CSE 417: Algorithms and Computational Complexity

Complexity:
More NP-completeness

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Steps to Proving Problem R is NP-complete

- Show **R** is **NP**-hard:
  - State:`Reduction is from **NP**-hard Problem **L**'
  - Show what the map **T** is
  - Argue that **T** is polynomial time
  - Argue correctness: two directions Yes for **L** implies Yes for **R** and vice versa.
- Show **R** is in **NP**
  - State what hint is and why it works
  - Argue that it is polynomial-time to check.

Problems we already know are NP-complete

- Satisfiability
- Independent-Set
- Clique
- Vertex-Cover
- There are 1000's of practical problems that are NP-complete, e.g. scheduling, optimal VLSI layout etc.

A particularly useful problem for proving NP-completeness

- **3-SAT**: Given a CNF formula **F** having **precisely 3 variables per clause** (i.e., in 3-CNF), is **F** satisfiable?
- **Claim: 3-SAT** is **NP**-complete
- **Proof:**
  - **3-SAT** $\in$ **NP**
    - Hint is a satisfying assignment
    - Just like Satisfiability it is polynomial-time to check the hint

Satisfiability $\le_P$ 3-SAT

- Reduction:
  - map CNF formula **F** to another CNF formula **G** that has precisely **3** variables per clause.
    - **G** has one or more clauses for each clause of **F**
    - **G** will have extra variables that don't appear in **F**
      - for each clause **C** of **F** there will be a different set of variables that are used only in the clauses of **G** that correspond to **C**

Satisfiability $\le_P$ 3-SAT

- Goal:
  - An assignment **a** to the original variables makes clause **C** true in **F** iff
    - there is an assignment to the extra variables that together with the assignment **a** will make all new clauses corresponding to **C** true.
- Define the reduction clause-by-clause
  - We'll use variable names z_j to denote the extra variables related to a single clause **C** to simplify notation
    - in reality, two different original clauses will not share z_j

Satisfiability $\le^P$ 3-SAT

- For each clause C in F:
  - If **C** has **3** variables:
    - Put **C** in **G** as is
  - If **C** has **2** variables, e.g. $C=(x_1 \vee \neg x_3)$
    - Use a new variable **z** and put two clauses in **G**
      $(x_1 \vee \neg x_3 \vee z) \wedge (x_1 \vee \neg x_3 \vee \neg z)$
    - If original **C** is true under assignment **a** then both new clauses will be true under **a**
    - If new clauses are both true under some assignment **b** then the value of **z** doesn't help in one of the two clauses so **C** must be true under **b**

Satisfiability $\le^P$ 3-SAT

- If **C** has **1** variable: e.g. $C=x_1$
  - Use two new variables z_1, z_2 and put **4** new clauses in **G**
    $(x_1 \vee \neg z_1 \vee \neg z_2) \wedge (x_1 \vee \neg z_1 \vee z_2) \wedge$
    $(x_1 \vee z_1 \vee \neg z_2) \wedge (x_1 \vee z_1 \vee z_2)$
  - If original **C** is true under assignment **a** then all new clauses will be true under **a**
  - If new clauses are all true under some assignment **b** then the values of z_1 and z_2 don't help in one of the 4 clauses so **C** must be true under **b**

Satisfiability $\le^P$ 3-SAT

- If **C** has $k \ge 4$ variables: e.g. $C=(x_1 \vee \ldots \vee x_k)$
  - Use **k-3** new variables $z_2,\ldots,z_{k-2}$ and put **k-2** new clauses in **G**
    $(x_1 \vee x_2 \vee z_2) \wedge (\neg z_2 \vee x_3 \vee z_3) \wedge (\neg z_3 \vee x_4 \vee z_4) \wedge \ldots$
    $\wedge (\neg z_{k-3} \vee x_{k-2} \vee z_{k-2}) \wedge (\neg z_{k-2} \vee x_{k-1} \vee x_k)$
  - If original **C** is true under assignment **a** then some x_i is true for $i \le k$. By setting z_j true for all $j<i$ and false for all $j \ge i$, we can extend **a** to make all new clauses true.
  - If new clauses are all true under some assignment **b** then some x_i must be true for $i \le k$ because $z_2 \wedge (\neg z_2 \vee z_3) \wedge \ldots \wedge (\neg z_{k-3} \vee z_{k-2}) \wedge \neg z_{k-2}$ is not satisfiable

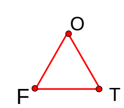
Graph Colorability

- Defn: Given a graph G=(V,E), and an integer k, a k-coloring of G is
  - an assignment of up to k different colors to the vertices of G so that the endpoints of each edge have different colors.
- 3-Color: Given a graph G=(V,E), does G have a 3-coloring?
- Claim: 3-Color is NP-complete
- Proof: 3-Color is in NP:
  - Hint is an assignment of red,green,blue to the vertices of G
  - Easy to check that each edge is colored correctly

3-SAT $\le^P$ 3-Color

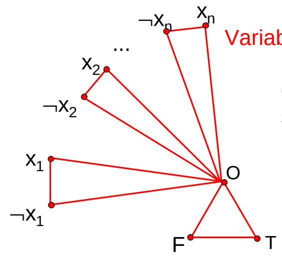
- Reduction:
  - We want to map a 3-CNF formula **F** to a graph **G** so that
    - **G** is 3-colorable iff **F** is satisfiable

