# Priority Queues

CSE 373
Data Structures & Algorithms
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## Today's Outline

- **Announcements**
  - **Homework #3 due Thurs, Oct 28, 11:45pm.**
- **Today's Topics:**
  - **Priority Queues**
    - **Binary Min Heap - buildheap**
    - **D-Heaps**
    - **Leftist Heaps**



## Facts about Binary Min Heaps

Observations:

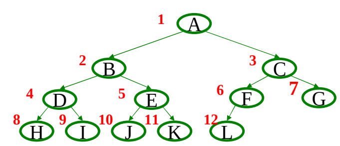
- finding a child/parent index is a multiply/divide by two
- operations jump widely through the heap
- each percolate step looks at only two new nodes
- inserts are *at least* as common as deleteMins

Realities:

- division/multiplication by *powers* of two are equally fast
- looking at only **two** new pieces of data: bad for cache!
- with huge data sets, disk accesses dominate



## Representing Complete Binary Trees in an Array



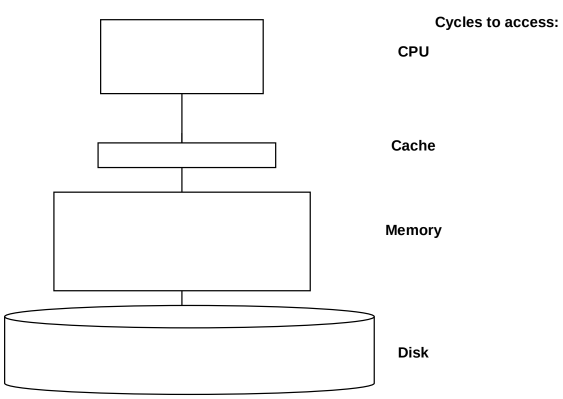

From node **i**:

left child:
right child:
parent:

implicit (array) implementation:

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | **A** | **B** | **C** | **D** | **E** | **F** | **G** | **H** | **I** | **J** | **K** | **L** | |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |

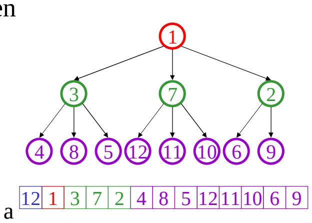







## A Solution: *d*-Heaps

- Each node has *d* children
- Still representable by array
- Good choices for *d*:
  - (choose a power of two for efficiency)
  - fit one set of children in a cache line
  - fit one set of children on a memory page/disk block



## Operations on *d*-Heap

- Insert : runtime =
- deleteMin: runtime =



## Priority Queues

(Leftist Heaps)



## One More Operation

- Merge two heaps. Ideas?



## New Operation: Merge

Given two heaps, merge them into one heap

- first attempt: insert each element of the smaller heap into the larger.
  *runtime:*
- second attempt: concatenate binary heaps' arrays and run buildHeap.
  *runtime:*



## Leftist Heaps

Idea:

Focus all heap maintenance work in one small part of the heap

Leftist heaps:

1. Most nodes are on the left
2. All the merging work is done on the right



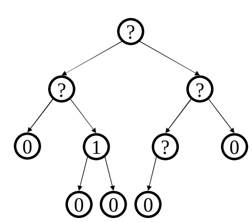
## Definition: Null Path Length

*null path length (npl)* of a node *x* = the number of nodes between *x* and a null in its subtree

OR

npl(x) = min distance to a descendant with 0 or 1 children

- *npl*(null) = -1
- *npl*(leaf, aka zero children) = 0
- *npl*(node with one child) = 0




Equivalent definitions:

1. *npl(x)* is the height of largest perfect subtree rooted at *x*
2. *npl(x)* = 1 + min{*npl*(left(x)), *npl*(right(x))}

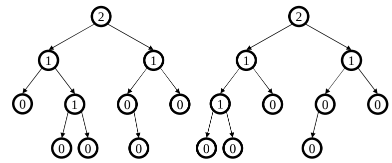
## Leftist Heap Properties

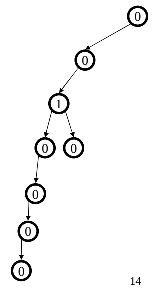
- Heap-order property
  - parent's priority value is ≤ to childrens' priority values
  - <u>result</u>: minimum element is at the root
- Leftist property
  - For every node $x$, $npl(\text{left}(x)) \geq npl(\text{right}(x))$
  - <u>result</u>: tree is at least as "heavy" on the left as the right

Are leftist trees…
complete?
balanced?



