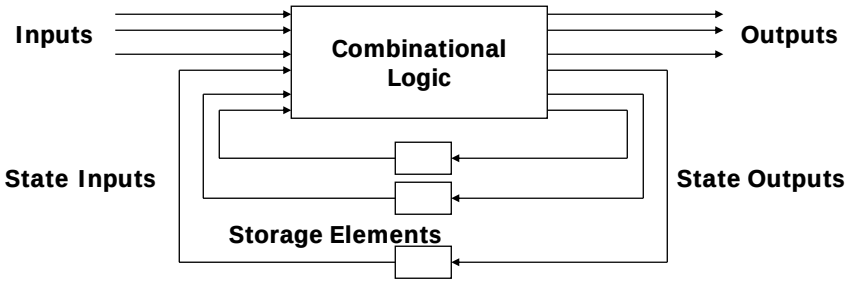
Finite State Machines

- Sequential circuits
  - primitive sequential elements
  - combinational logic
- Models for representing sequential circuits
  - finite-state machines (Moore and Mealy)
- Basic sequential circuits revisited
  - shift registers
  - counters
- Design procedure
  - state diagrams
  - state transition table
  - next state functions
- Hardware description languages

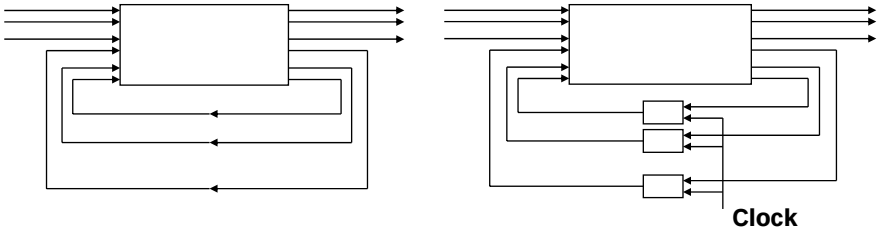
Abstraction of state elements

- Divide circuit into combinational logic and state
- Localize the feedback loops and make it easy to break cycles
- Implementation of storage elements leads to various forms of sequential logic



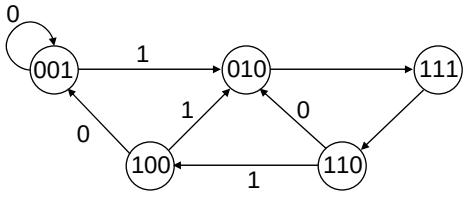
Forms of sequential logic

- Asynchronous sequential logic – state changes occur whenever state inputs change (elements may be simple wires or delay elements)
- Synchronous sequential logic – state changes occur in lock step across all storage elements (using a periodic waveform - the clock)



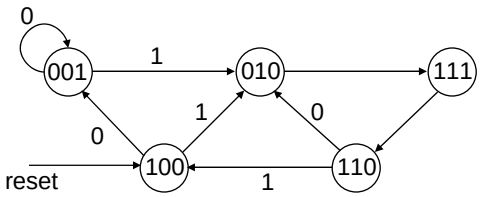
Finite state machine representations

- States: determined by possible values in sequential storage elements
- Transitions: change of state
- Clock: controls when state can change by controlling storage elements
- Sequential logic
  - sequences through a series of states
  - based on sequence of values on input signals
  - clock period defines elements of sequence



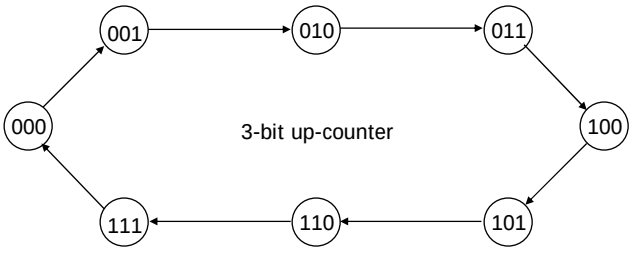
Example finite state machine diagram

- 5 states
- 8 other transitions between states
  - 6 conditioned by input
    - 1 self-transition (on 0 from 001 to 001)
  - 2 independent of input
- 1 reset transition (from all states) to state 100



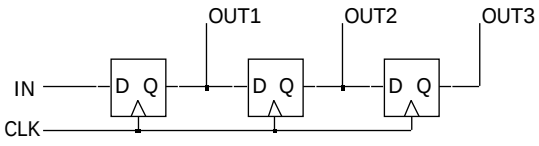
Counters are simple finite state machines