3-SAT $\le^P$ 3-Color



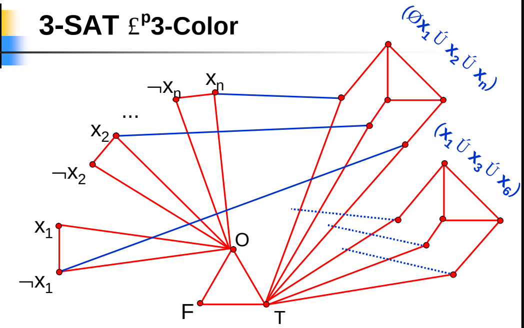
Base Triangle

3-SAT £^{P}3-Color



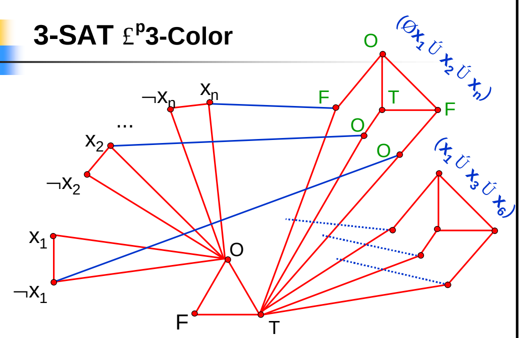
Variable Part:
in 3-coloring, variable colors correspond to some truth assignment (same color as T or F)

3-SAT £^{P}3-Color



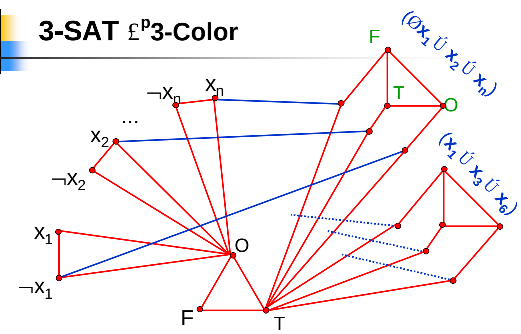
Clause Part:
Add one 6 vertex gadget per clause connecting its ‘outer vertices’ to the literals in the clause

3-SAT £^{P}3-Color



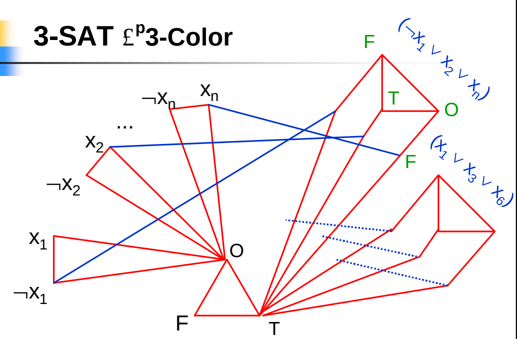
Any truth assignment satisfying the formula can be extended to a 3-coloring of the graph

3-SAT £^{P}3-Color



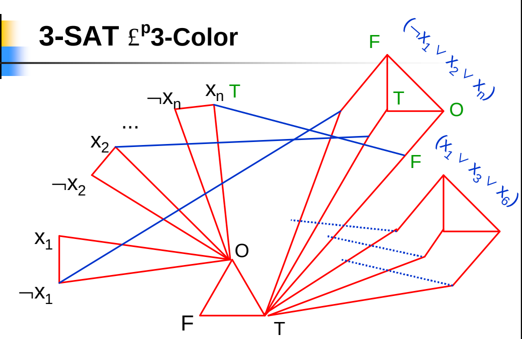
Any 3-coloring of the graph colors each gadget triangle using each color

3-SAT £^{P}3-Color



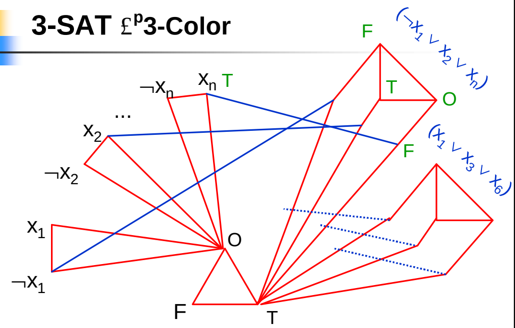
Any 3-coloring of the graph has an F opposite the O color in the triangle of each gadget

3-SAT £^{P}3-Color



Any 3-coloring of the graph has T at the other end of the blue edge connected to the F

3-SAT £^{P}3-Color



Any 3-coloring of the graph yields a satisfying assignment to the formula

Another NP-complete problem

- Knapsack problem
  - Same problem as described on the midterm
    - Given **n** integers $a_1,\ldots,a_n$ and integer **K**
    - Is there a subset of the **n** input integers that adds up to exactly **K**?
- **O(nK)** solution possible but if **K** and each a_i can be **n** bits long then this is exponential time

Is NP as bad as it gets?

- NO! **NP**-complete problems are frequently encountered, but there's worse:
  - Some problems provably require exponential time.
    - Ex: Does **P** halt on **x** in $2^{|x|}$ steps?
  - Some require 2^n, 2^{2^n}, $2^{2^{2^n}}$, ... steps
  - And of course, some are just plain uncomputable

Summary

- Big-O(n^2) – good
- P – good
- Exp – bad
- Hints help? NP
- NP-hard, NP-complete – bad (I bet)