## Are These Leftist?




**Every subtree of a leftist tree is leftist!**



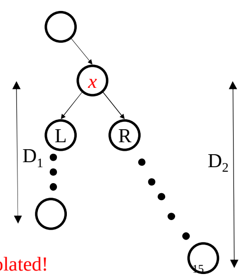
## Right Path in a Leftist Tree is Short (#1)

<u>Claim</u>: The right path is as short as *any* in the tree.
<u>Proof</u>: (By contradiction)

Pick a shorter path: $D_1 < D_2$
Say it diverges from right path at $x$

$npl(L) \leq D_1-1$ because of the path of length $D_1-1$ to null

$npl(R) \geq D_2-1$ because every node on right path is leftist




Leftist property at $x$ violated!



## Right Path in a Leftist Tree is Short (#2)

<u>Claim</u>: If the right path has **r** nodes, then the tree has at least **$2^r$-1** nodes.

<u>Proof</u>: (By induction)

**Base case** : **r=1**. Tree has at least **$2^1$-1 = 1** node

**Inductive step** : assume true for **r' < r**. Prove for tree with right path at least **r**.

1. Right subtree: right path of **r-1** nodes
   ⇒ **$2^{r-1}$-1** right subtree nodes (by induction)
2. Left subtree: also right path of length at least **r-1** (by previous slide) ⇒ **$2^{r-1}$-1** left subtree nodes (by induction)

Total tree size: $(2^{r-1}-1) + (2^{r-1}-1) + 1 = 2^r-1$



## Why do we have the leftist property?

Because it guarantees that:

- the *right path is really short* compared to the number of nodes in the tree
- A leftist tree of N nodes, has a right path of at most **log (N+1)** nodes

**Idea** – perform all work on the right path

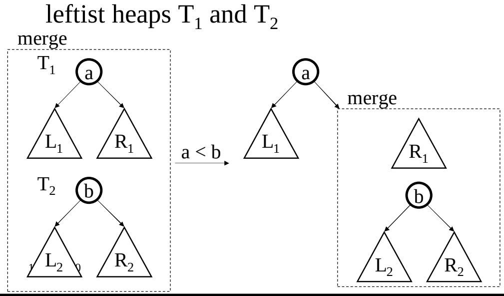


## Merge two heaps (basic idea)

- Put the smaller root as the new root,
- Hang its left subtree on the left.
- <u>Recursively</u> merge its right subtree and the other tree.

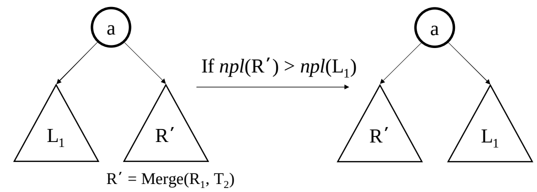
## Merging Two Leftist Heaps

- merge($T_1$,$T_2$) returns one leftist heap containing all elements of the two (distinct) leftist heaps $T_1$ and $T_2$





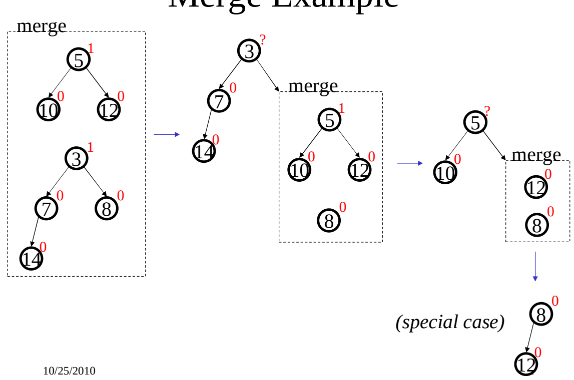

## Merge Continued




runtime:

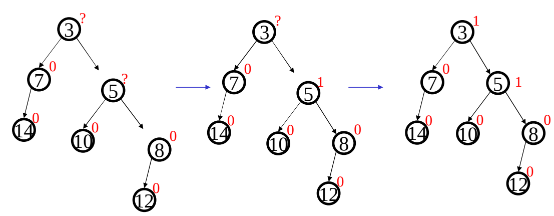


## Merge Example





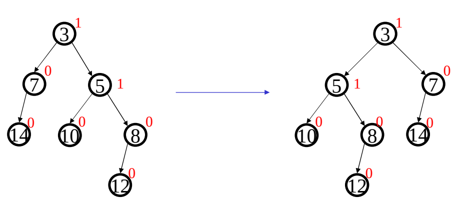

## Sewing Up the Example



Done?