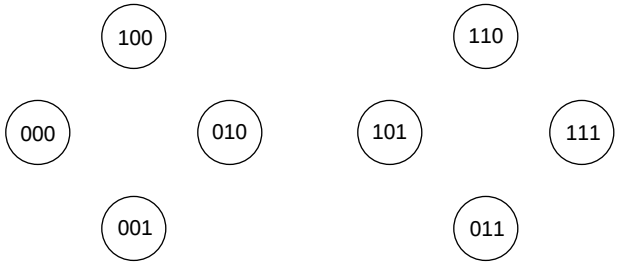
- Counters
  - proceed through well-defined sequence of states in response to enable
- Many types of counters: binary, BCD, Gray-code
  - 3-bit up-counter: 000, 001, 010, 011, 100, 101, 110, 111, 000, ...
  - 3-bit down-counter: 111, 110, 101, 100, 011, 010, 001, 000, 111, ...



Can any sequential system be represented with a state diagram?

- Shift register
  - input value shown on transition arcs
  - output values shown within state node





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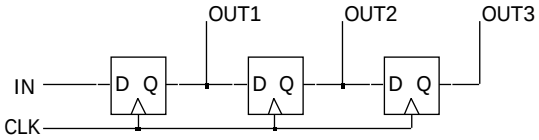
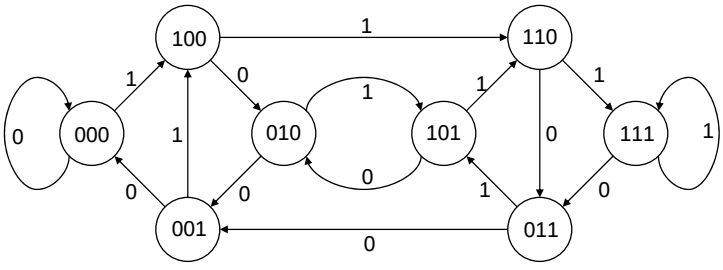
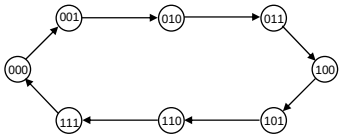
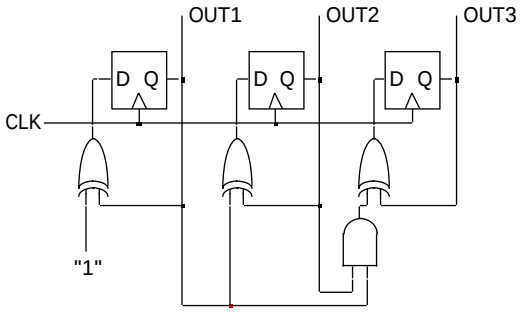




How do we turn a state diagram into logic?

- Counter
  - 3 flip-flops to hold state
  - logic to compute next state
  - clock signal controls when flip-flop memory can change
    - wait long enough for combinational logic to compute new value
    - don't wait too long as that is low performance



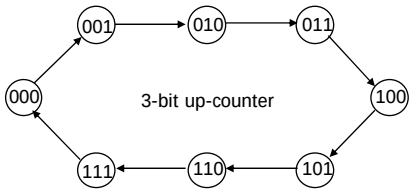


FSM design procedure

- We started with counters
  - simple because the output is just its state
  - simple because there is no input used to choose next state
- State diagram to state transition table
  - tabular form of state diagram
  - like a truth-table
- State encoding
  - decide on representation of states
  - for counters it is simple: just its value
- Implementation
  - flip-flop for each state bit
  - combinational logic based on encoding

FSM design procedure: state diagram to encoded state transition table

- Tabular form of state diagram
- Like a truth-table (specify output for all input combinations)
- Encoding of states: easy for counters – just use value



current state		next state	
0	000	001	1
1	001	010	2
2	010	011	3
3	011	100	4
4	100	101	5
5	101	110	6
6	110	111	7
7	111	000	0

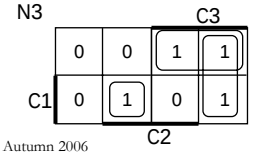
Implementation

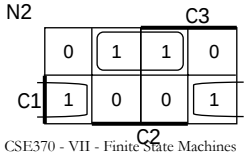
- D flip-flop for each state bit
- Combinational logic based on encoding

