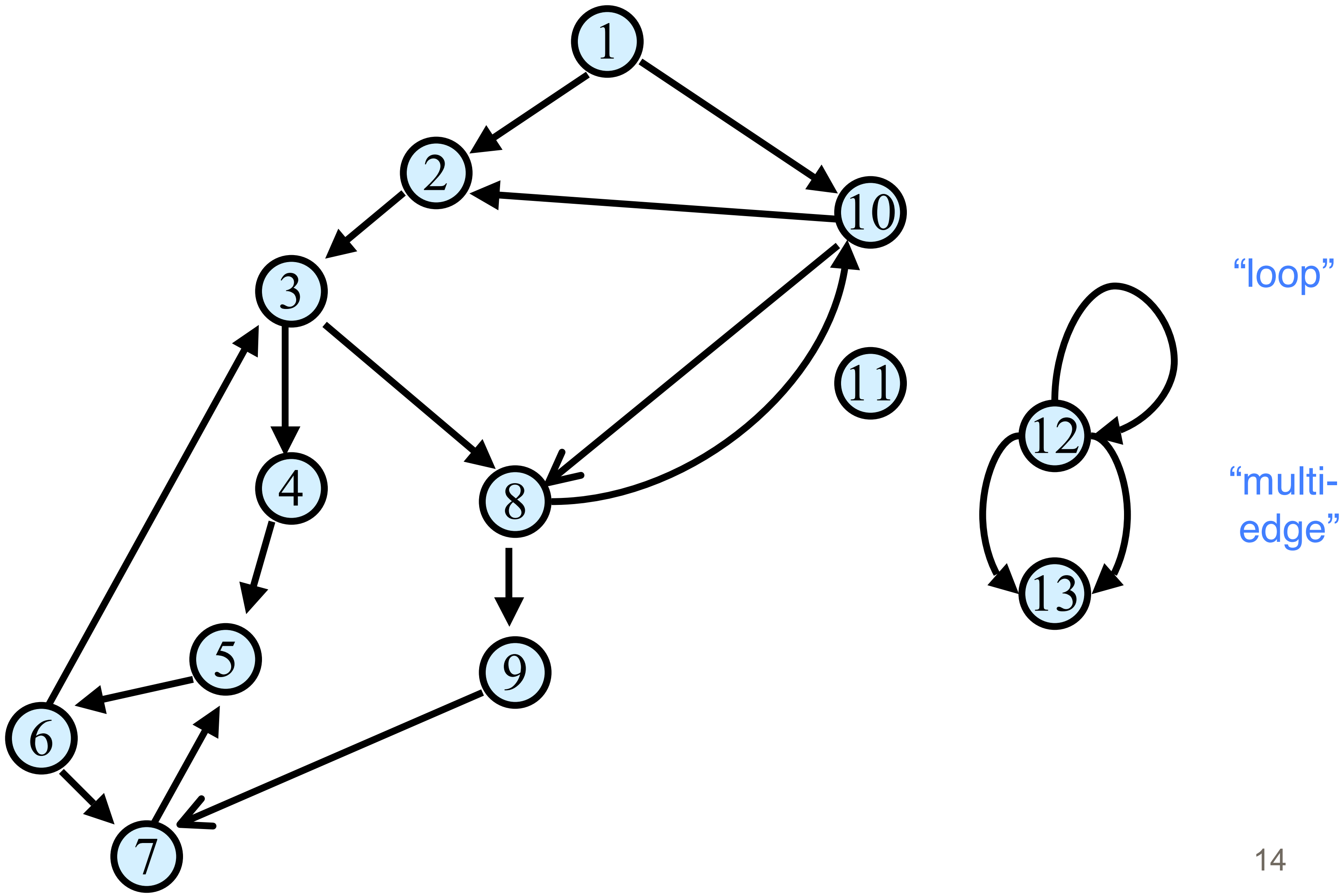
Directed Graph G = (V,E)



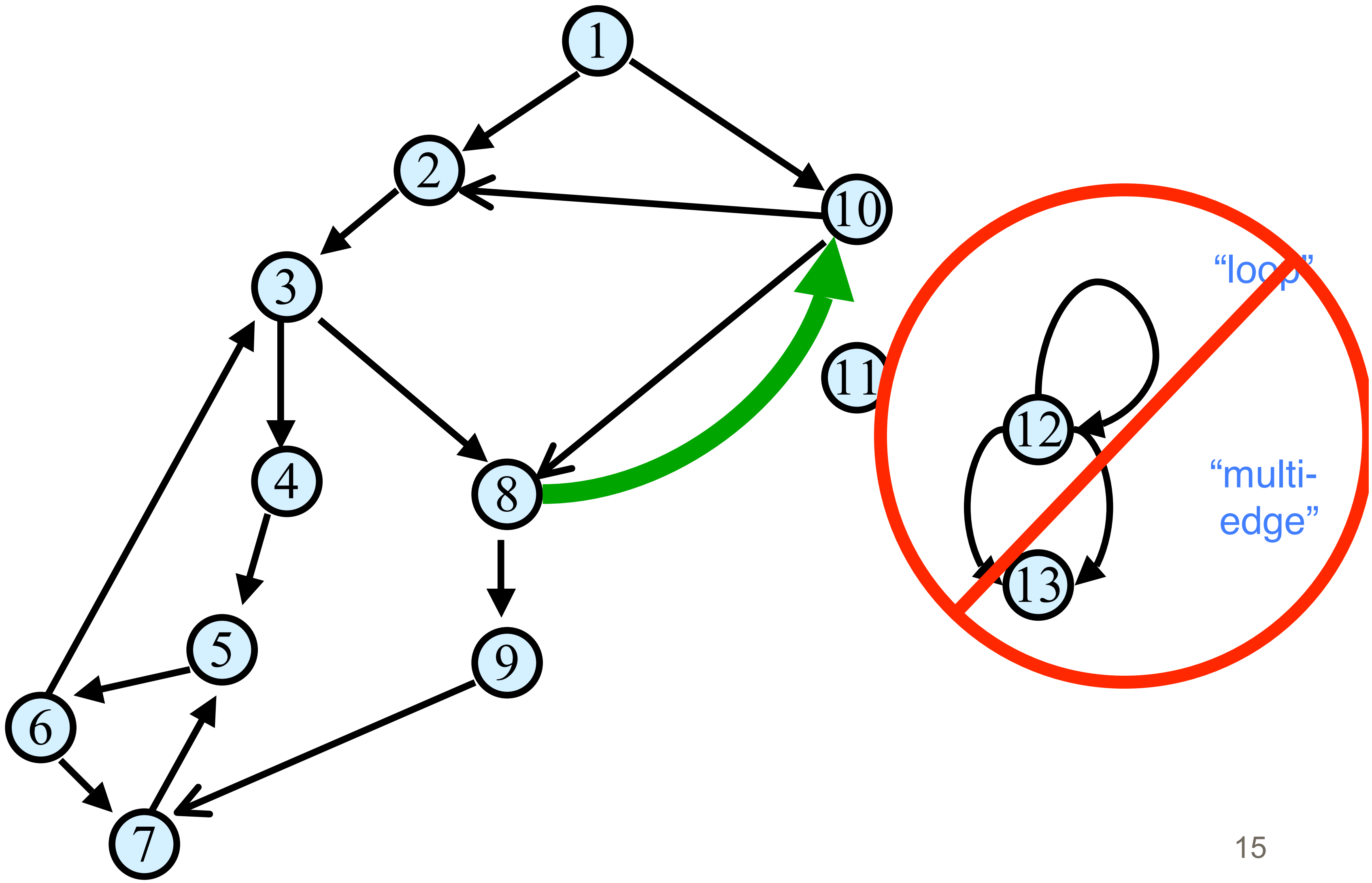
Directed Graph G = (V,E)



