Trees

CSE 373
Data Structures & Algorithms
Ruth Anderson
Spring 2008

Today's Outline

- **Admin**:
  - HW #1 due thurs 4/10 at 11:59pm,
  - Bring printouts to class Friday 4/11
- **Math**
- **Trees!**

Math Fundamentals & Asymptotic Analysis

Powers of 2

- Many of the numbers we use in Computer Science are powers of 2
- Binary numbers (base 2) are easily represented in digital computers
  - each "bit" is a 0 or a 1
  - an n-bit wide field can represent how many different things?

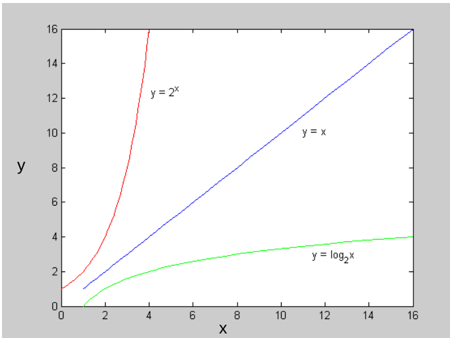
0000000000101011

Unsigned binary numbers

- For **unsigned** numbers in a fixed width field
  - the minimum value is 0
  - the maximum value is 2^n-1, where n is the number of bits in the field
  - The value is $\sum_{i=0}^{i=n-1} a_i 2^i$
- Each bit position represents a power of 2 with $a_i = 0$ or $a_i = 1$

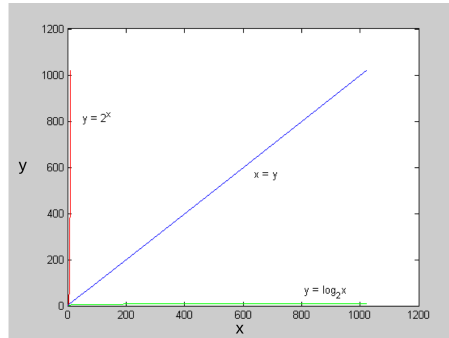
Logs and exponents

- Definition: $\log_2 x = y$ means $x = 2^y$
  - $8 = 2^3$, so $\log_2 8 = 3$
  - $65536 = 2^{16}$, so $\log_2 65536 = 16$
- Notice that $\log_2 x$ tells you how many bits are needed to hold x values
  - 8 bits holds 256 numbers: 0 to 2^8-1 = 0 to 255
  - $\log_2 256 = 8$




```
x = 0:.1:4
y = 2.^x
plot(x,y,'r')
hold on
plot(y,x,'g')
plot(y,y,'b')
```

x, 2^x and $\log_2 x$




```
x = 0:10
y = 2.^x
plot(x,y,'r')
hold on
plot(y,x,'g')
plot(y,y,'b')
```

2^x and $\log_2 x$

Floor and Ceiling

$\lfloor X \rfloor$ Floor function: the largest integer $\leq$ X

$\lfloor 2.7 \rfloor = 2 \qquad \lfloor -2.7 \rfloor = -3 \qquad \lfloor 2 \rfloor = 2$

$\lceil X \rceil$ Ceiling function: the smallest integer $\geq$ X

$\lceil 2.3 \rceil = 3 \qquad \lceil -2.3 \rceil = -2 \qquad \lceil 2 \rceil = 2$

Facts about Floor and Ceiling

1. $X - 1 < \lfloor X \rfloor \leq X$
2. $X \leq \lceil X \rceil < X + 1$
3. $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$ if n is an integer

Properties of logs

- We will assume logs to base 2 unless specified otherwise
- log AB = log A + log B
  - $A = 2^{\log_2 A}$ and $B = 2^{\log_2 B}$
  - $AB = 2^{\log_2 A} \bullet 2^{\log_2 B} = 2^{\log_2 A + \log_2 B}$
  - so $\log_2 AB = \log_2 A + \log_2 B$
  - [note: log AB $\neq$ log A•log B]

Other log properties

- log A/B = log A – log B
- log (A^B) = B log A
- log log X < log X < X for all X > 0
  - log log X = Y means $2^{2^Y} = X$
  - log X grows slower than X
    - called a “sub-linear” function