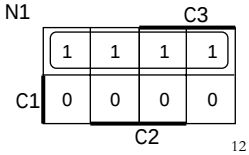
C3	C2	C1	N3	N2	N1
0	0	0	0	0	1
0	0	1	0	1	0
0	1	0	0	1	1
0	1	1	1	0	0
1	0	0	1	0	1
1	0	1	1	1	0
1	1	0	1	1	1
1	1	1	0	0	0

Verilog notation to show function represents an input to D-FF

```
N1 <= C1'
N2 <= C1C2' + C1'C2
   <= C1 xor C2
N3 <= C1C2C3' + C1'C3 + C2'C3
   <= (C1C2)C3' + (C1' + C2')C3
   <= (C1C2)C3' + (C1C2)'C3
   <= (C1C2) xor C3
```



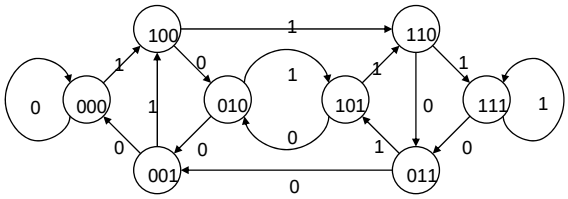




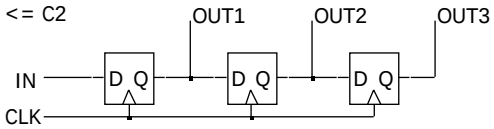
Back to the shift register

- Input determines next state

In	C1	C2	C3	N1	N2	N3
0	0	0	0	0	0	0
0	0	0	1	0	0	0
0	0	1	0	0	0	1
0	0	1	1	0	0	1
0	1	0	0	0	1	0
0	1	0	1	0	1	0
0	1	1	0	0	1	1
0	1	1	1	0	1	1
1	0	0	0	1	0	0
1	0	0	1	1	0	0
1	0	1	0	1	0	1
1	0	1	1	1	0	1
1	1	0	0	1	1	0
1	1	0	1	1	1	0
1	1	1	0	1	1	1
1	1	1	1	1	1	1

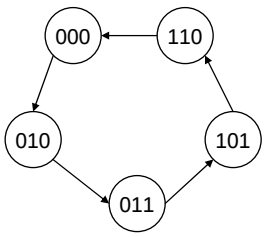


N1 <= In
N2 <= C1
N3 <= C2



More complex counter example

- Complex counter
  - repeats 5 states in sequence
  - not a binary number representation
- Step 1: derive the state transition diagram
  - count sequence: 000, 010, 011, 101, 110
- Step 2: derive the state transition table from the state transition diagram

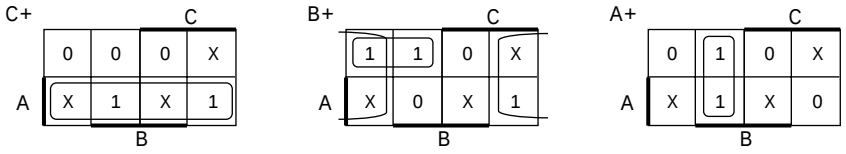


Present State			Next State		
C	B	A	C+	B+	A+
0	0	0	0	1	0
0	0	1	–	–	–
0	1	0	0	1	1
0	1	1	1	0	1
1	0	0	–	–	–
1	0	1	1	1	0
1	1	0	0	0	0
1	1	1	–	–	–

note the don't care conditions that arise from the unused state codes

More complex counter example (cont'd)

- Step 3: K-maps for next state functions



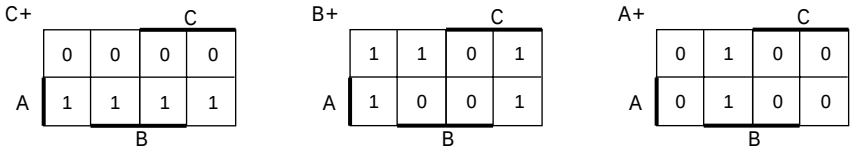
C+ <= A

B+ <= B' + A'C'

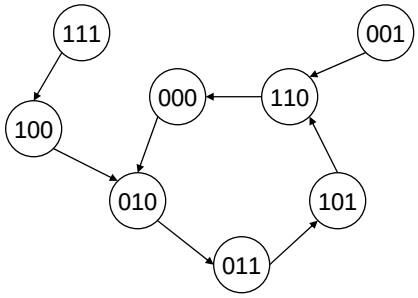
A+ <= BC'

