CSE 417: Algorithms and Computational Complexity

Winter 2001

Lecture 19

Instructor: Paul Beame

Halting Problem

- Given: the code of a program P and an input x for P, i.e. given <P,x>
- Output: 1 if P halts on input x and 0 if P does not halt on input x
- Theorem (Turing): There is no program that solves the halting problem

  "The halting problem is undecidable"

Undecidability of the Halting Problem

- Suppose that there is a program H that computes the answer to the Halting Problem
- We'll build a table with all the possible programs down one side and all the possible inputs along the other and do a diagonal flip to produce a contradiction

input

program code	ε	0	1	00	01	10	11	000	001	010	011	....
ε	0	1	1	0	1	1	1	0	0	0	1	....
0	1	1	0	1	0	1	1	0	1	1	1	....
1	1	0	1	0	0	0	0	0	0	0	1	....
00	0	1	1	0	1	0	1	1	0	1	0	....
01	0	1	1	1	1	1	1	0	0	0	1	....
10	1	1	0	0	0	1	1	0	1	1	1	....
11	1	0	1	1	0	0	0	0	0	0	1	....
000	0	1	1	1	1	0	1	1	0	1	0	....
001	.	.	.	.	.	.	.	.	.	.	.	
.	.	.	.	.	.	.	.	.	.	.	.	
.												

Entries are 1 if program P given by the code halts on input x and 0 if it runs forever

input

program code	ε	0	1	00	01	10	11	000	001	010	011	....
ε	1	1	1	0	1	1	1	0	0	0	1	....
0	1	0	0	1	0	1	1	0	1	1	1	....
1	1	0	0	0	0	0	0	0	0	0	1	....
00	0	1	1	1	1	0	1	1	0	1	0	....
01	0	1	1	1	0	1	1	0	0	0	1	....
10	1	1	0	0	0	0	1	0	1	1	1	....
11	1	0	1	1	0	0	1	0	0	0	1	....
000	0	1	1	1	1	0	1	0	0	1	0	....
001	.	.	.	.	.	.	.	.	.	.	.	
.	.	.	.	.	.	.	.	.	.	.	.	
.												

Want to create a new program whose halting properties are given by the flipped diagonal

Diagonal construction

- Suppose H exists
- Now define a new program D such that
  - D on input x:
    - runs H checking if the program P whose code is x halts when given x as input; i.e. does P halt on input <P>
    - if H outputs 1 then D goes into an infinite loop
    - if H outputs 0 then D halts.
- The row for the program D would be like the flip of the diagonal

Code for D assuming subroutine for H

- Function D(x):
  - **if** H(x,x)=1 **then**
    - **while** (true); /* loop forever */
  - **else**
    - **no-op**; /* do nothing and halt */
  - **endif**

7

Finishing the argument

- D must be different from any program in the list:
- Suppose it has code <D> then
  - D halts on input <D>
  - iff (by definition of D)
  - H outputs 0 given program D and input <D>
  - iff (by definition of H)
  - D runs forever on input <D>
- Contradiction!

8

Relating hardness of problems

- We have one problem that we know is impossible to solve
  - Halting problem
- Showing this took serious effort
- We'd like to use this fact to derive that other problems are impossible to solve
  - don't want to go back to square one to do it

9

Reductions

- Given two problems to solve, L and R.
  - (think Left and Right)
- Suppose you had a translation program T so that the following would correctly solve L (if you happened to have code for R handy)
  - Function L(x)
    - Run program T to translate input x for L into an input y for R
    - Call a subroutine for problem R on input y
    - Output the answer produced by R(y)

10

Property that makes this correct

- It better be the case that no matter what x is

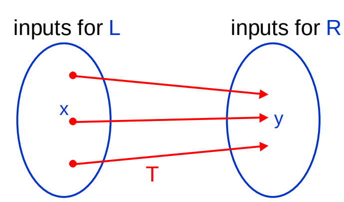
  L(x)=R(y)

  i.e. L(x)=R(T(x))

- T is called a reduction from problem L to problem R
- If such a T exists we write L ≤ R.

11

Reduction L ≤ R



L(x)=R(T(x))

Intuition: L is at least as easy as R or, equivalently, R is at least as hard as L

12

Example: BFS $\le$ Shortest-Path

- BFS: Given a graph G and a vertex v, output the BFS tree of G started at v
- Shortest-Paths: Given a graph G with non-negative weights on its edges, and a vertex v output the shortest-path tree of G from v
- Reduction T: Given G and v, create weights for all edges in G giving each edge weight 1
  - <G,v> $\xrightarrow[T]{}$ <G,weights,v>

Properties of reductions

- Given that I have any reduction T such that L(x)=R(T(x))
  - If I had a program that solves R then I would have a program that solves L
- Therefore
  - If there is no program that solves L then there cannot be any program that solves R!
  - (statement is just equivalent to one above)

Another undecidable problem

- 1's problem: Given the code of a program M does M output 1 on input 1? If so, answer 1 else answer 0.
- Claim: the 1's problem is undecidable
- Proof: by reduction from the Halting Problem

What we want for the reduction

- Halting problem takes as input a pair <P,x>
- 1's problem takes as input <M>
- Given <P,x> can we create an <M> so that M outputs 1 on input 1 exactly when P halts on input x?

Yes

- Here is all that we need to do to create M
  - modify the code of P so that instead of reading x, x is hard-coded as the input to P and get rid of all output statements in P
  - add a new statement at the end of P that outputs 1.
- We can write another program T that can do this transformation from <P,x> to <M>

Finishing things off

- Therefore we get a reduction
  - Halting Problem $\le$ 1's problem
- Since there is no program solving the Halting Problem there must be no program solving the 1's problem.

Why the name reduction?

- Weird: it maps an easier problem into a harder one
- Same sense as saying Maxwell reduced the problem of analyzing electricity & magnetism to solving partial differential equations
  - solving partial differential equations in general is a much harder problem than solving E&M problems

19

A geek joke

- An engineer
  - is placed in a kitchen with an empty kettle on the table and told to boil water; she fills the kettle with water, puts it on the stove, turns on the gas and boils water.
  - she is next confronted with a kettle full of water sitting on the counter and told to boil water; she puts it on the stove, turns on the gas and boils water.
- A mathematician
  - is placed in a kitchen with an empty kettle on the table and told to boil water; he fills the kettle with water, puts it on the stove, turns on the gas and boils water.
  - he is next confronted with a kettle full of water sitting on the counter and told to boil water: he empties the kettle in the sink, places the empty kettle on the table and says, "I've reduced this to an already solved problem".

20