Directed Graph G = (V,E)

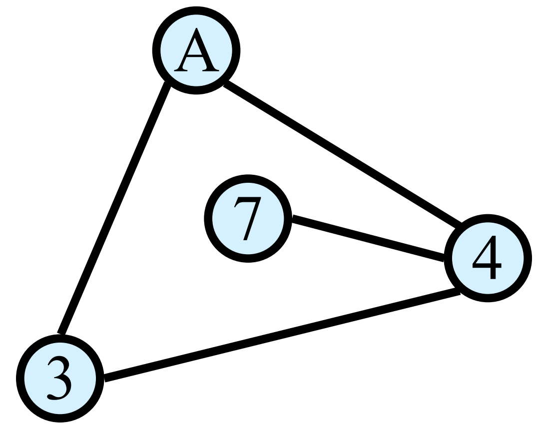


Directed Graph G = (V,E)



Specifying undirected graphs as input

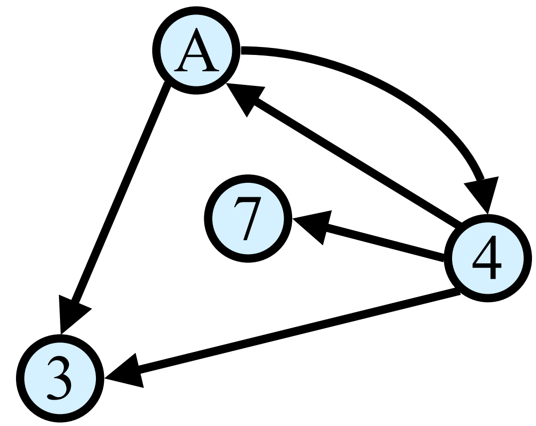
- What are the vertices?
  - Explicitly list them: {“A”, “7”, “3”, “4”}
- What are the edges?
  - Either, set of edges {{A,3}, {7,4}, {4,3}, {4,A}}
  - Or, (symmetric) adjacency matrix:



	A	7	3	4
A	0	0	1	1
7	0	0	0	1
3	1	0	0	1
4	1	1	1	0

Specifying directed graphs as input

- What are the vertices
  - Explicitly list them: {“A”, “7”, “3”, “4”}
- What are the edges
  - Either, set of directed edges: {(A,4), (4,7), (4,3), (4,A), (A,3)}
  - Or, (nonsymmetric) adjacency matrix:



	A	7	3	4
A	0	0	1	1
7	0	0	0	0
3	0	0	0	0
4	1	1	1	0

Vertices vs # Edges

- Let G be an undirected graph with n vertices and m edges
- How are n and m related?
- Since
  - every edge connects two *different* vertices (no loops), and
  - no two edges connect the *same* two vertices (no multi-edges),

  it must be true that: $\boxed{0 \le m \le n(n-1)/2 = O(n^2)}$

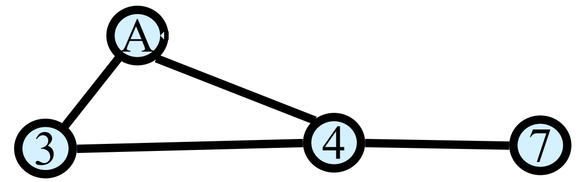
More Cool Graph Lingo

- A graph is called *sparse* if $m \ll n^2$, otherwise it is *dense*
  - Boundary is somewhat fuzzy; $O(n)$ edges is certainly sparse, $\Omega(n^2)$ edges is dense.
- Sparse graphs are common in practice
  - E.g., all planar graphs are sparse
- Q: which is a better run time, $O(n+m)$ or $O(n^2)$?

A: $O(n+m) = O(n^2)$, but $n+m$ usually way better!

Representing Graph $G = (V,E)$

internally, indp of input format



- Vertex set $V = \{v_1, \ldots, v_n\}$
- Adjacency Matrix A
  - $A[i,j] = 1$ iff $(v_i, v_j) \in E$
  - Space is n^2 bits

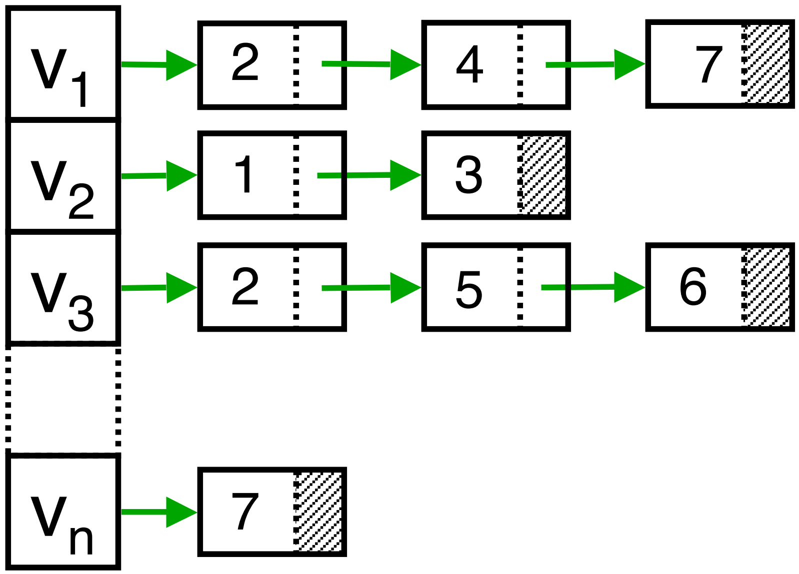
	A	7	3	4
A	0	0	1	1
7	0	0	0	1
3	1	0	0	1
4	1	1	1	0

- Advantages:
  - $O(1)$ test for presence or absence of edges.
- Disadvantages: inefficient for sparse graphs, both in storage and access

