CSE373: Data Structures & Algorithms

Lecture 12: Amortized Analysis and Memory Locality

Lauren Milne

Spring 2015

Announcements

- Homework 3 due on Wednesday at 11pm
- Catie back Monday

Amortized Analysis

- In amortized analysis, the time required to perform a sequence of data structure operations is averaged over all the operations performed.

- Typically used to show that the average cost of an operation is small for a sequence of operations, even though a single operation can cost a lot

Amortized Analysis

- Recall our plain-old stack implemented as an array that doubles its size if it runs out of room
  - Can we claim **`push`** is $O(\mathbf{1})$ time if resizing is $O(\mathbf{n})$ time?
  - We *can't*, but we *can* claim it's an $O(\mathbf{1})$ amortized operation

Amortized Complexity

We get an upperbound T(n) on the total time of a sequence of n operations. The average time per operation is then T(n)/n, which is also the amortized time per operation.

If a sequence of $\mathbf{n}$ operations takes $O(\mathbf{n\ f(n)})$ time, we say the amortized runtime is $O(\mathbf{f(n)})$

- If $\mathbf{n}$ operations take $O(\mathbf{n})$, what is amortized time per operation?
  - $O(\mathbf{1})$ per operation
- If $\mathbf{n}$ operations take $O(\mathbf{n}^3)$, what is amortized time per operation?
  - $O(\mathbf{n}^2)$ per operation

The worst case time for an operation can be larger than $\mathbf{f(n)}$, but amortized guarantee ensures the average time per operation for any sequence is $O(\mathbf{f(n)})$

“Building Up Credit”

- Can think of preceding “cheap” operations as building up “credit” that can be used to “pay for” later “expensive” operations

- Because any sequence of operations must be under the bound, enough “cheap” operations must come *first*
  - Else a prefix of the sequence would violate the bound

Example #1: Resizing stack

A stack implemented with an array where we double the size of the array if it becomes full

Claim: Any sequence of **`push`**/**`pop`**/**`isEmpty`** is amortized $O(\mathbf{1})$ per operation

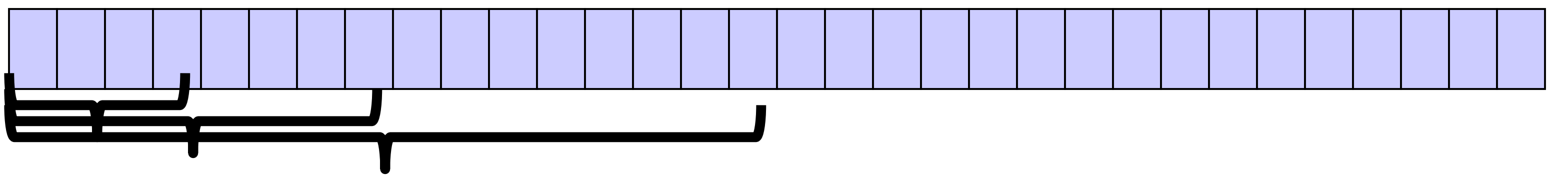
Need to show any sequence of **`M`** operations takes time $O(\mathbf{M})$

- Recall the non-resizing work is $O(\mathbf{M})$ (i.e., $\mathbf{M}*O(\mathbf{1})$)
- The resizing work is proportional to the total number of element copies we do for the resizing
- So it suffices to show that:

  After **`M`** operations, we have done **`<`** **`2M`** total element copies

  (So average number of copies per operation is bounded by a constant)

Amount of copying



Claim: after $\mathbf{M}$ operations, we have done $<$ $\mathbf{2M}$ total element copies

Let $\mathbf{n}$ be the size of the array after $\mathbf{M}$ operations

– Then we have done a total of:

$$\mathbf{n/2 + n/4 + n/8 + \ldots\ INITIAL\_SIZE < n}$$

element copies

– Because we must have done at least enough **push** operations to cause resizing up to size $\mathbf{n}$:

$$\mathbf{M \geq n/2}$$

– So

$$\mathbf{2M \geq n >} \textit{ number of element copies}$$

Other approaches

- If array grows by a constant amount (say 1000), operations are not amortized $O(\mathtt{1})$
  - After $O(\mathbf{M})$ operations, you may have done $\Theta(\mathbf{M}^2)$ copies

