CSE 417: Algorithms and Computational Complexity

Dynamic Programming

Autumn 2002
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Reading assignment

- Read sections 3.1-3.2 of *The ALGORITHM Design Manual*

Some Algorithm Design Techniques, I

- General overall idea
  - Reduce solving a problem to a smaller problem or problems of the same type
- Greedy algorithms
  - Used when one needs to build something a piece at a time
  - Repeatedly make the **greedy** choice - the one that looks the best right away
    - e.g. closest pair in TSP search
  - Usually fast if they work (but often don't)

Some Algorithm Design Techniques, II

- Divide & Conquer
  - Reduce problem to one or more sub-problems of the same type
  - Typically, each sub-problem is at most a constant fraction of the size of the original problem
    - e.g. Mergesort, Binary Search, Strassen's Algorithm (we'll see this later), Quicksort (kind of)

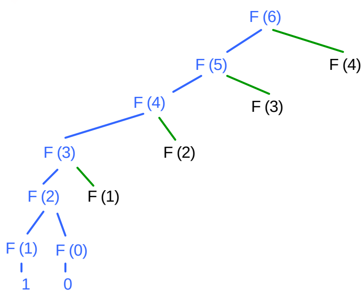
Some Algorithm Design Techniques, III

- Dynamic Programming
  - Give a solution of a problem using smaller sub-problems where all the possible sub-problems are determined in advance
  - Useful when the same sub-problems show up again and again in the solution

A simple case: Computing Fibonacci Numbers

- Recall $F_n=F_{n-1}+F_{n-2}$ and $F_0=0$, $F_1=1$
- Recursive algorithm:
  - Fibo(**n**)
    if **n=0** then return(**0**)
    else if **n=1** then return(**1**)
    else return(Fibo(**n-1**)+Fibo(**n-2**))

Call tree - start



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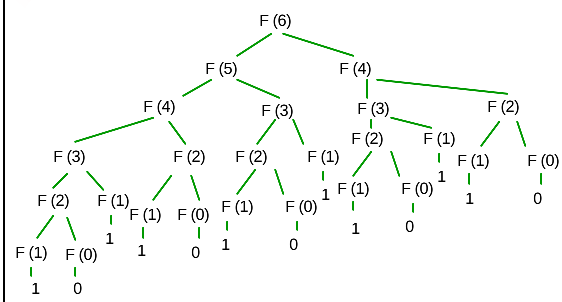
Full call tree



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Memo-ization (Caching)

- Remember all values from previous recursive calls
- Before recursive call, test to see if value has already been computed
- Dynamic Programming
  - Convert memo-ized algorithm from a recursive one to an iterative one

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Fibonacci - Dynamic Programming Version

- FiboDP (**n**):

```
F[0] ← 0
F[1] ← 1
for i=2 to n do
    F[i]=F[i-1]+F[i-2]
endfor
return(F[n])
```

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Dynamic Programming

- Useful when
  - same recursive sub-problems occur repeatedly
  - Can anticipate the parameters of these recursive calls
  - The solution to whole problem can be figured out with knowing the internal details of how the sub-problems are solved
    - principle of optimality
      "Optimal solutions to the sub-problems suffice for optimal solution to the whole problem"

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List partition problem

- Given: a sequence of n positive integers $s_1,\ldots,s_n$ and a positive integer k
- Find: a partition of the list into up to k blocks:
  $s_1,\ldots,s_{i_1}|s_{i_1+1}\ldots s_{i_2}|s_{i_2+1}\ldots s_{i_{k-1}}|s_{i_{k-1}+1}\ldots s_n$
  so that the sum of the numbers in the largest block is as small as possible.
  i.e. find spots for up to k-1 dividers

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Greedy approach

- Ideal size would be $P = \sum_{i=1}^{n} s_i / k$
- Greedy: walk along until what you have so far adds up to P then insert a divider
- Problem: it may not be exact (or correct)

  100 200 400 500 900 700 600 800 600

  - sum is 4800 so if k=3 size must be at least 1600.
  - Greedy? Best?

Recursive solution

- Try all possible values for the position of the last divider
- For each position of this last divider
  - there are k-2 other dividers that must divide the list of numbers prior to the last divider as evenly as possible
    - $s_1,\ldots,s_{i_1} | s_{i_1+1} \ldots s_{i_2} | s_{i_2+1} \ldots s_{i_{k-1}} | s_{i_{k-1}+1} \ldots s_n$
  - recursive sub-problem of the same type

Recursive idea

- Let M[n,k] the smallest cost (size of largest block) of any partition of the first n #'s into k pieces.
- If best position for last divider lies between the i^{th} and $i+1^{st}$ then

  max cost of 1st k-1 blocks — cost of last block

  $$M[n,k] = \max\left(M[i,k-1] ,\ \sum_{j=i+1}^{n} s_j \right)$$
- In general

  $$M[n,k] = \min_{i<n} \max\left(M[i,k-1] ,\ \sum_{j=i+1}^{n} s_j \right)$$
- Base case(s)?

Time-saving - prefix sums

- Computing the costs of the blocks may be expensive and involved repeated work
- Idea: Pre-compute prefix sums
- Length of block

  $s_{i+1} + \ldots + s_j$

  is just

  $p[j]-p[i]$
- Cost: n additions

$p[1]=s_1$
$p[2]=s_1+s_2$
$p[3]=s_1+s_2+s_3$
...
$p[n]=s_1+s_2+\ldots+s_n$

Linear Partition Algorithm

```
Partition(S,k):
   p[0]←0;
   for i=1 to n do p[i] ←p[i-1]+s_i
   for i=1 to n do M[i,1] ←p[i]
   for j=1 to k do M[1,j] ← s_1
   for i=2 to n do
       for j=2 to k do
           M[i,j] ← min_{pos<i}{max(M[pos,j-1], p[i]-p[pos])}
           D[i,j] ← value of pos where min is achieved
```


Linear Partition Algorithm

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   for i=1 to n do M[i,1] ←p[i]
   for j=1 to k do M[1,j] ← s_1
   for i=2 to n do
       for j=2 to k do
           M[i,j]←¥
           for pos=1 to i-1 do
               s←max(M[pos,j-1], p[i]-p[pos])
               if M[i,j]>s then
                   M[i,j] ←s ; D[i,j] ←pos
```

Example:

	1	2	3
100			
200			
400			
500			
900			
700			
600			
800			
600			

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                     is achieved
```


Example:

	1	2	3
100	100	100	100
200	300		
400	700		
500	1200		
900	2100		
700	2800		
600	3400		
800	4200		
600	4800		

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Example:

	1	2	3
100	100	100	100
200	300	200	200
400	700	400	400
500	1200	700	500
900	2100	1200	900
700	2800	1600	1200
600	3400	2100	
800	4200	2100	
600	4800	2700	

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Example:

	1	2	3
100	100	100	100
200	300	200	200
400	700	400	400
500	1200	700	500
900	2100	1200	900
700	2800	1600	1200
600	3400	2100	1300
800	4200	2100	1600
600	4800	2700	2000

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