$m << n^2$

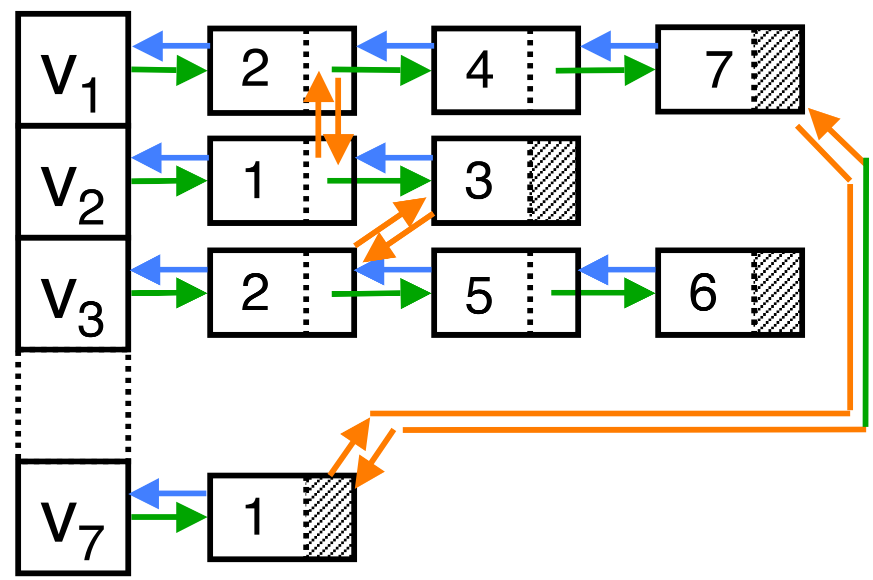
Representing Graph G=(V,E)
n vertices, m edges

- Adjacency List:
  - O(n+m) words
- Advantages:
  - Compact for sparse graphs
  - Easily see all edges
- Disadvantages
  - More complex data structure
  - no O(1) edge test



Representing Graph G=(V,E)
n vertices, m edges

- Adjacency List:
  - O(n+m) words



- Back- and cross pointers more work to build, but allow easier traversal and deletion of edges, *if needed,* (don't bother if not)

Graph Traversal

- Learn the basic structure of a graph
- "Walk," via edges, from a fixed starting vertex v to all vertices reachable from v

- Three states of vertices
  - **undiscovered**
  - **discovered**
  - **fully-explored**

Breadth-First Search

- Completely explore the vertices in order of their distance from v

- Naturally implemented using a queue

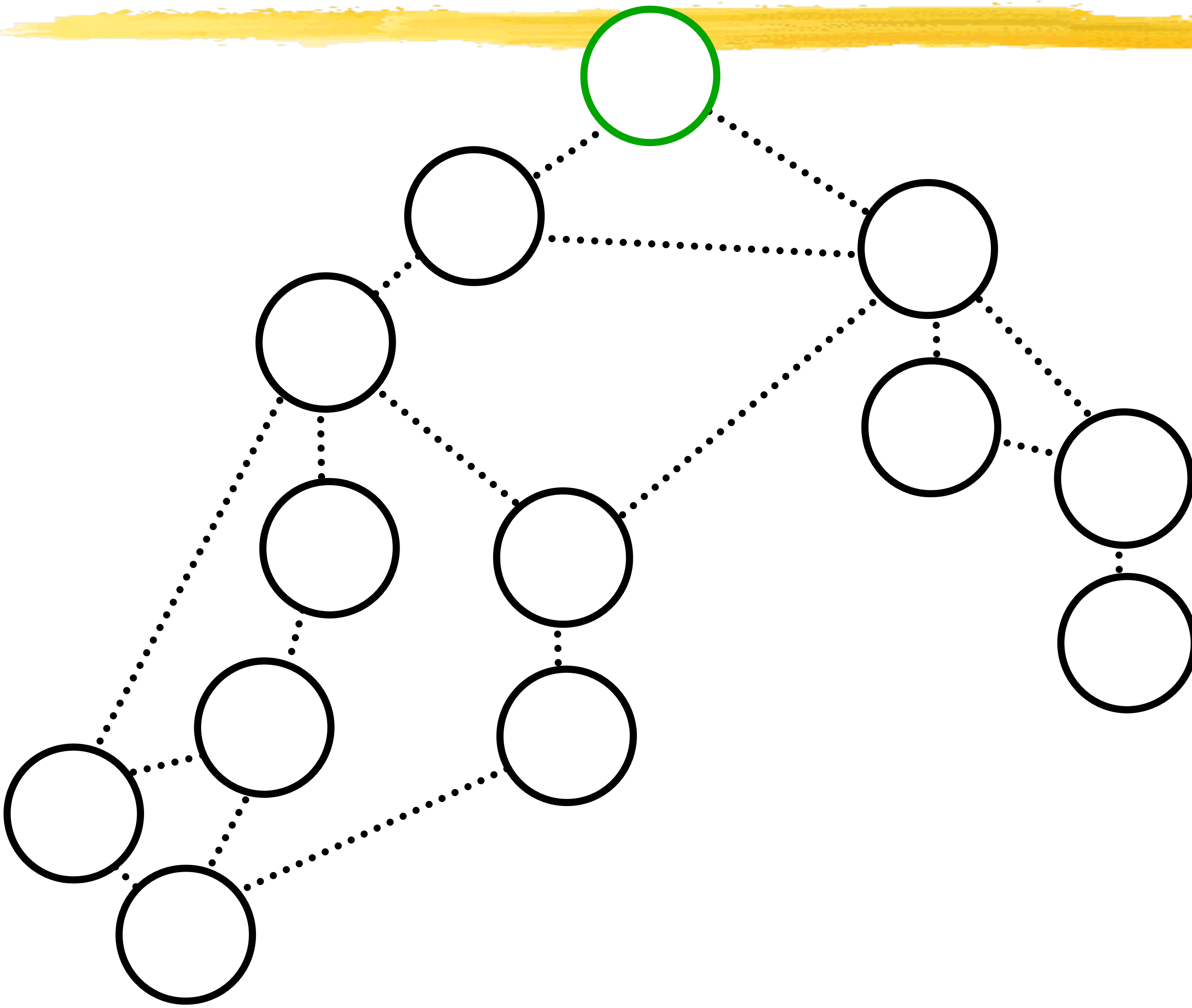
BFS(v)

Global initialization: mark all vertices "undiscovered"