- If array shrinks when 1/2 empty, operations are not amortized $O(\mathtt{1})$
  - Terrible case: **`pop`** once and shrink, **`push`** once and grow, **`pop`** once and shrink, …

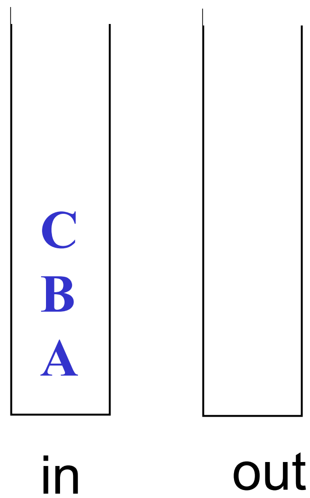
- If array shrinks when 3/4 empty, it is amortized $O(\mathtt{1})$
  - Proof is more complicated, but basic idea remains: by the time an expensive operation occurs, many cheap ones occurred

Example #2: Queue with two stacks

A clever and simple queue implementation using only stacks

```
class Queue<E> {
  Stack<E> in  = new Stack<E>();
  Stack<E> out = new Stack<E>();
  void enqueue(E x){ in.push(x); }
  E dequeue(){
    if(out.isEmpty()) {
      while(!in.isEmpty()) {
        out.push(in.pop());
      }
    }
    return out.pop();
  }
}
```

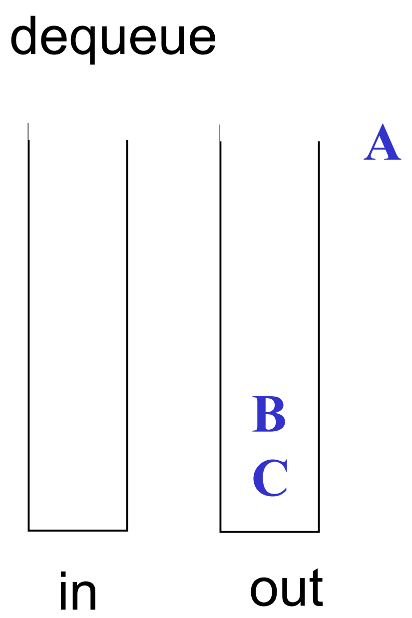
enqueue: A, B, C



Example #2: Queue with two stacks

A clever and simple queue implementation using only stacks

```
class Queue<E> {
  Stack<E> in  = new Stack<E>();
  Stack<E> out = new Stack<E>();
  void enqueue(E x){ in.push(x); }
  E dequeue(){
    if(out.isEmpty()) {
      while(!in.isEmpty()) {
        out.push(in.pop());
      }
    }
    return out.pop();
  }
}
```

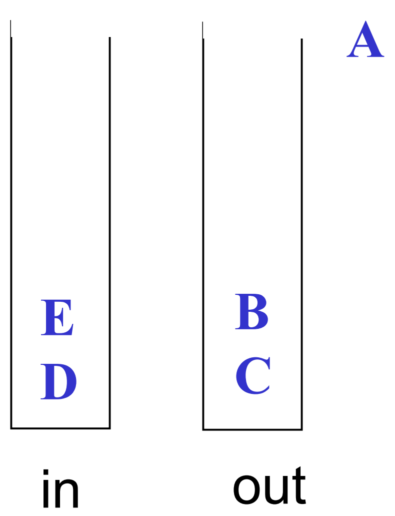


Example #2: Queue with two stacks

A clever and simple queue implementation using only stacks

```
class Queue<E> {
  Stack<E> in  = new Stack<E>();
  Stack<E> out = new Stack<E>();
  void enqueue(E x){ in.push(x); }
  E dequeue(){
    if(out.isEmpty()) {
      while(!in.isEmpty()) {
        out.push(in.pop());
      }
    }
    return out.pop();
  }
}
```