Self-starting counters (cont'd)

- Re-deriving state transition table from don't care assignment

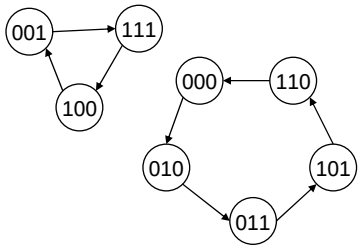


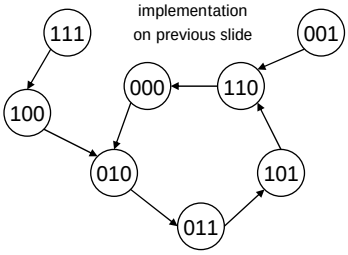
Present State			Next State		
C	B	A	C+	B+	A+
0	0	0	0	1	0
0	0	1	1	1	0
0	1	0	0	1	1
0	1	1	1	0	1
1	0	0	0	1	0
1	0	1	1	1	0
1	1	0	0	0	0
1	1	1	1	0	0



Self-starting counters

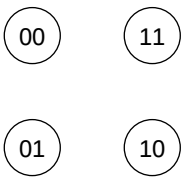
- Start-up states
  - at power-up, counter may be in an unused or invalid state
  - designer must guarantee that it (eventually) enters a valid state
- Self-starting solution
  - design counter so that invalid states eventually transition to a valid state
  - may limit exploitation of don't cares





Activity

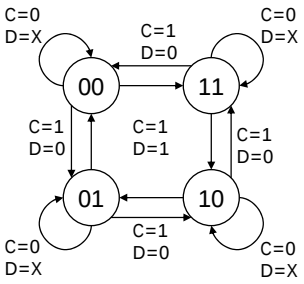
- 2-bit up-down counter (2 inputs)
  - direction: D = 0 for up, D = 1 for down
  - count: C = 0 for hold, C = 1 for count



S1	S0	C	D	N1	N0

Activity

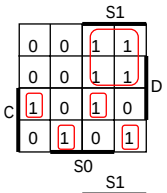
- 2-bit up-down counter (2 inputs)
  - direction: D = 0 for up, D = 1 for down
  - count: C = 0 for hold, C = 1 for count



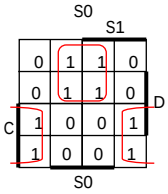
S1	S0	C	D	N1	N0
0	0	0	0	0	0
0	0	0	1	0	0
0	0	1	0	0	1
0	0	1	1	1	1
0	1	0	0	0	1
0	1	0	1	0	1
0	1	1	0	1	0
0	1	1	1	0	0
1	0	0	0	1	0
1	0	0	1	1	0
1	0	1	0	1	1
1	0	1	1	0	1
1	1	0	0	1	1
1	1	0	1	1	1
1	1	1	0	0	0
1	1	1	1	1	0

Activity (cont'd)

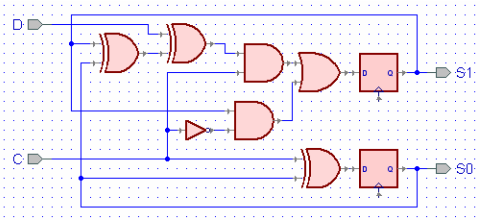
S1	S0	C	D	N1	N0
0	0	0	0	0	0
0	0	0	1	0	0
0	0	1	0	0	1
0	0	1	1	1	1
0	1	0	0	0	1
0	1	0	1	0	1
0	1	1	0	1	0
0	1	1	1	0	0
1	0	0	0	1	0
1	0	0	1	1	0
1	0	1	0	1	1
1	0	1	1	0	1
1	1	0	0	1	1
1	1	0	1	1	1
1	1	1	0	0	0
1	1	1	1	1	0



N1 = C'S1
+ CDS0'S1' + CDS0S1
+ CD'S0S1' + CD'S0'S1
= C'S1
+ C(D'(S1 ⊕ S0) + D(S1 ≡ S0))
= C'S1 + C (D ⊕ (S1 ⊕ S0))

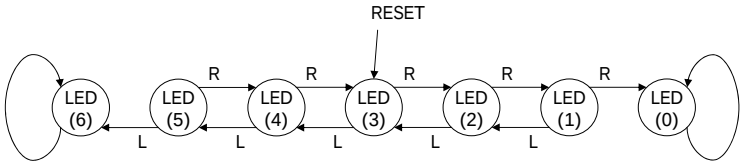


N0 = CS0' + C'S0

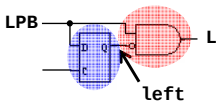


Tug-of-War Game FSM

- Tug of War game
  - 7 LEDs, 2 push buttons (L, R)