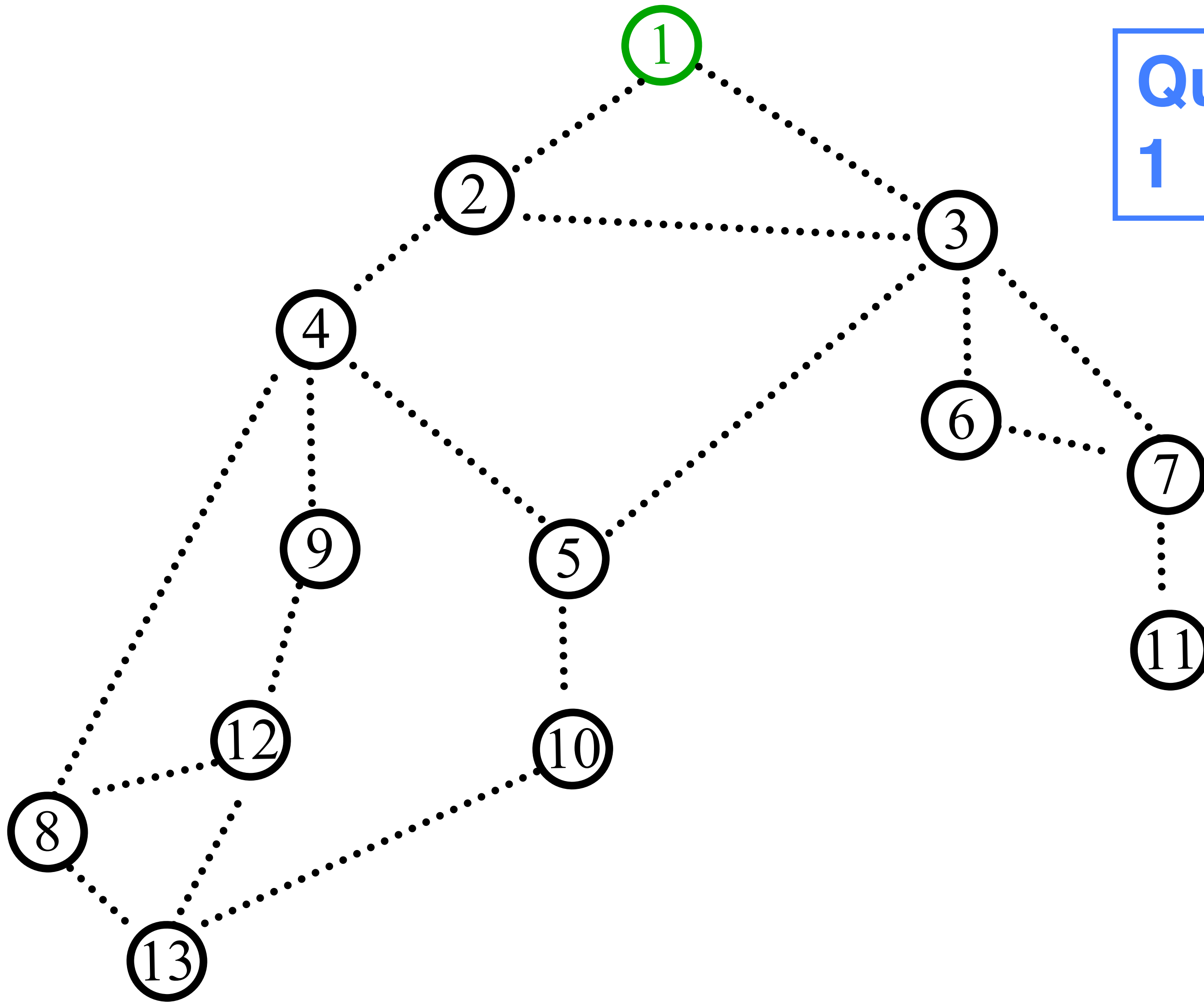
```
BFS(v)
    mark  v "discovered"
    queue = v
    while queue not empty
        u = remove_first(queue)
        for each edge {u,x}
            if (x is undiscovered)
                mark x discovered
                append x on queue
        mark u completed
```

Exercise: modify code to number vertices & compute level numbers

BFS(v)

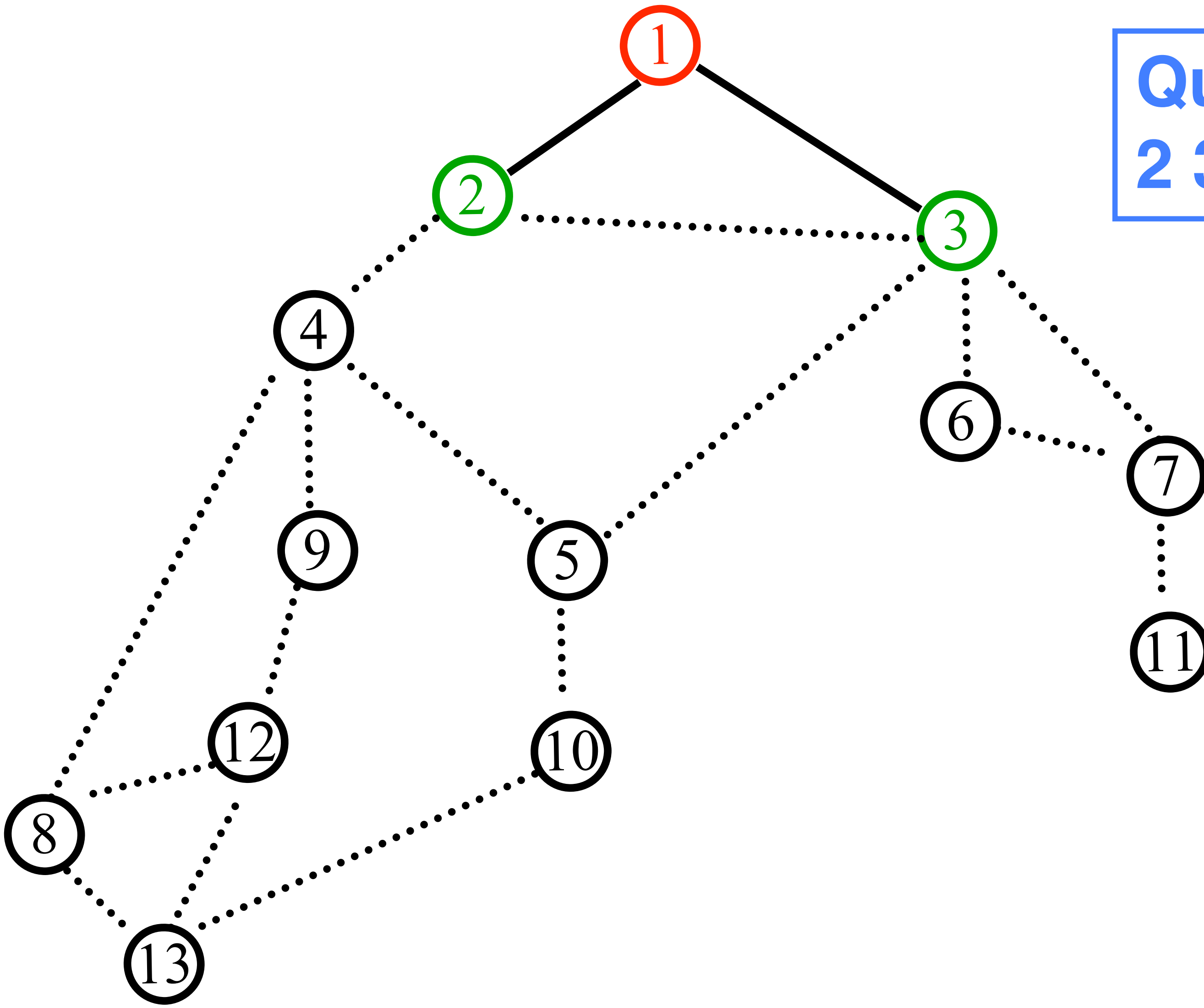


BFS(v)



