Lecture 3: Boolean Algebra

- Boolean algebra
- Axioms
- Useful laws and theorems
- Examples

The big picture

- Part of the combinational logic topics (memoryless)
  - Different from sequential logic (can store information)
- Axioms and theorems allow you to…
  - … design logic functions
  - … know how to combine different logic gates
  - … simplify or optimize complex operations

Boolean algebra

- A Boolean algebra consists of…
  - a set of elements *B*
  - binary operators (+ , •)
  - unary operator (' or ¯)

Boolean algebra axioms

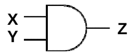
1. Closure:
   a+*b* is in *B*
   a•*b* is in *B*
2. Commutative:
   a+*b* = *b*+*a*
   a•*b* = *b*•*a*
3. Associative:
   a+(*b*+*c*) = (*a*+b)+c
   a•(*b*•*c*) = (*a*•*b*)•*c*
4. Identity:
   a+0 = *a*
   a•1 = *a*
5. Distributive:
   a+(*b*•*c*) = (a+b)•(*a*+*c*)
   a•(*b*+*c*) = (a•b)+(*a*•*c*)
6. Complementarity:
   a+*a*' = 1
   a•*a*' = 0

Binary logic

- Axioms hold for binary logic where
  - *B* = {0, 1}
  - • → AND
  - \+ → OR
  - ' → NOT
- A *Boolean function* maps some number of inputs over {0, 1} into an output set {0, 1}

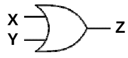
Logic gates and truth tables

- AND X•Y XY



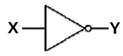
X	Y	Z
0	0	0
0	1	0
1	0	0
1	1	1

- OR X + Y



X	Y	Z
0	0	0
0	1	1
1	0	1
1	1	1

- NOT X' $\overline{X}$



X	Y
0	1
1	0

Boolean expressions

- *Any* logic function that is expressible as a truth table can be written in Boolean algebra.

Z=X•Y

X	Y	Z
0	0	0
0	1	0
1	0	0
1	1	1

Z=X'•Y

X	Y	X'	Z
0	0	1	0
0	1	1	1
1	0	0	0
1	1	0	0

Z=(X•Y)+(X'•Y')

X	Y	X'	Y'	X • Y	X' •Y'	Z
0	0	1	1	0	1	1
0	1	1	0	0	0	0
1	0	0	1	0	0	0
1	1	0	0	1	0	1

Precedence

1. Parentheses
2. NOT
3. AND
4. OR

Example: $\overline{A}$+B•$\overline{C}$ = ($\overline{A}$)+(B•($\overline{C}$))

Duality

- Duality (a meta-theorem—*a theorem about theorems*)
  - All Boolean expressions have logical duals
  - Any theorem that can be proved is also proved for its dual
  - Replace: • with +, + with •, 0 with 1, and 1 with 0
  - Leave the variables unchanged
- Example:
  The dual of *X*+0=*X* is *X*•1=*X*

Useful laws and theorems

- Identity
  X+0 = *X* Dual: *X*•1 = *X*
- Null
  X+1 = 1 Dual: *X*•0 = 0
- Idempotent
  X+*X* = *X* Dual: *X*•*X* = *X*
- Involution
  (*X*')' = *X*
- Complementarity
  X + *X*' = 1 Dual: *X*•*X*' = 0

Useful laws and theorems

- Commutative
  X+*Y* = *Y*+*X* Dual: *X*•*Y* = *Y*•*X*
- Associative
  X+(*Y*+*Z*) = (*X*+*Y*)+*Z* Dual: X•(*Y*•*Z*) = (*X*•*Y*)•*Z*
- Distributive
  X•(*Y*+*Z*) = (*X*•*Y*)+(*X*•*Z*) Dual: X+(*Y*•*Z*) = (*X*+*Y*)•(*X*+*Z*)
- Uniting
  X•*Y*+*X*•*Y*' = *X* Dual: (*X*+*Y*)•(*X*+*Y*') = *X*

Useful laws and theorems

- Absorption
  X+*X*•*Y* = *X* Dual: *X*•(*X*+*Y*) = *X*
  (*X*+*Y*')•*Y* = *X*•*Y* Dual: (*X*•*Y*')+*Y*=*X*+*Y*
- Consensus
  X•*Y*+*Y*•*Z*+*X*'•*Z* = *X*•*Y*+*X*'•*Z*
  Dual: (*X*+*Y*)•(*Y*+*Z*)•(*X*'+*Z*) = (*X*+*Y*)•(*X*'+*Z*)
- Multiplying and factoring
  (*X*+*Y*)•(*X*'+*Z*) = *X*•*Z*+*X*'•*Y*
  Dual: *X*•*Y*+*X*'•*Z* = (*X*+*Z*)•(*X*'+*Y*)

DeMorgan's law

- Procedure for complementing Boolean functions
  - Replace: • with +, + with •, 0 with 1, and 1 with 0
  - Replace all variables with their complements
- Look familiar?
  - Duality: "Leave the variables unchanged"
  - However, duality and DeMorgan's are **NOT** the same thing!
- Example:
  The complement of $F = X \bullet \overline{Y}$ is $\overline{F} = \overline{X} + Y$

Proving theorems

- Example 1: Uniting theorem

$X \bullet Y + X \bullet Y'$	$= X \bullet (Y + Y')$	Distributive
	$= X \bullet (1)$	Complementarity
	$= X$	Identity

- Example 2: Absorption

$X + X \bullet Y$	$= X \bullet 1 + X \bullet Y$	Identity
	$= X \bullet (1 + Y)$	Distributive
	$= X \bullet (1)$	Null
	$= X$	Identity

Proving theorems

- Example 3: Consensus

$X \bullet Y + Y \bullet Z + X' \bullet Z$

$= XY + (1)YZ + X'Z$	Identity
$= XY + (X + X')YZ + X'Z$	Complementarity
$= XY + XYZ + X'YZ + X'Z$	Distributive
$= XY + X'YZ + X'Z$	Absorption
$\{AB + A = A\}$ with $A = XY$ and $B = Z$	
$= XY + X'Z$	Absorption
$\{AB + A = A\}$ with $A = X'Z$ and $B = Y$	

Applying DeMorgan's

- Find the complement of F=(A+B)•(A'+C).
- Answer: F'=(A'•B')+(A•C')

A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

A	B	C	F'
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

Logic simplification

Z = A'BC + AB'C' + AB'C + ABC' + ABC	
= A'BC + AB'(C' + C) + AB(C' + C)	Distributive
= A'BC + AB'(1) + AB(1)	Complementarity
= A'BC + AB' + AB	Identity
= A'BC + A(B' + B)	Distributive
= A'BC + A(1)	Complementarity
= A'BC + A	Identity
= BC + A	Absorption
$\{(X \bullet Y') + Y = X + Y\}$ with $X = BC$ and $Y = A$	