A log is a log is a log

- Any base x log is equivalent to base 2 log within a constant factor

$$\log_x B = \log_x B$$

$$B = 2^{\log_2 B} \qquad x = 2^{\log_2 x}$$

substitution

$$x^{\log_x B} = B \qquad x^{\log_x B} = B \text{ by def. of logs}$$

$$(2^{\log_2 x})^{\log_x B} = 2^{\log_2 B}$$

$$2^{\log_2 x \log_x B} = 2^{\log_2 B}$$

$$\log_2 x \log_x B = \log_2 B$$

$$\boxed{\log_x B = \frac{\log_2 B}{\log_2 x}}$$

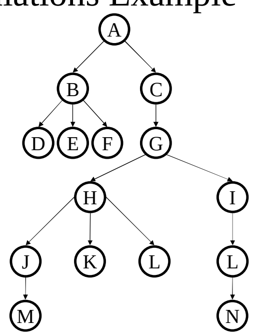
Trees

BSTs, and AVL Trees

Chapter 4 in Weiss

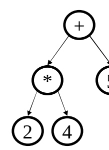
Tree Calculations Example

How high is this tree?



More Recursive Tree Calculations: Tree Traversals

A *traversal* is an order for visiting all the nodes of a tree



(an expression tree)

Three types:

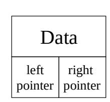
- <u>Pre-order</u>: Root, left subtree, right subtree
- <u>In-order</u>: Left subtree, root, right subtree

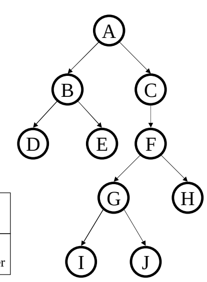
Traversals

```
void traverse(BNode t){
  if (t != NULL)
    traverse (t.left);
    print t.element;
    traverse (t.right);
  }
}
```

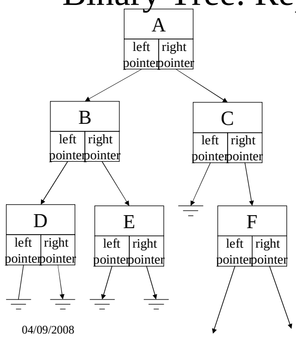

Binary Trees

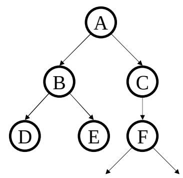
- Binary tree is
  - a root
  - left subtree *(maybe empty)*
  - right subtree *(maybe empty)*
- Representation:



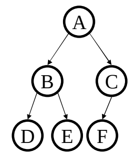


Binary Tree: Representation

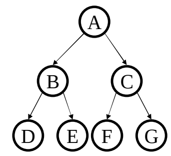




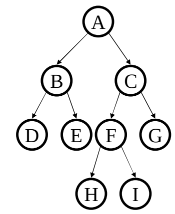
Binary Tree: Special Cases



Complete Tree



Perfect Tree



Full Tree

ADTs Seen So Far

- Stack
  - Push
  - Pop
- Queue
  - Enqueue
  - Dequeue

The Dictionary ADT

- Data:
  - a set of (key, value) pairs
- Operations:
  - Insert (key, value)
  - Find (key)
  - Remove (key)

insert(rea, ….)

find(devynp)

- devynp
  Devy Pranowo
  …

- rea
  Ruth Anderson
  OH: M 12:30-1:30
  CSE 360
- sang
  Tian Sang,
  OH: W & TH 4:30-5:30
  CSE 220
- ericm6
  Eric McCambridge
  OH: Th 1:30-2:30
  CSE 218
- devynp
  Devy Pranowo
  OH: W 1:30-2:30
  CSE 218

*The Dictionary ADT is sometimes called the "**Map ADT**"*

A Modest Few Uses

- Student, Customer records
- Networks : Router tables
- Operating systems : Page tables
- Compilers : Symbol tables

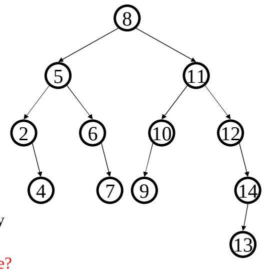
Probably the most widely used ADT!