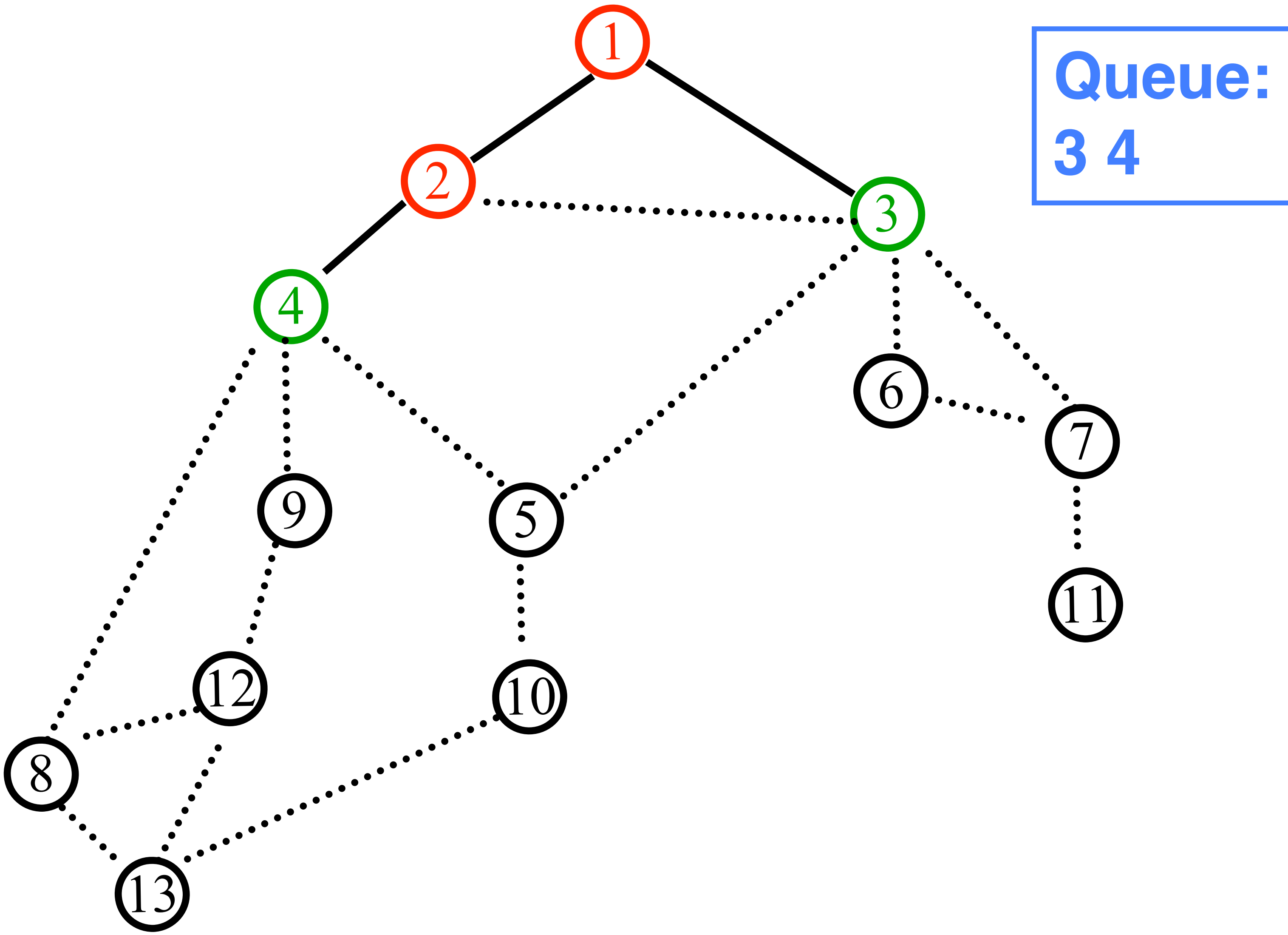
Queue:
1

BFS(v)

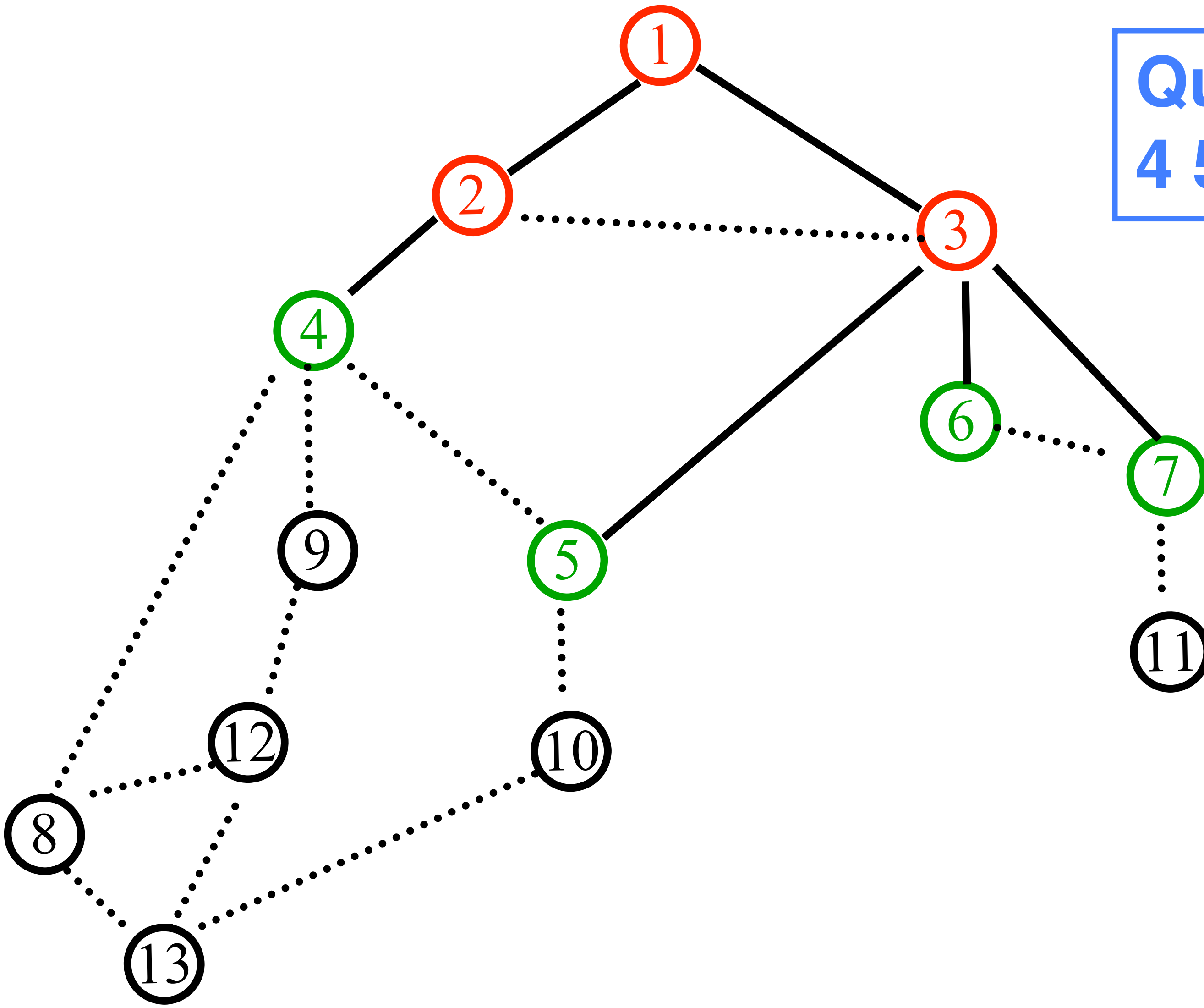


Queue:
2 3

BFS(v)