enqueue D, E

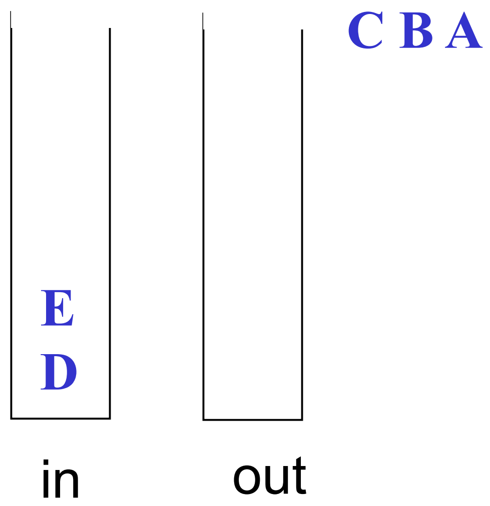


Example #2: Queue with two stacks

A clever and simple queue implementation using only stacks

```
class Queue<E> {
  Stack<E> in  = new Stack<E>();
  Stack<E> out = new Stack<E>();
  void enqueue(E x){ in.push(x); }
  E dequeue(){
    if(out.isEmpty()) {
      while(!in.isEmpty()) {
        out.push(in.pop());
      }
    }
    return out.pop();
  }
}
```

dequeue twice

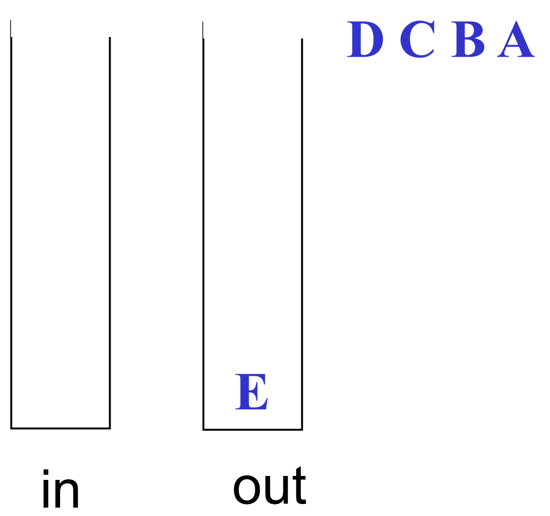


Example #2: Queue with two stacks

A clever and simple queue implementation using only stacks

```
class Queue<E> {
  Stack<E> in  = new Stack<E>();
  Stack<E> out = new Stack<E>();
  void enqueue(E x){ in.push(x); }
  E dequeue(){
    if(out.isEmpty()) {
      while(!in.isEmpty()) {
        out.push(in.pop());
      }
    }
    return out.pop();
  }
}
```

dequeue again



Analysis

- **`dequeue`** is not $O(\mathbf{1})$ worst-case because **out** might be empty and **in** may have lots of items

- But if the stack operations are (amortized) $O(\mathbf{1})$, then any sequence of queue operations is amortized $O(\mathbf{1})$

  – The total amount of work done per element is 1 **`push`** onto **in**, 1 **`pop`** off of **in**, 1 **`push`** onto **out**, 1 **`pop`** off of **out**

  – When you reverse **`n`** elements, there were **`n`** earlier $O(\mathbf{1})$ **`enqueue`** operations to average with

When is Amortized Analysis Useful?

- When the average per operation is all we care about (i.e., sum over all operations), amortized is perfectly fine

- If we need every operation to finish quickly (e.g., in a web server), amortized bounds may be too weak

Not always so simple

- Proofs for amortized bounds can be much more complicated
- Example: Splay trees are dictionaries with amortized O(**`log n`**) operations
  - See Chapter 4.5 if curious
- For more complicated examples, the proofs need much more sophisticated invariants and “potential functions” to describe how earlier cheap operations build up “energy” or “money” to “pay for” later expensive operations
  - See Chapter 11 if curious
- But complicated *proofs* have nothing to do with the code (which may be easy!)