Light Game FSM Verilog



positive edge detector

```
module Light_Game (LEDS, LPB, RPB, CLK, RESET);

   input LPB ;
   input RPB ;
   input CLK ;
   input RESET;
   output [6:0] LEDS ;

   reg [6:0] position;
   reg left;
   reg right;
```

combinational logic

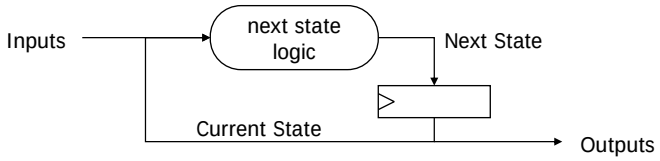
```
   wire L, R;
   assign L = ~left && LPB;
   assign R = ~right && RPB;
   assign LEDS = position;
```

sequential logic

```
   always @(posedge CLK)
        begin
             left <= LPB;
             right <= RPB;
             if (RESET) position <= 7'b0001000;
             else if ((position == 7'b0000001) || (position == 7'b1000000)) ;
             else if (L) position <= position << 1;
             else if (R) position <= position >> 1;
        end

endmodule
```

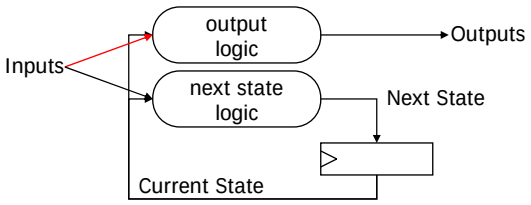
Counter/shift-register model

- Values stored in registers represent the state of the circuit
- Combinational logic computes:
  - next state
    - function of current state and inputs
  - outputs
    - values of flip-flops



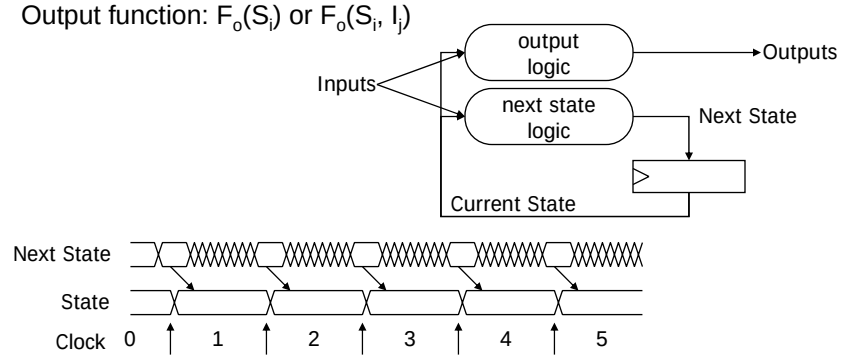
General state machine model

- Values stored in registers represent the state of the circuit
- Combinational logic computes:
  - next state
    - function of current state and inputs
  - outputs
    - function of current state and inputs (Mealy machine)
    - function of current state only (Moore machine)



State machine model (cont'd)

- States: S_1, S_2, ..., S_k
- Inputs: I_1, I_2, ..., I_m
- Outputs: O_1, O_2, ..., O_n
- Transition function: $F_s(S_i, I_j)$
- Output function: $F_o(S_i)$ or $F_o(S_i, I_j)$

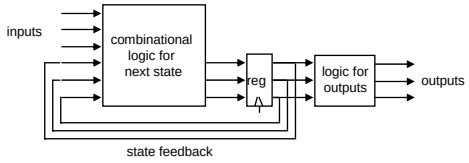


Comparison of Mealy and Moore machines

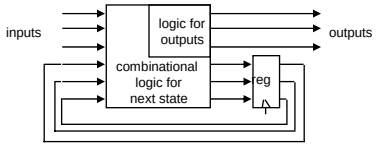
- Mealy machines tend to have less states
  - different outputs on arcs (n2) rather than states (n)
- Moore machines are safer to use
  - outputs change at clock edge (always one cycle later)
  - in Mealy machines, input change can cause output change as soon as logic is done – a big problem when two machines are interconnected – asynchronous feedback may occur if one isn't careful
- Mealy machines react faster to inputs
  - react in same cycle – don't need to wait for clock
  - in Moore machines, more logic may be necessary to decode state into outputs – more gate delays after clock edge