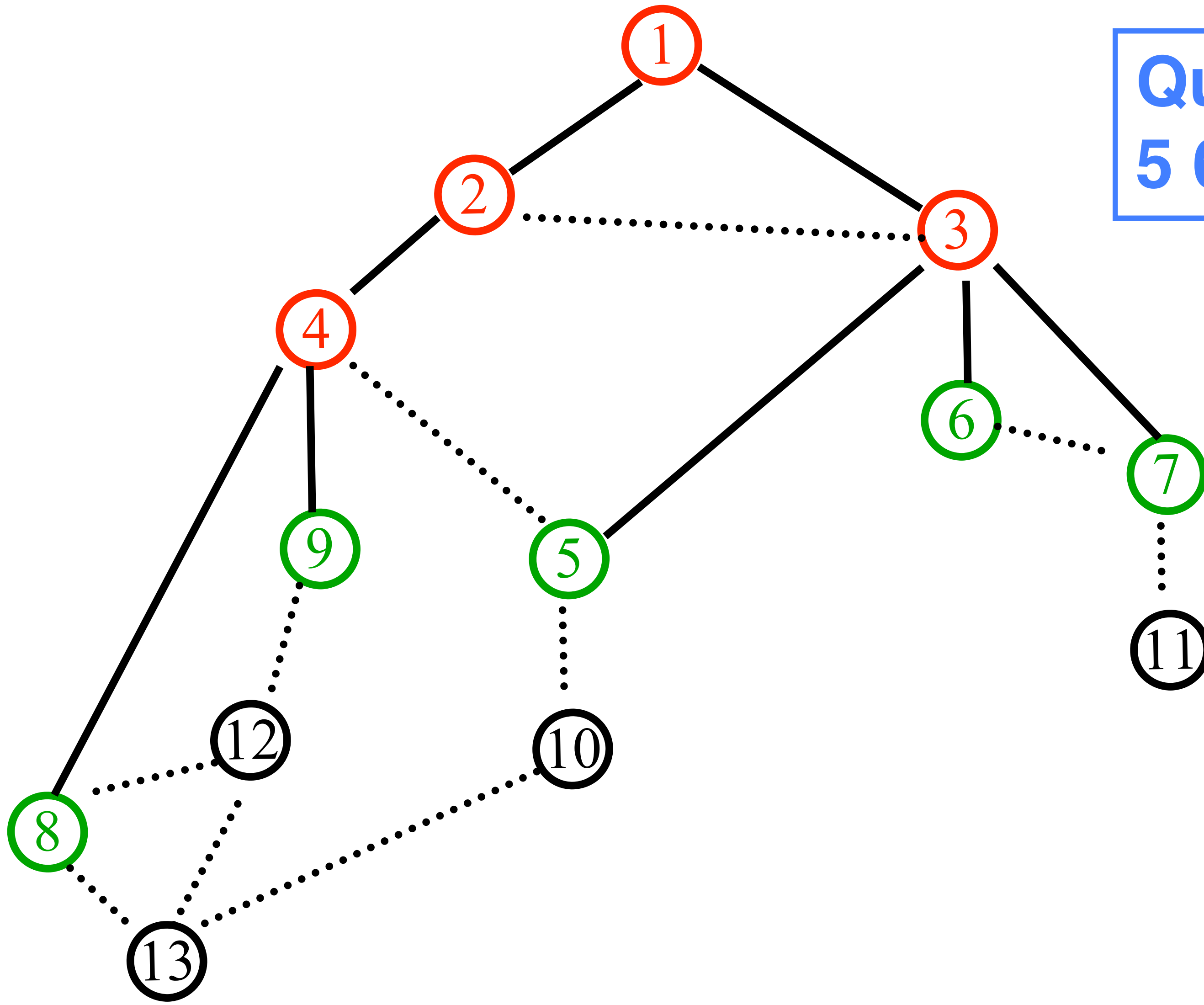


BFS(v)



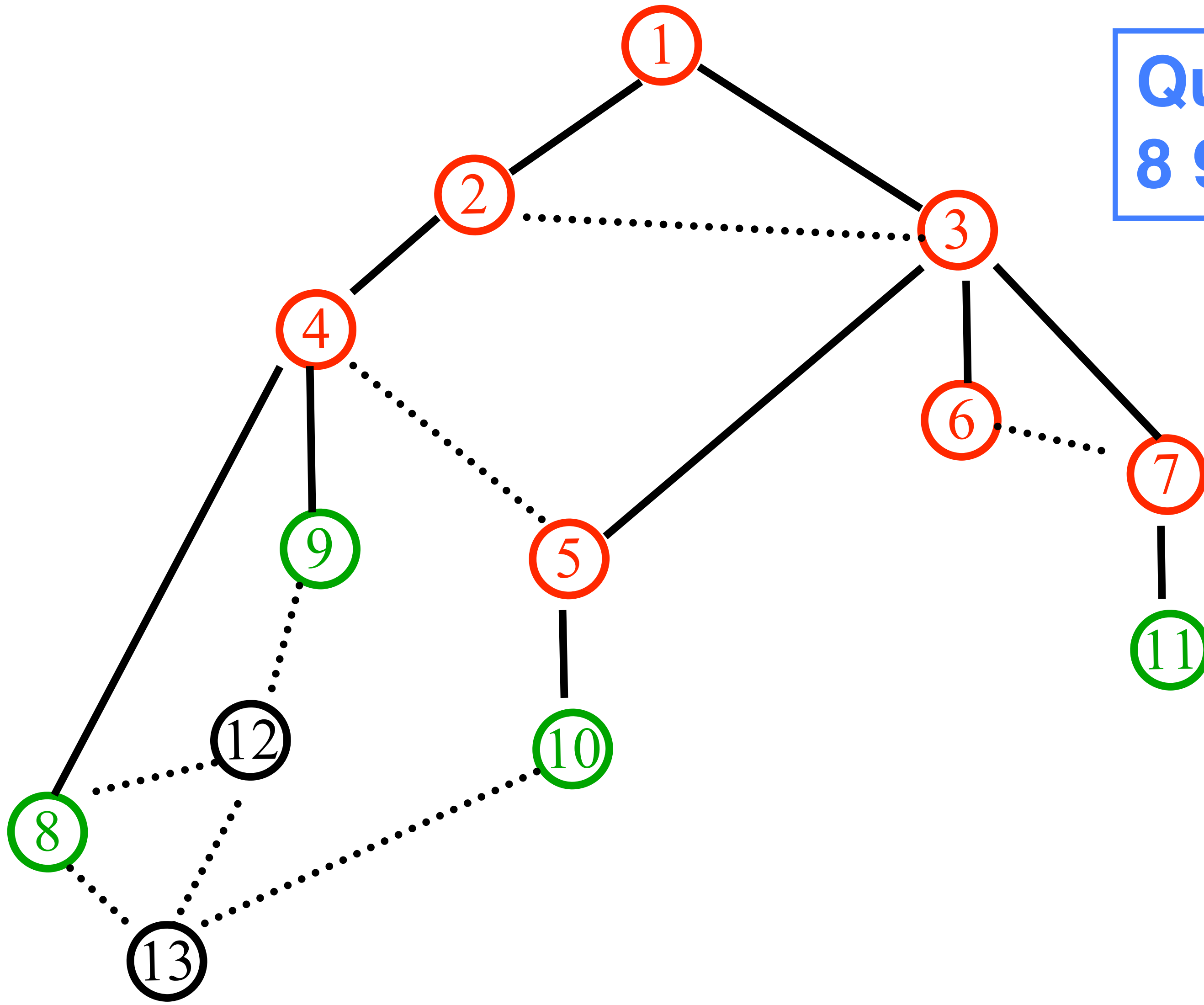
Queue:
4 5 6 7

BFS(v)