Switching gears…

- Memory hierarchy/locality

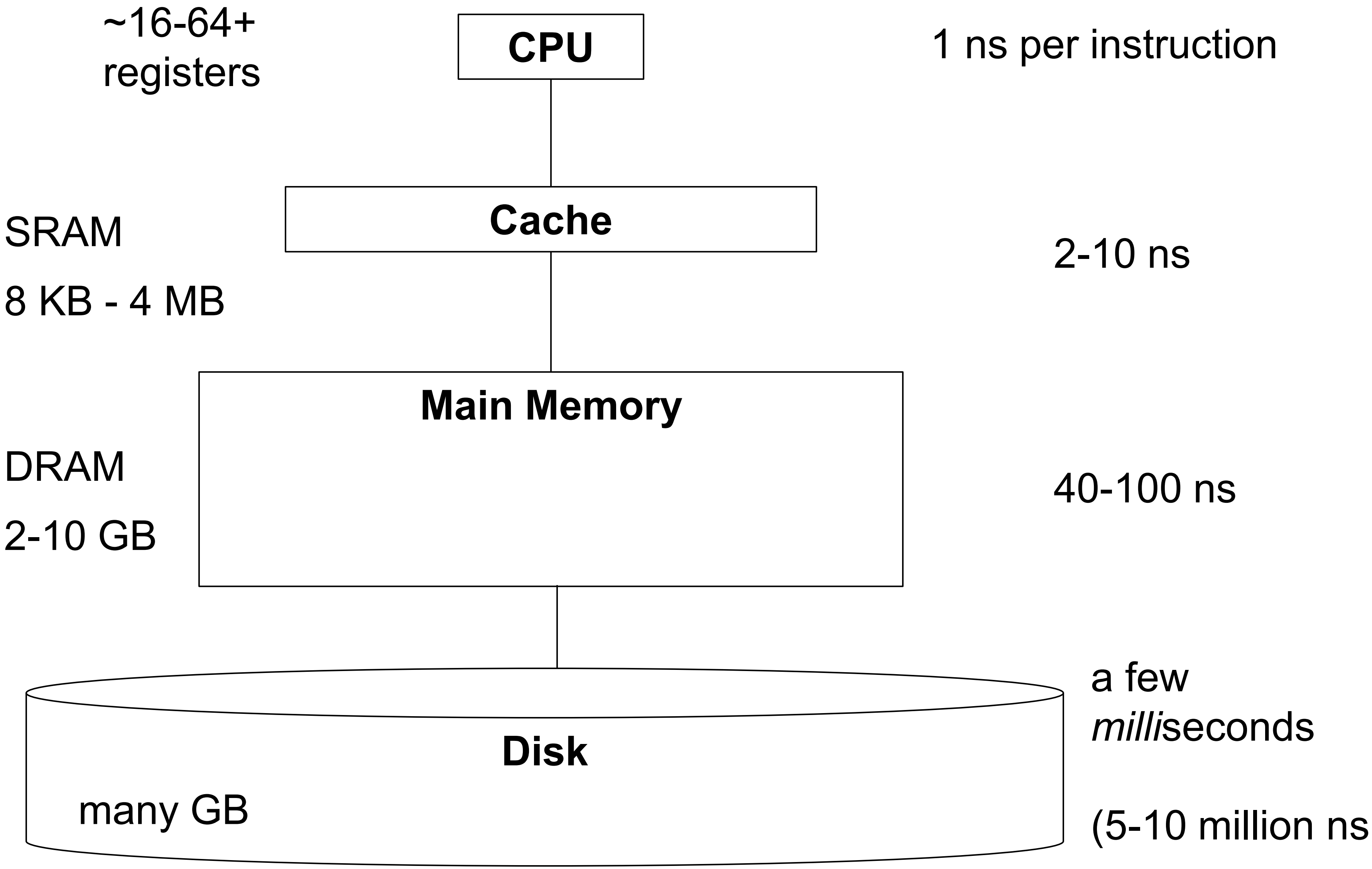
Why do we need to know about the memory hierarchy/locality?

- One of the assumptions that Big-O makes is that *all operations take the same amount of time*
- Is this really true?