Comparison of Mealy and Moore machines (cont'd)

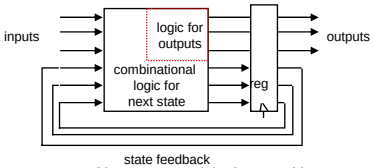
- Moore



- Mealy

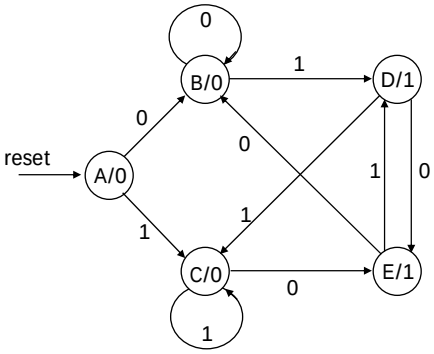


- Synchronous Mealy



Specifying outputs for a Moore machine

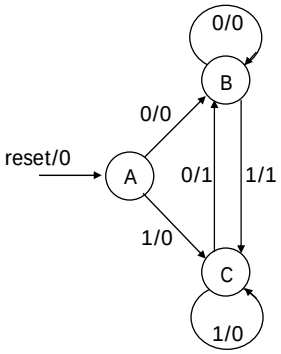
- Output is only function of state
  - specify in state bubble in state diagram
  - example: sequence detector for 01 or 10



reset	input	current state	next state	output
1	–	–	A	
0	0	A	B	0
0	1	A	C	0
0	0	B	B	0
0	1	B	D	0
0	0	C	E	0
0	1	C	C	0
0	0	D	E	1
0	1	D	C	1
0	0	E	B	1
0	1	E	D	1

Specifying outputs for a Mealy machine

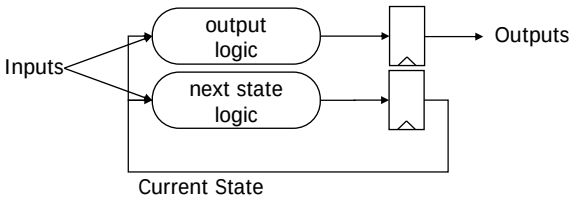
- Output is function of state and inputs
  - specify output on transition arc between states
  - example: sequence detector for 01 or 10



reset	input	current state	next state	output
1	–	–	A	0
0	0	A	B	0
0	1	A	C	0
0	0	B	B	0
0	1	B	C	1
0	0	C	B	1
0	1	C	C	0

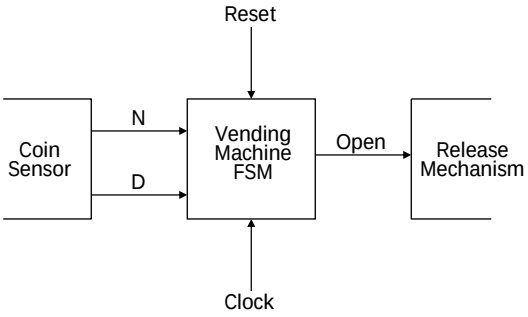
Registered Mealy machine (really Moore)

- Synchronous (or registered) Mealy machine
  - registered state AND outputs
  - avoids 'glitchy' outputs
  - easy to implement in PLDs
- Moore machine with no output decoding
  - outputs computed on transition to next state rather than after entering
  - view outputs as expanded state vector



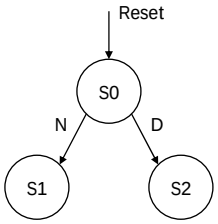
Example: vending machine

- Release item after 15 cents are deposited
- Single coin slot for dimes, nickels
- No change



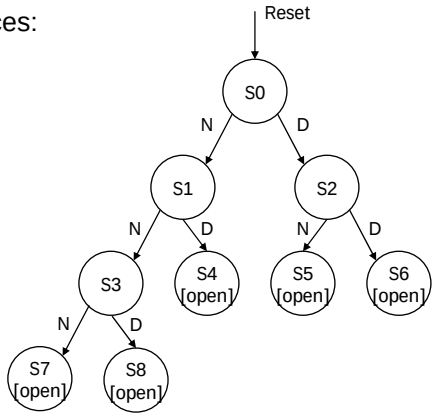
Example: vending machine (cont'd)