Implementations

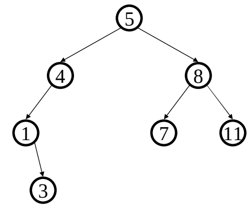
	insert	find	delete
• Unsorted Linked-list			
• Unsorted array			
• Sorted array			

Binary Search Tree Data Structure

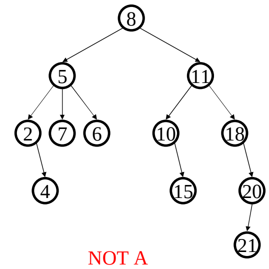
- Structural property
  - each node has ≤ 2 children
  - result:
    - storage is small
    - operations are simple
    - average depth is small
- Order property
  - all keys in left subtree smaller than root's key
  - all keys in right subtree larger than root's key
  - result: easy to find any given key
- What must I know about what I store?



Example and Counter-Example

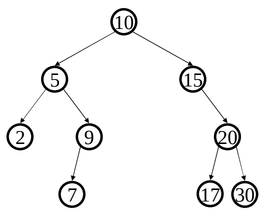


BINARY SEARCH TREE



NOT A
BINARY SEARCH TREE

Find in BST, Recursive

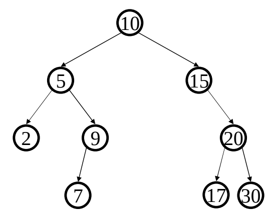



```
Node Find(Object key,
            Node root) {
  if (root == NULL)
    return NULL;

  if (key < root.key)
    return Find(key,
                root.left);
  else if (key > root.key)
    return Find(key,
                root.right);
  else
    return root;
}
```

Runtime:

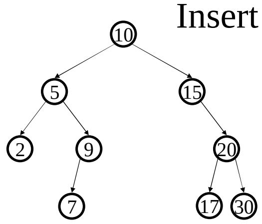
Find in BST, Iterative

```
Node Find(Object key,
            Node root) {

  while (root != NULL &&
         root.key != key) {
    if (key < root.key)
      root = root.left;
    else
      root = root.right;
  }

  return root;
}
```



Runtime:

Insert in BST



Insert(13)
Insert(8)
Insert(31)

Insertions happen only at the leaves – easy!

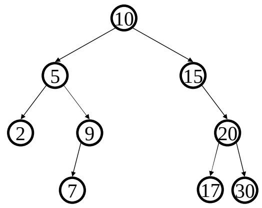
Runtime:

BuildTree for BST

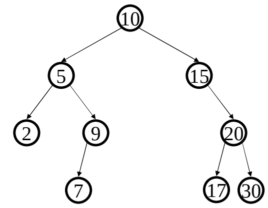
- Suppose keys 1, 2, 3, 4, 5, 6, 7, 8, 9 are inserted into an initially empty BST.
  Runtime depends on the order!
  - in given order
  - in reverse order
  - median first, then left median, right median, etc.

Bonus: FindMin/FindMax

- Find minimum
- Find maximum



Deletion in BST



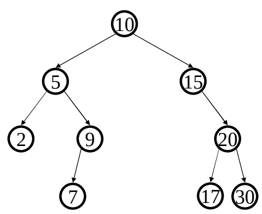
Why might deletion be harder than insertion?

Lazy Deletion

Instead of physically deleting nodes, just mark them as deleted

+ simpler
+ physical deletions done in batches
+ some adds just flip deleted flag

– extra memory for deleted flag
– many lazy deletions slow finds
– some operations may have to be modified (e.g., min and max)

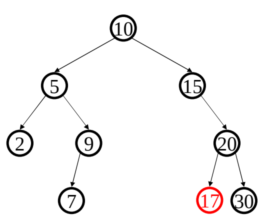


Non-lazy Deletion

- Removing an item disrupts the tree structure.
- Basic idea: find the node that is to be removed. Then "fix" the tree so that it is still a binary search tree.
- Three cases:
  - – node has no children (leaf node)
  - – node has one child
  - – node has two children

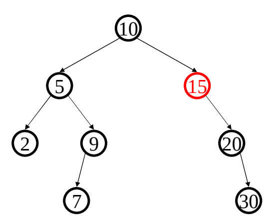
Non-lazy Deletion – The Leaf Case

Delete(17)



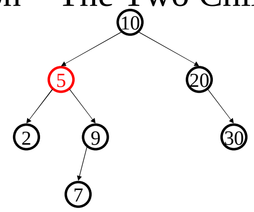
Deletion – The One Child Case

Delete(15)



Deletion – The Two Child Case

Delete(5)



What can we replace 5 with?