Definitions

- A cycle (for our purposes) is the time it takes to execute a single simple instruction (e.g. adding two registers together)
- Memory latency is the time it takes to access memory

Time to access:

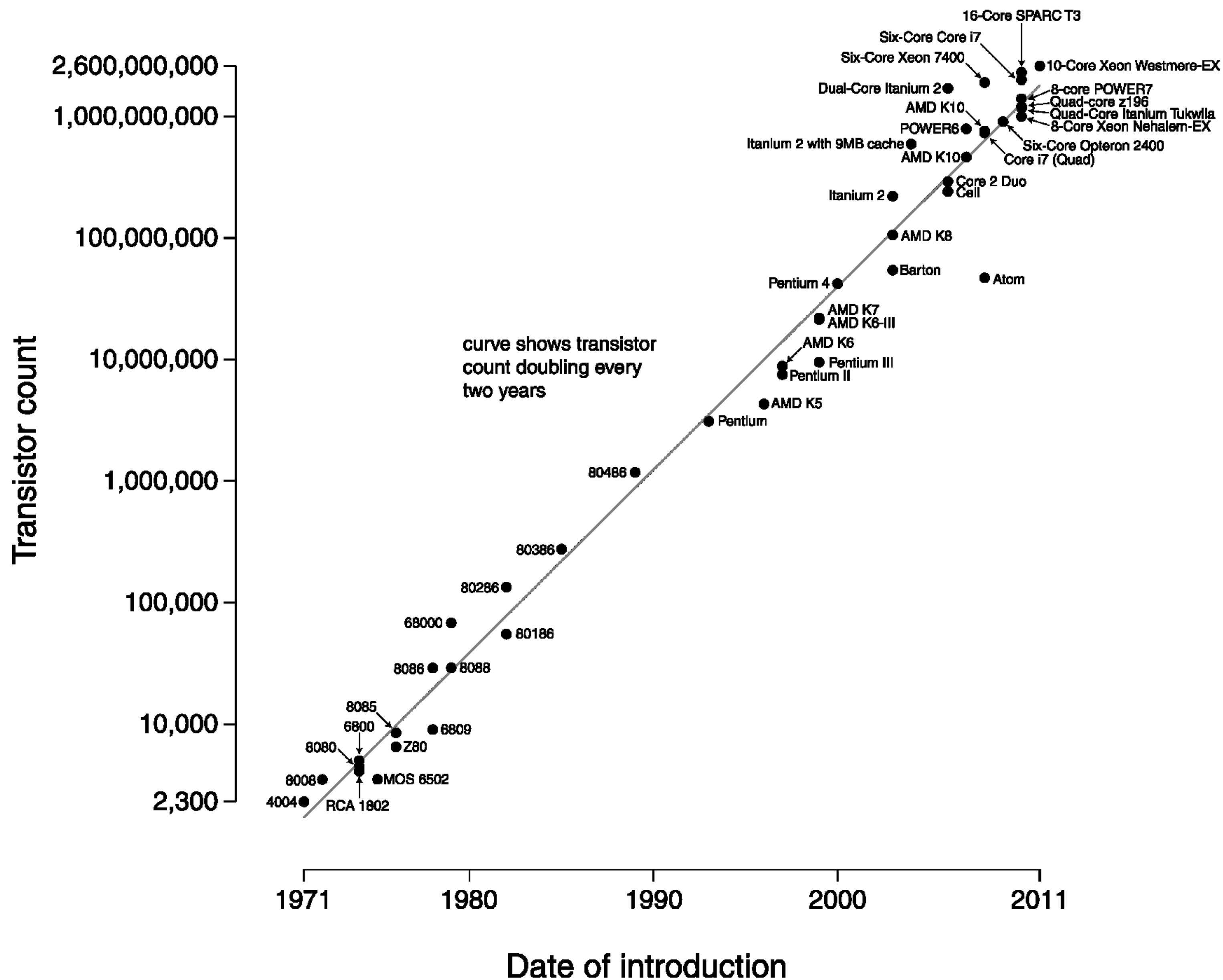


What does this mean?