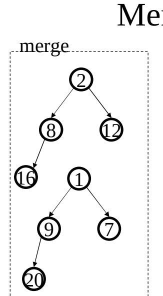


## Finally…





## Merge Two Leftist Heaps




Student Activity

## Other Heap Operations

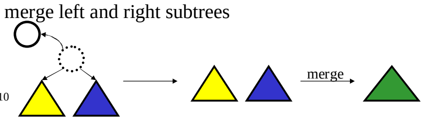
- insert ?
- deleteMin ?



## Operations on Leftist Heaps

- merge with two trees of total size n: O(log n)
- insert with heap size n: O(log n)
  - pretend node is a size 1 leftist heap
  - insert by merging original heap with one node heap


- deleteMin with heap size n: O(log n)
  - remove and return root
  - merge left and right subtrees






## Leftist Heaps: Summary

Good

- 
- 

Bad

- 
- 



## Amortized Time

am·or·tized time:

**Running time limit resulting from "writing off" expensive runs of an algorithm over multiple cheap runs of the algorithm, usually resulting in a lower overall running time than indicated by the worst possible case.**

If M operations take total O(M log N) time, *amortized* time per operation is O(log N)

Difference from **average time:**



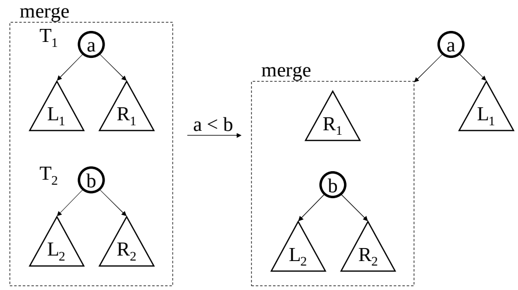
## Skew Heaps

Problems with leftist heaps
- extra storage for npl
- extra complexity/logic to maintain and check npl
- right side is "often" heavy and requires a switch

Solution: skew heaps
- "blindly" adjusting version of leftist heaps
- merge *always* switches children when fixing right path
- amortized time for: merge, insert, deleteMin = O(log $n$)
- however, worst case time for all three = O($n$)

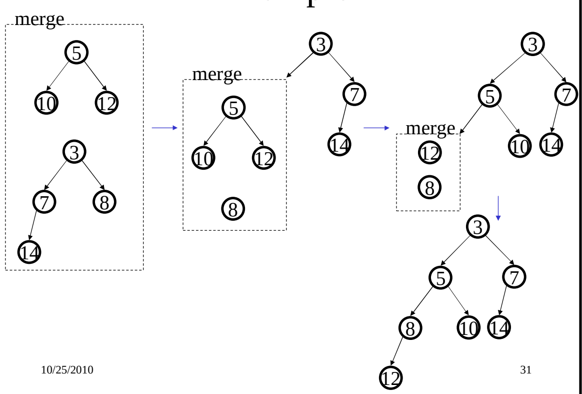


## Merging Two Skew Heaps




**Only one step per iteration, with children *always* switched**

## Example






## Skew Heap Code

```
void merge(heap1, heap2) {
  case {
     heap1 == NULL: return heap2;
     heap2 == NULL: return heap1;
     heap1.findMin() < heap2.findMin():
          temp = heap1.right;
          heap1.right = heap1.left;
          heap1.left = merge(heap2, temp);
          return heap1;
     otherwise:
          return merge(heap2, heap1);
  }
}
```



## Runtime Analysis: Worst-case and Amortized

- No worst case guarantee on right path length!
- All operations rely on merge

  $\Rightarrow$ worst case complexity of all ops =
- Amortized Analysis (Chapter 11)
- Result: $M$ merges take time $M \log n$

  $\Rightarrow$ amortized complexity of all ops =



## Comparing Priority Queues

| | |
|---|---|
| • Binary Heaps | • Leftist Heaps |
| • d-Heaps | • Skew Heaps |



Student Activity