Deletion – The Two Child Case

Idea: Replace the deleted node with a value guaranteed to be between the two child subtrees!

Options:

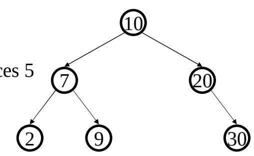
- *succ* from right subtree: findMin(t.right)
- *pred* from left subtree : findMax(t.left)

Now delete the original node containing *succ* or *pred*

- Leaf or one child case – easy!

Finally…

7 replaces 5



Original node containing
7 gets deleted

Balanced BST

Observation

- BST: the shallower the better!
- For a BST with *n* nodes
  - Average height is O(log *n*)
  - Worst case height is O(*n*)
- Simple cases such as insert(1, 2, 3, ..., n) lead to the worst case scenario

Solution: Require a **Balance Condition** that

1. ensures depth is O(log *n*) – strong enough!
2. is easy to maintain – not too strong!

Potential Balance Conditions

1. Left and right subtrees of the root have equal number of nodes

2. Left and right subtrees of the root have equal *height*

Potential Balance Conditions

3. Left and right subtrees of *every node* have equal number of nodes

4. Left and right subtrees of *every node* have equal *height*

The AVL Balance Condition

Left and right subtrees of *every node*
have equal *heights* **differing by at most 1**

Define: **balance**(x) = height(x.left) – height(x.right)

AVL property: **–1 ≤ balance(x) ≤ 1, for every node x**

- Ensures small depth
  - Will prove this by showing that an AVL tree of height h must have a lot of (i.e. $O(2^h)$) nodes
- Easy to maintain
  - Using single and double rotations

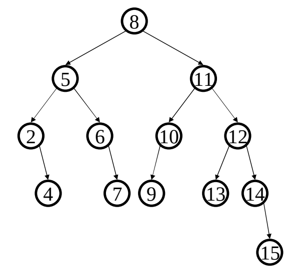
The AVL Tree Data Structure

Structural properties

1. Binary tree property
2. Balance property:
   balance of every node is between -1 and 1

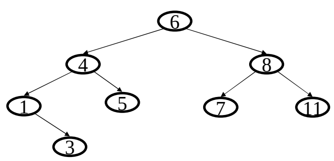
Result:

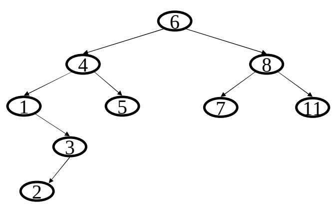
Worst case depth is O(log n)



Ordering property

– Same as for BST





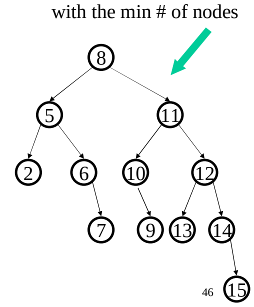
Proving Shallowness Bound

Let **S**(h) be the min # of nodes in an AVL tree of height h

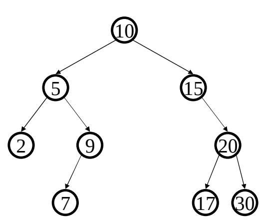
Claim: **S**(h) = **S**(h-1) + **S**(h-2) + 1

Solution of recurrence: **S**(h) = $O(2^h)$ (like Fibonacci numbers)

AVL tree of height h=4 with the min # of nodes



Testing the Balance Property



We need to be able to:

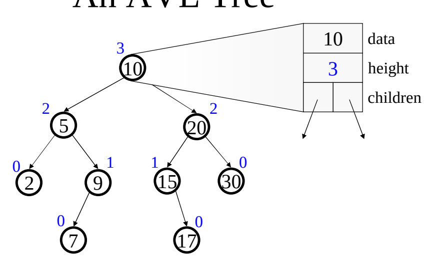
1.

2.

3.

`NULL`s have height **`-1`**

An AVL Tree



AVL trees: find, insert

- **AVL find**:
  - same as BST find.
- **AVL insert**:
  - same as BST insert, *except* may need to "fix" the AVL tree after inserting new value.