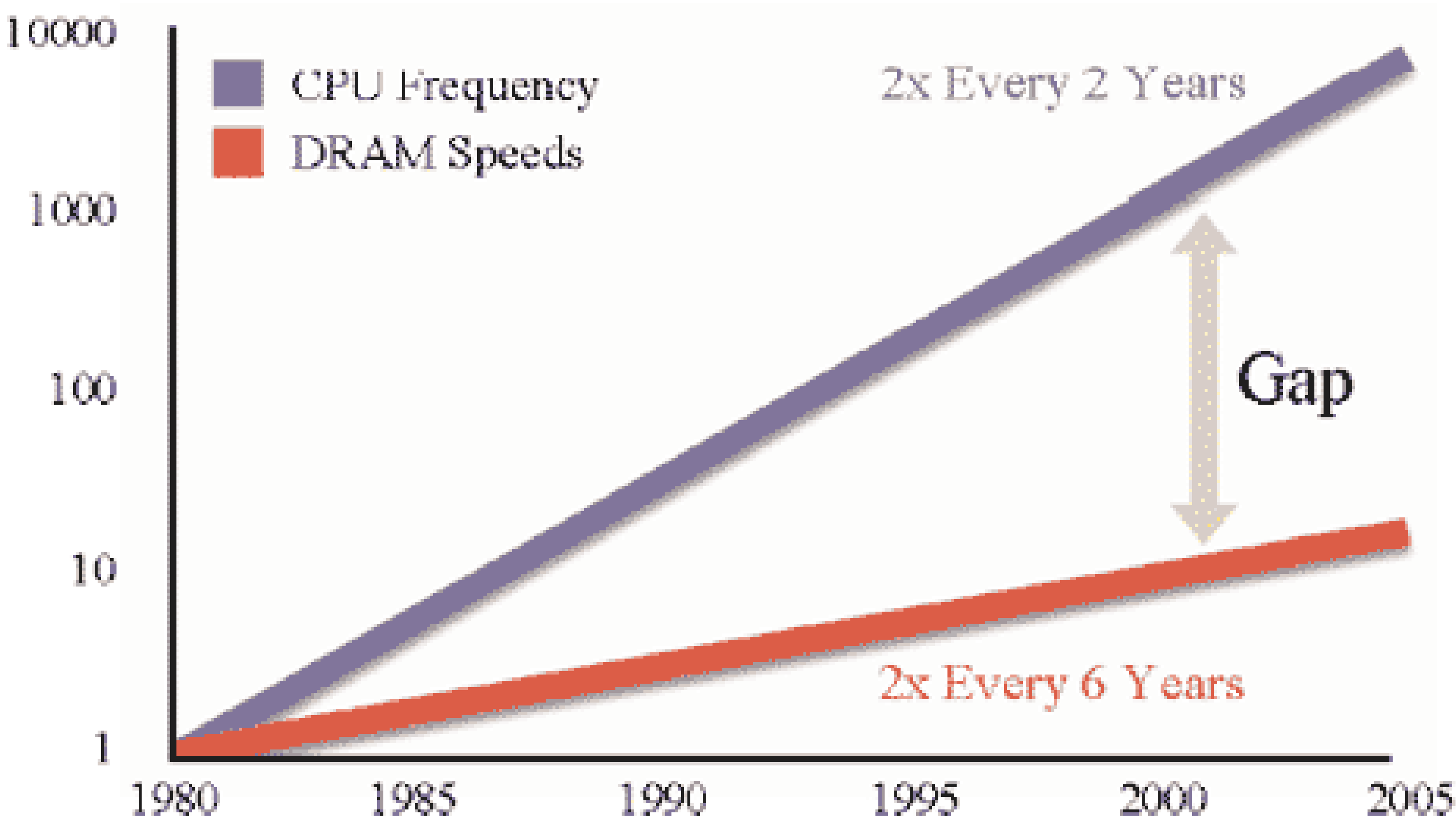
• It is much faster to do:	Than:
5 million arithmetic ops	1 disk access
2500 L2 cache accesses	1 disk access
400 main memory accesses	1 disk access

- Why are computers build this way?
  - Physical realities (speed of light, closeness to CPU)
  - Cost (price per byte of different storage technologies)
  - Under the right circumstances, this kind of hierarchy can simulate storage with access time of highest (fastest) level and size of lowest (largest) level

Microprocessor Transistor Counts 1971-2011 & Moore's Law



Processor-Memory Performance Gap



What can be done?

