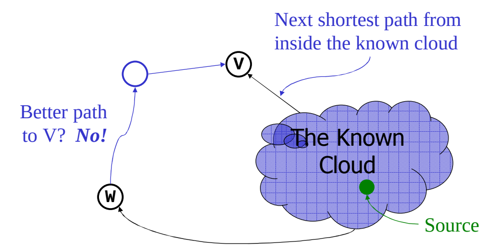
Graphs: More on Shortest Paths, Plus Minimum Spanning Trees

CSE 373
Data Structures and Algorithms

Correctness: The Cloud Proof



How does Dijkstra's decide which vertex to add to the Known set next?

- If path to **V** is shortest, path to **W** must be *at least as long*
  (or else we would have picked **W** *as the next vertex)*
- So the path through **W** to **V** cannot be any shorter!

Correctness: Inside the Cloud

Prove by induction on # of nodes in the cloud:

Initial cloud is just the source with shortest path 0

Assume: Everything inside the cloud has the correct shortest path

Inductive step: Only when we prove the shortest path to some node ***v*** (which is *not* in the cloud) is correct, we add it to the cloud

When does Dijkstra's algorithm not work?

Dijkstra's vs BFS

At each step:
1) Pick closest unknown vertex
2) Add it to finished vertices
3) Update distances

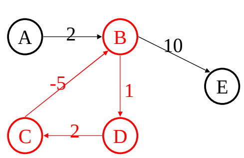
Dijkstra's Algorithm

At each step:
1) Pick vertex from queue
2) Add it to visited vertices
3) Update queue with neighbors

Breadth-first Search

Some Similarities:

The Trouble with Negative Weight Cycles



What's the shortest path from A to E?

Problem?

Minimum Spanning Trees

Given an undirected graph **G**=(**V**,**E**), find a graph **G'=(V, E')** such that:

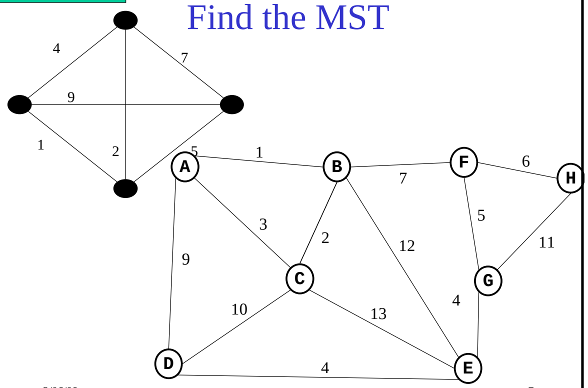
- E' is a subset of E
- |E'| = |V| - 1
- G' is connected
- $\sum_{(u,v)\in E'} c_{uv}$ is minimal

G' is a **minimum spanning tree.**

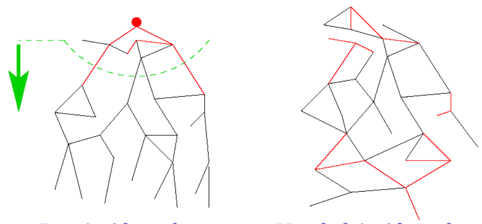
Applications: wiring a house, power grids, Internet connections

Student Activity

Find the MST



Two Different Approaches

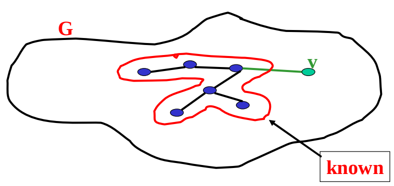


Prim's Algorithm
Almost identical to Dijkstra's

Kruskals's Algorithm
Completely different!

Prim's algorithm

Idea: Grow a tree by adding an edge from the "known" vertices to the "unknown" vertices. Pick the edge with the smallest weight.



Prim's Algorithm for MST

A *node-based* greedy algorithm

Builds MST by greedily adding nodes

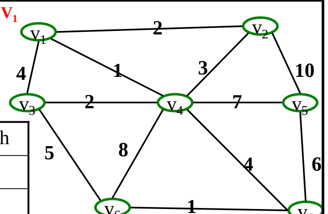
1. Select a node to be the "root"
   - mark it as known
   - Update cost of all its neighbors
2. While there are unknown nodes left in the graph
   a. Select an unknown node *b* with the smallest cost from some *known* node *a*
   b. Mark *b* as known
   c. Add (*a*, *b*) to MST
   d. Update cost of all nodes adjacent to *b*

Student Activity

Start with V_1

Find MST using Prim's

V	Kwn	Distance	path
v1			
v2			
v3			
v4			
v5			
v6			
v7			



Order Declared Known:

V_1

Prim's Algorithm Analysis

Running time:

Same as Dijkstra's: O(|E| log |V|)

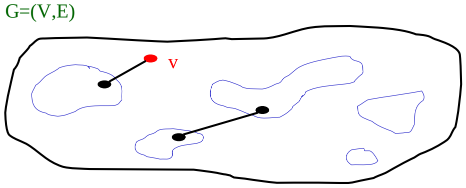
Correctness:

Proof is similar to Dijkstra's

Kruskal's MST Algorithm

Idea: Grow a forest out of edges that do not create a cycle. Pick an edge with the smallest weight.

G=(V,E)



Kruskal's Algorithm for MST

An *edge-based* greedy algorithm

Builds MST by greedily adding edges

1. Initialize with
   - empty MST
   - all vertices marked unconnected
   - all edges unmarked
2. While there are still unmarked edges
   a. Pick the lowest cost edge `(u,v)` and mark it
   b. If `u` and `v` are not already connected, add `(u,v)` to the MST and mark `u` and `v` as connected to each other

Doesn't it sound familiar?

Kruskal code

```
void Graph::kruskal(){
  int edgesAccepted = 0;
  DisjSet s(NUM_VERTICES);

  while (edgesAccepted < NUM_VERTICES – 1){
    e = smallest weight edge not deleted yet;
    // edge e = (u, v)
    uset = s.find(u);
    vset = s.find(v);
    if (uset != vset){
      edgesAccepted++;
      s.unionSets(uset, vset);
    }
  }
}
```

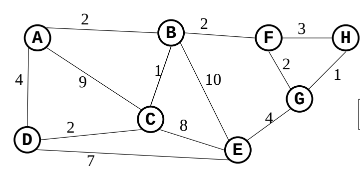
|E| heap ops

2|E| finds

|V| unions

Student Activity

Find MST using Kruskal's



Total Cost:

- Now find the MST using Prim's method.
- Under what conditions will these methods give the same result?