AVL tree insert

Let *x* be the node where an imbalance occurs.

Four cases to consider. The insertion is in the

1. left subtree of the left child of *x*.
2. right subtree of the left child of *x*.
3. left subtree of the right child of *x*.
4. right subtree of the right child of *x*.

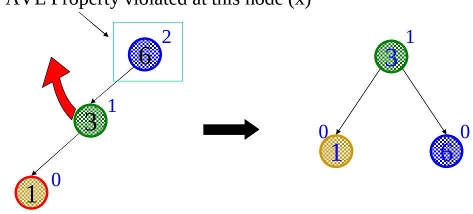
Idea: Cases 1 & 4 are solved by a single rotation.
Cases 2 & 3 are solved by a double rotation.

Bad Case #1

Insert(6)
Insert(3)
Insert(1)

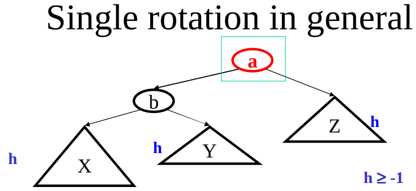
Fix: Apply Single Rotation

AVL Property violated at this node (x)

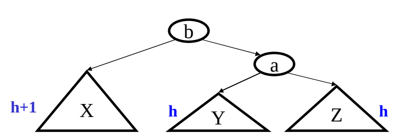


Single Rotation:
1. Rotate between x and child

Single rotation in general

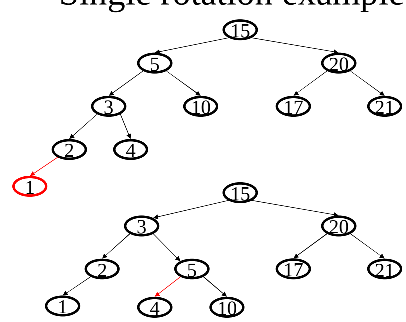


X < b < Y < a < Z



Height of tree before? Height of tree after? Effect on Ancestors?

Single rotation example

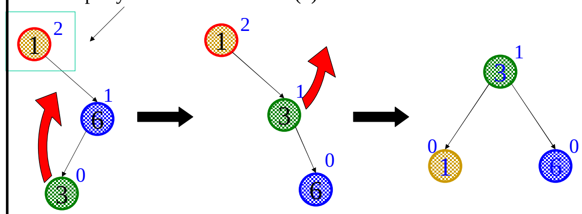


Bad Case #2

Insert(1)
Insert(6)
Insert(3)

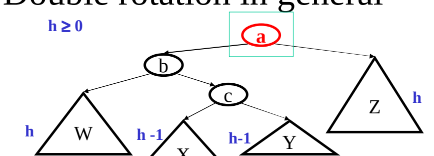
Fix: Apply Double Rotation

AVL Property violated at this node (x)

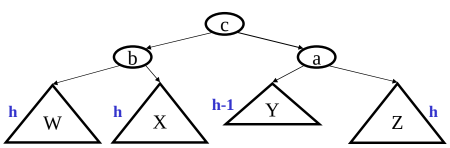


Double Rotation
1. Rotate between x's child and grandchild
2. Rotate between x and x's new child

Double rotation in general

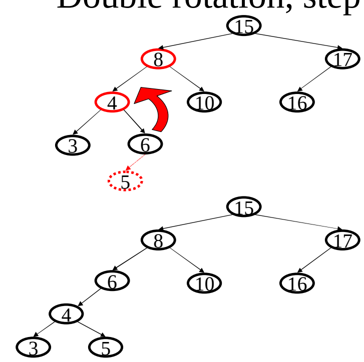


W < b <X < c < Y < a < Z

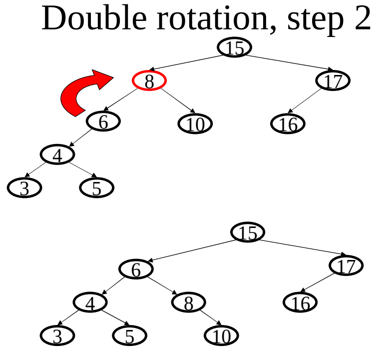


Height of tree before? Height of tree after? Effect on Ancestors?