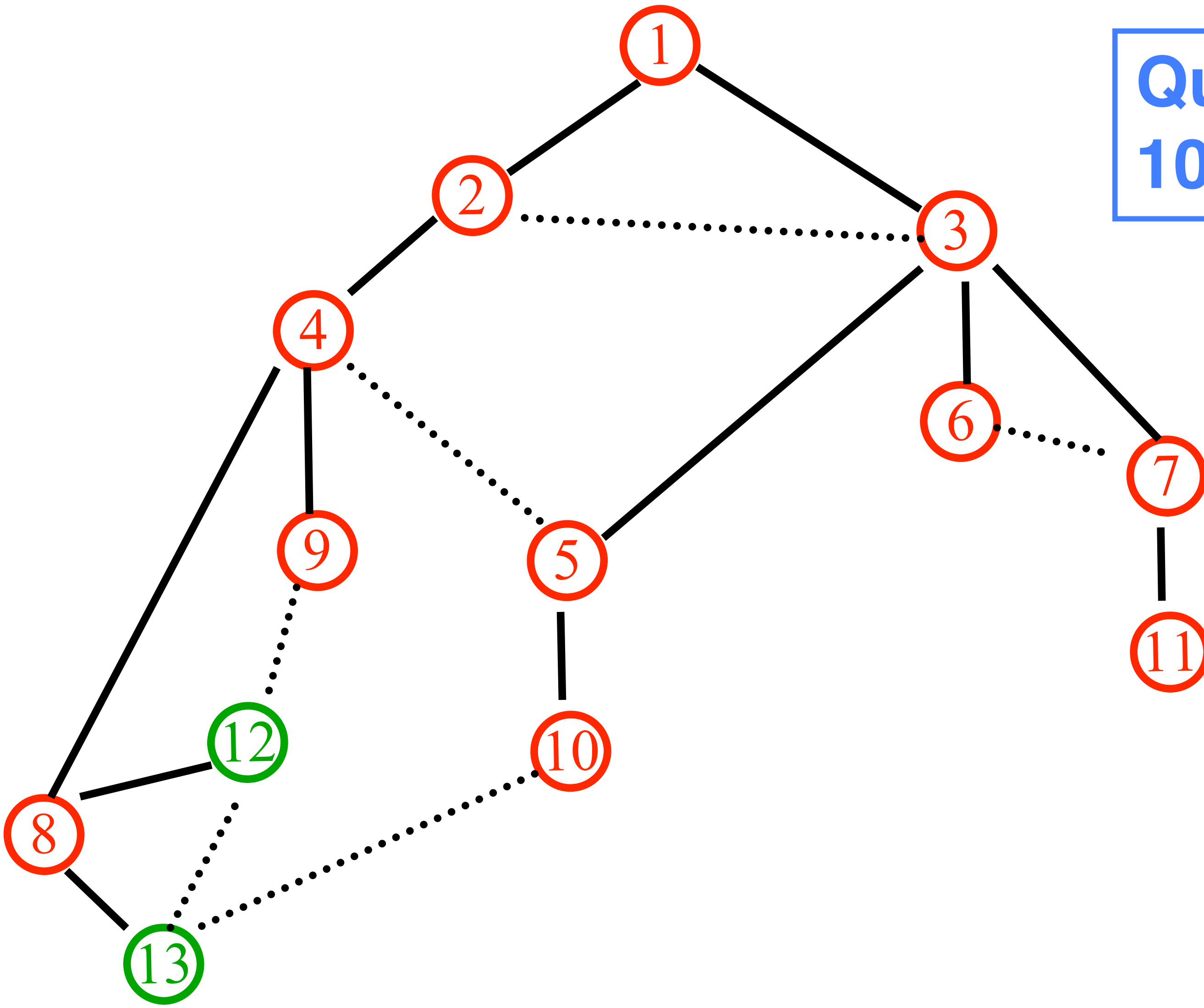
Queue:
5 6 7 8 9

BFS(v)