- Suitable abstract representation
  - tabulate typical input sequences:
    - 3 nickels
    - nickel, dime
    - dime, nickel
    - two dimes
  - draw state diagram:
    - inputs: N, D, reset
    - output: open chute
  - assumptions:
    - assume N and D asserted for one cycle
    - each state has a self loop for N = D = 0 (no coin)



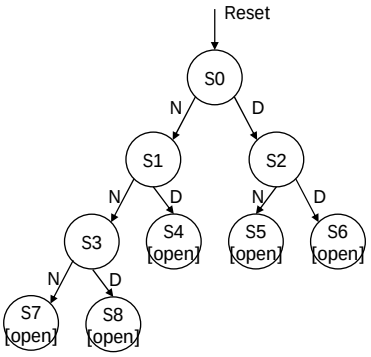
Example: vending machine (cont'd)

- Suitable abstract representation
  - tabulate typical input sequences:
    - 3 nickels
    - nickel, dime
    - dime, nickel
    - two dimes
  - draw state diagram:
    - inputs: N, D, reset
    - output: open chute
  - assumptions:
    - assume N and D asserted for one cycle
    - each state has a self loop for N = D = 0 (no coin)



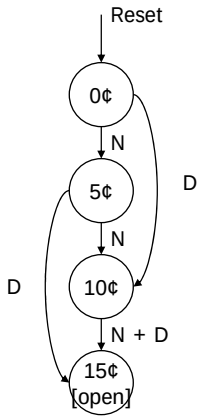
Activity: reuse states

- Redraw the state diagram using as few states as possible



Example: vending machine (cont'd)

- Minimize number of states - reuse states whenever possible



present state	inputs D	N	next state	output open
0¢	0	0	0¢	0
	0	1	5¢	0
	1	0	10¢	0
	1	1	–	–
5¢	0	0	5¢	0
	0	1	10¢	0
	1	0	15¢	0
	1	1	–	–
10¢	0	0	10¢	0
	0	1	15¢	0
	1	0	15¢	0
	1	1	–	–
15¢	–	–	15¢	1

symbolic state table

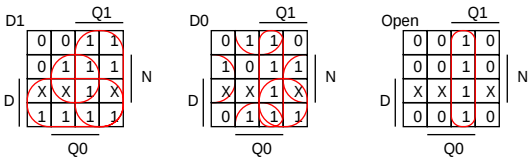
Example: vending machine (cont'd)

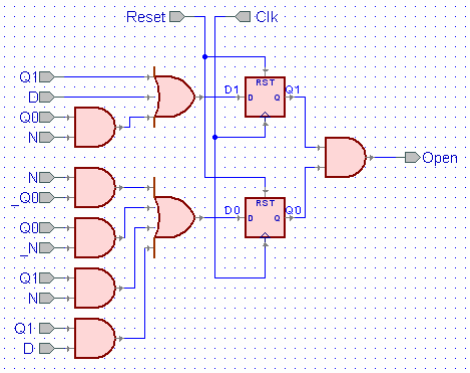
- Uniquely encode states

present state Q1	Q0	inputs D	N	next state D1	D0	output open
0	0	0	0	0	0	0
		0	1	0	1	0
		1	0	1	0	0
		1	1	–	–	–
0	1	0	0	0	1	0
		0	1	1	0	0
		1	0	1	1	0
		1	1	–	–	–
1	0	0	0	1	0	0
		0	1	1	1	0
		1	0	1	1	0
		1	1	–	–	–
1	1	–	–	1	1	1

Example: Moore implementation

- Mapping to logic





D1 = Q1 + D + Q0 N

D0 = Q0' N + Q0 N' + Q1 N + Q1 D

OPEN = Q1 Q0

Example: vending machine (cont'd)

- One-hot encoding

present state Q3 Q2 Q1 Q0	inputs D N	next state D3 D2 D1 D0	output open
0 0 0 1	0 0	0 0 0 1	0
	0 1	0 0 1 0	0
	1 0	0 1 0 0	0
	1 1	- - - -	-
0 0 1 0	0 0	0 0 1 0	0
	0 1	0 1 0 0	0
	1 0	1 0 0 0	0
	1 1	- - - -	-
0 1 0 0	0 0	0 1 0 0	0
	0 1	1 0 0 0	0
	1 0	1 0 0 0	0
	1 1	- - - -	-
1 0 0 0	- -	1 0 0 0	1