- **Goal**: attempt to reduce the accesses to slower levels

So, what can <u>we</u> do?

- The hardware automatically moves data from main memory into the caches for you
  - Replacing items already there
  - Algorithms are much faster if “data fits in cache” (often does)

- Disk accesses are done by software (e.g. ask operating system to open a file or database to access some records)

- So most code “just runs,” but sometimes it’s worth designing algorithms / data structures with knowledge of memory hierarchy
  - To do this, we need to understand locality

Locality

- Temporal Locality (locality in time)
  - If an item (a location in memory) is referenced, **that same location** will tend to be referenced again soon.

- Spatial Locality (locality in space)
  - If an item is referenced, items **whose addresses are close by** tend to be referenced soon.

How does data move up the hierarchy?

- Moving data up the hierarchy is slow because of *latency* (think distance to travel)
  - Since we're making the trip anyway, might as well carpool
    - Get a **block** of data in the same time we could get a byte
  - Sends *nearby memory* because
    - It's easy
    - Likely to be asked for soon (think fields/arrays)
- Once a value is in cache, may as well keep it around for a while; accessed once, a value is more likely to be accessed again in the near future (as opposed to some random other value)

Spatial Locality

Temporal Locality

Cache Facts

- Definitions:
  - Cache hit – address requested is in the cache
  - Cache miss – address requested is NOT in the cache
  - Block or page size – the number of contiguous bytes moved from disk to memory
  - Cache line size – the number of contiguous bytes moved from memory to cache

Examples

```
x = a + 6
```
miss

```
y = a + 5
```
hit

```
z = 8 * a
```
hit

Examples

`x = a + 6` miss `x = a[0] + 6` miss

`y = a + 5` hit `y = a[1] + 5` hit

`z = 8 * a` hit `z = 8 * a[2]` hit

Examples

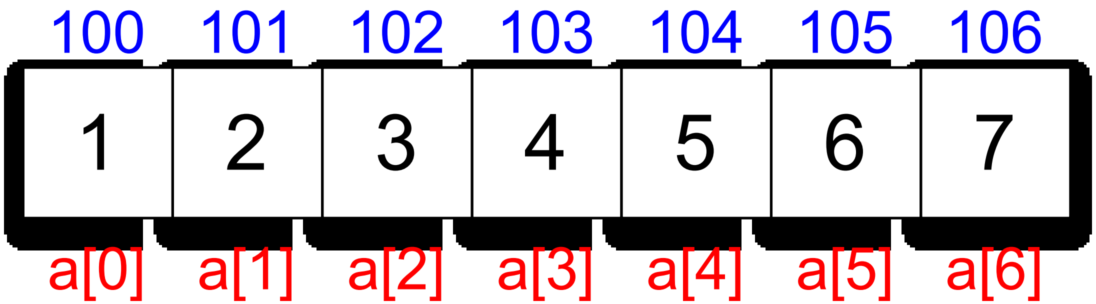
`x = a + 6` miss	`x = a[0] + 6` miss
`y = a + 5` hit	`y = a[1] + 5` hit
`z = 8 * a` hit	`z = 8 * a[2]` hit