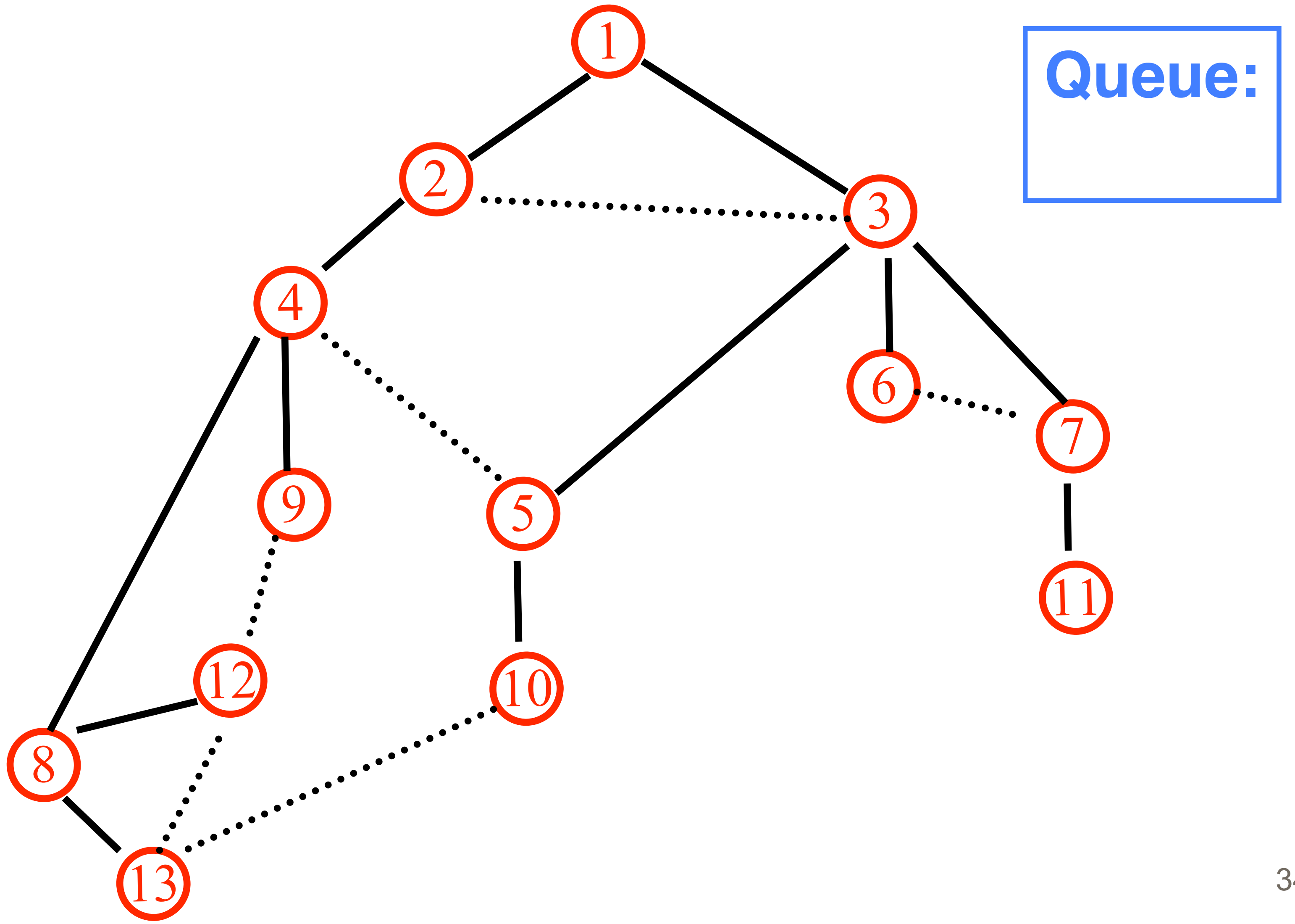
Queue:
8 9 10 11

BFS(v)



Queue:
10 11 12 13

BFS(v)



BFS analysis

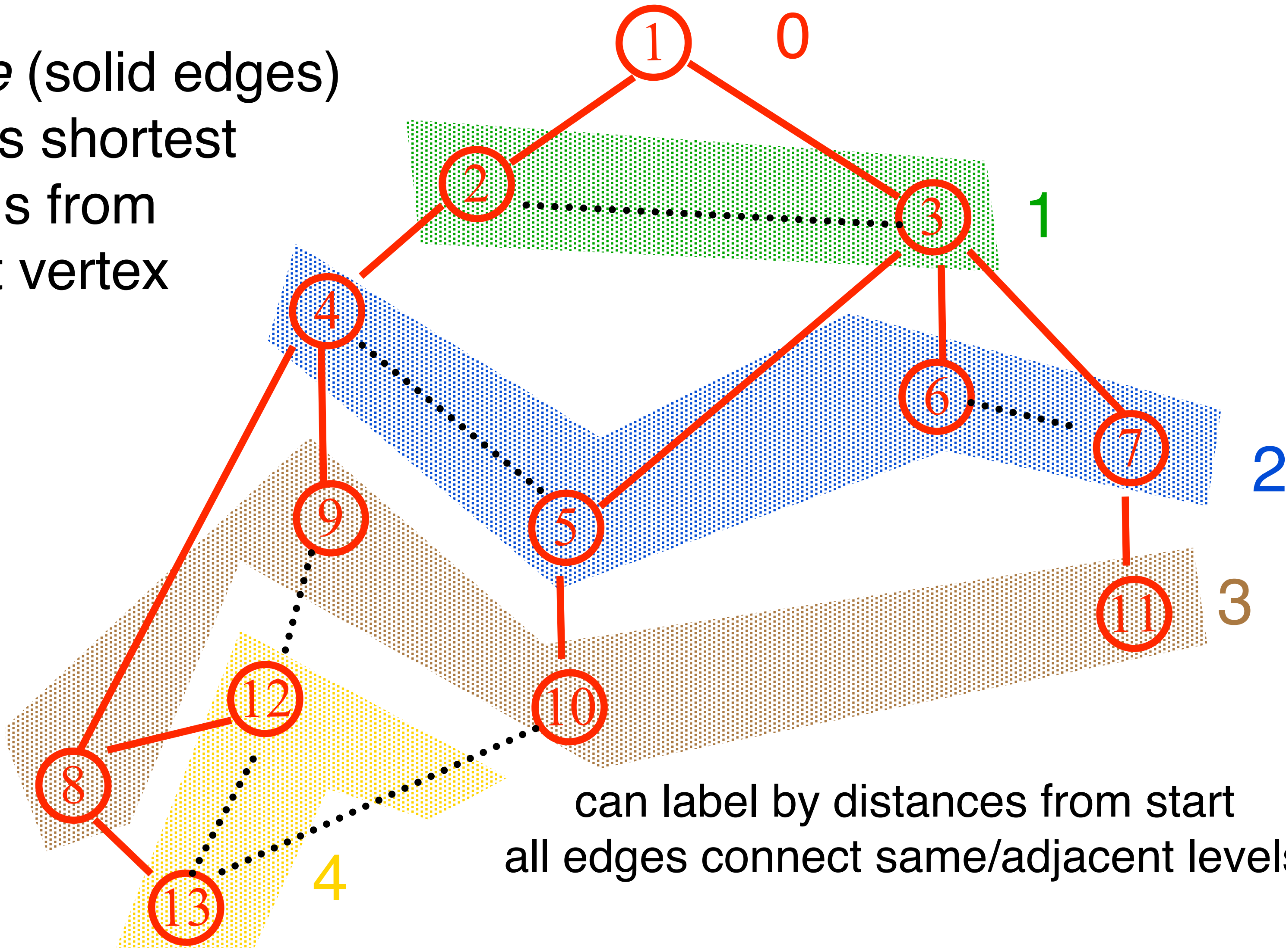
- Each edge is explored once from each end-point (at most)
- Each vertex is discovered by following a different edge
- Total cost O(m) where m=# of edges

Properties of (Undirected) BFS(v)

- BFS(v) visits x if and only if there is a path in G from v to x.
- Edges into then-undiscovered vertices define a ***tree*** – the "breadth first spanning tree" of G
- Level i in this tree are exactly those vertices u such that the shortest path (in G, not just the tree) from the root v is of length i.
- ***All*** non-tree edges join vertices on the same or adjacent levels

BFS Application: Shortest Paths

Tree (solid edges) gives shortest paths from start vertex



can label by distances from start
all edges connect same/adjacent levels

Why fuss about trees?

- Trees are simpler than graphs
- Ditto for algorithms on trees vs on graphs
- So, this is often a good way to approach a graph problem: find a “nice” tree in the graph, i.e., one such that non-tree edges have some simplifying structure
- E.g., BFS finds a tree s.t. level-jumps are minimized
- DFS (next) finds a different tree, but it also has interesting structure…

Graph Search Application: Connected Components

- Want to answer questions of the form:
  - given vertices u and v, is there a path from u to v?

Q: Why not create 2-d array Path[u,v]?

- Idea: create array A such that

  A[u] = smallest numbered vertex that is connected to u

- question reduces to whether A[u]=A[v]?

Graph Search Application: Connected Components

- initial state: all v undiscovered
  for v=1 **to** n **do**
    if state(v) != fully-explored **then**
      BFS(v): **setting** A[u] $\leftarrow$v **for each** u **found**
      (and marking u discovered/fully-explored)
    endif
  endfor

- Total cost: O(n+m)
  - each edge is touched a constant number of times
  - works also with DFS

Depth-First Search

- Follow the first path you find as far as you can go
- Back up to last unexplored edge when you reach a dead end, then go as far you can

- Naturally implemented using recursive calls or a stack