Double rotation, step 1



Double rotation, step 2



Imbalance at node X

Single Rotation
1. Rotate between x and child

Double Rotation
1. Rotate between x's child and grandchild
2. Rotate between x and x's new child

Insert into an AVL tree: a b e c d

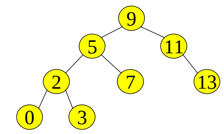
Student Activity

Circle your final answer

Single and Double Rotations:

Inserting what integer values
would cause the tree to need a:

1. single rotation?



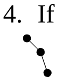
2. double rotation?

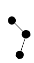
3. no rotation?

Student Activity

Insertion into AVL tree

1. Find spot for new key
2. Hang new node there with this key
3. Search back up the path for imbalance
4. If there is an imbalance:

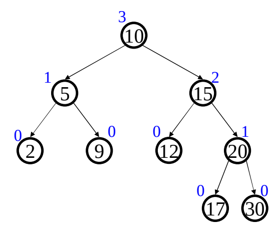

case #1: Perform single rotation and exit


case #2: Perform double rotation and exit

Both rotations keep the subtree height unchanged.
Hence only one rotation is sufficient!

Easy Insert

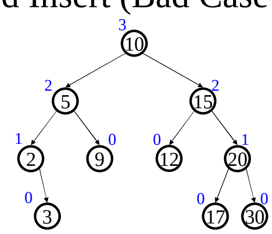
Insert(3)



Unbalanced?

Hard Insert (Bad Case #1)

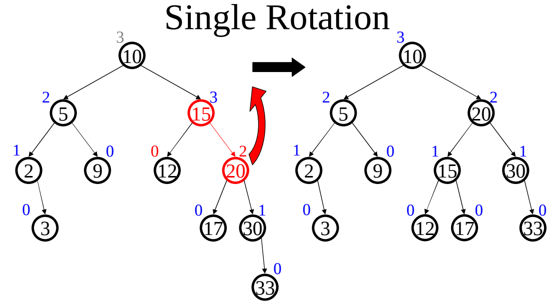
Insert(33)



Unbalanced?

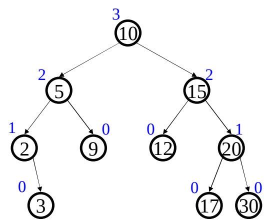
How to fix?

Single Rotation



Hard Insert (Bad Case #2)

Insert(18)

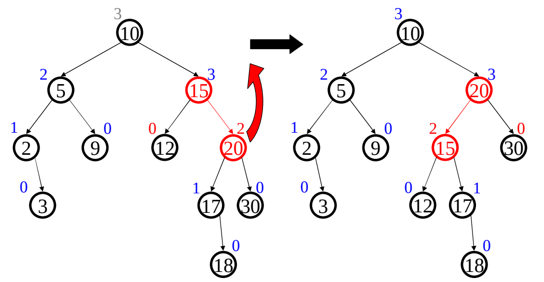


Unbalanced?

How to fix?

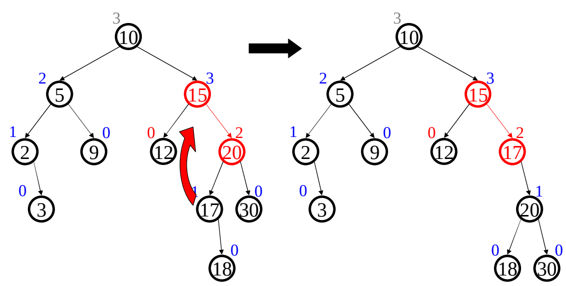
04/09/2008 67

Single Rotation (oops!)



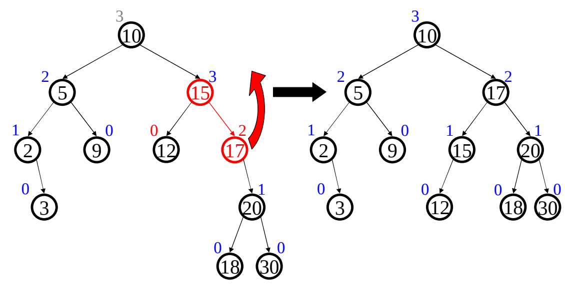
04/09/2008 68

Double Rotation (Step #1)



04/09/2008 69

Double Rotation (Step #2)



04/09/2008 70