temporal locality

spatial locality

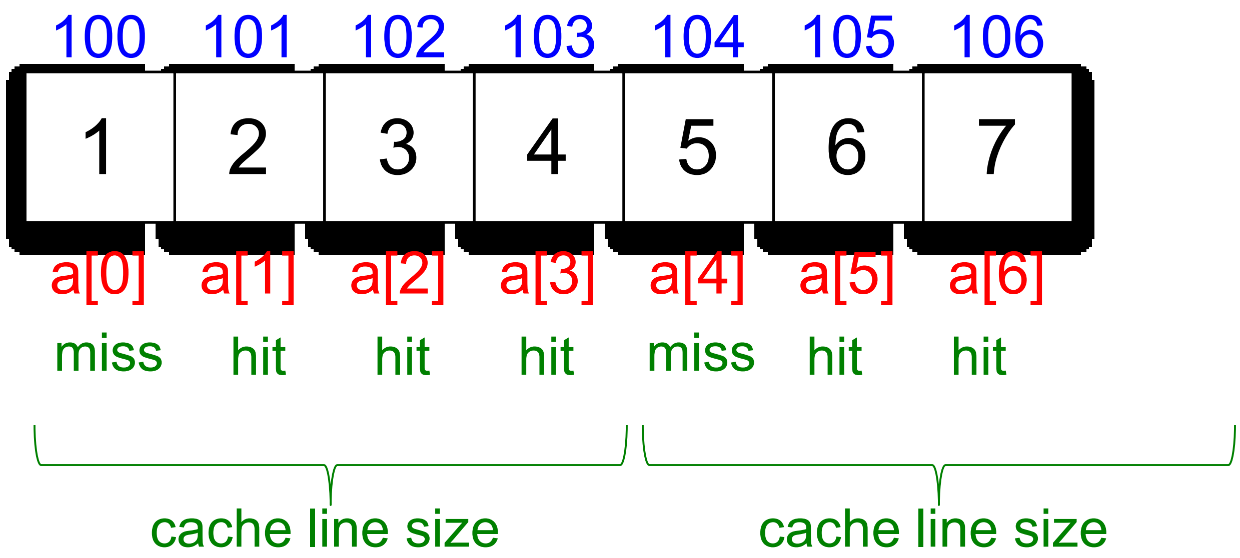
Locality and Data Structures

- Which has (at least the potential) for better spatial locality, arrays or linked lists?



Locality and Data Structures

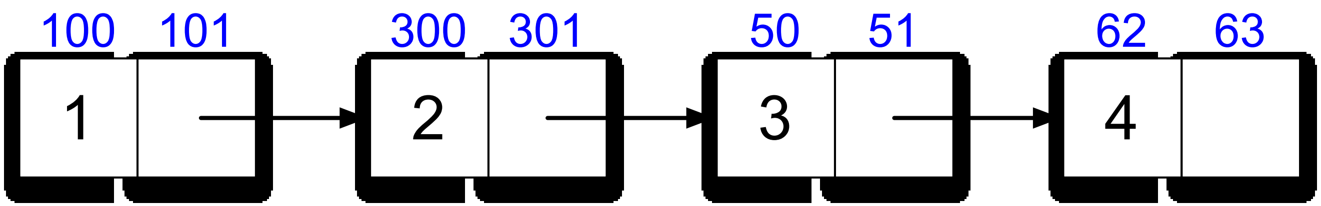
- Which has (at least the potential) for better spatial locality, arrays or linked lists?
  - e.g. traversing elements



- Only miss on first item in a cache line

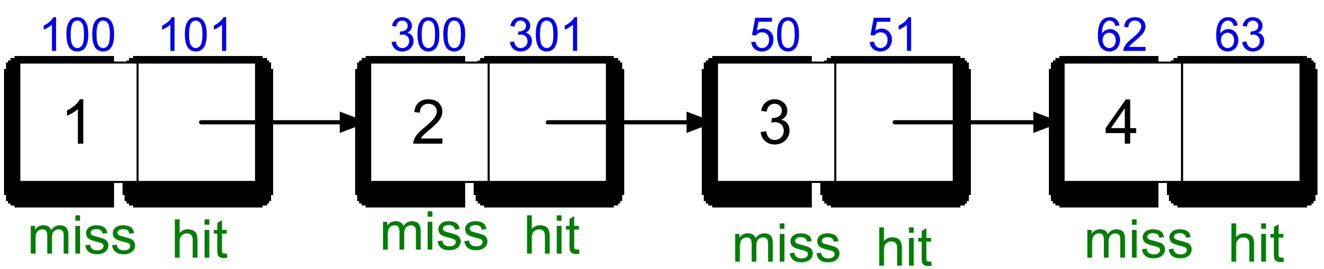
Locality and Data Structures

- Which has (at least the potential) for better spatial locality, arrays or linked lists?
  - e.g. traversing elements



Locality and Data Structures

- Which has (at least the potential) for better spatial locality, arrays or linked lists?
  - e.g. traversing elements



- Miss on **every** item (unless more than one randomly happen to be in the same cache line)