D0 = Q0 D' N'

D1 = Q0 N + Q1 D' N'

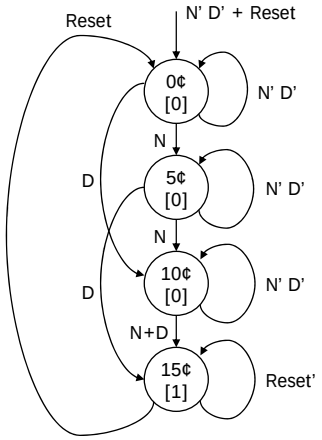
D2 = Q0 D + Q1 N + Q2 D' N'

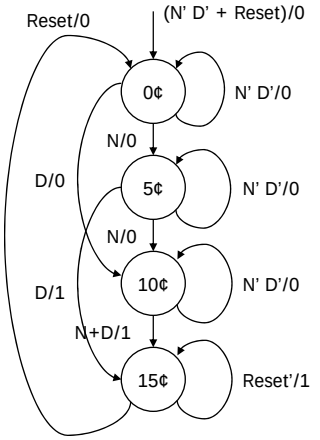
D3 = Q1 D + Q2 D + Q2 N + Q3

OPEN = Q3

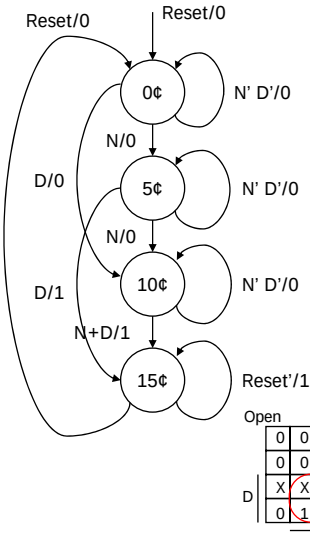
Equivalent Mealy and Moore state diagrams

- Moore machine
  - outputs associated with state
- Mealy machine
  - outputs associated with transitions

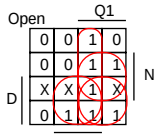




Example: Mealy implementation



present state Q1	Q0	inputs D	N	next state D1	D0	output open
0	0	0	0	0	0	0
		0	1	0	1	0
		1	0	1	0	0
		1	1	–	–	–
0	1	0	0	0	1	0
		0	1	1	0	0
		1	0	1	1	1
		1	1	–	–	–
1	0	0	0	1	0	0
		0	1	1	1	1
		1	0	1	1	1
		1	1	–	–	–
1	1	–	–	1	1	1



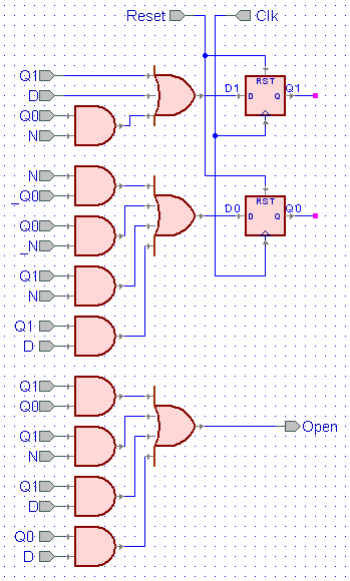
D0 = Q0'N + Q0N' + Q1N + Q1D

D1 = Q1 + D + Q0N

OPEN = Q1Q0 + Q1N + Q1D + Q0D

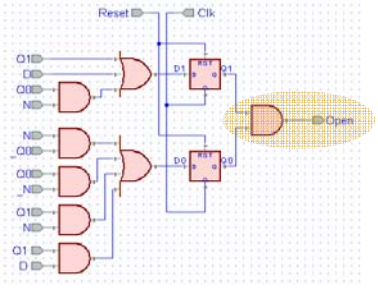
Example: Mealy implementation

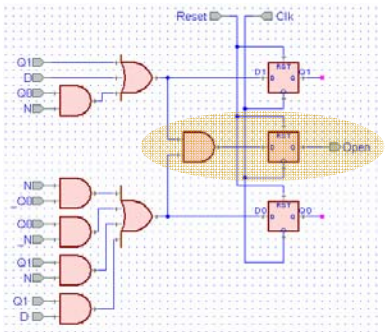
D0 = Q0'N + Q0N' + Q1N + Q1D
D1 = Q1 + D + Q0N
OPEN = Q1Q0 + Q1N + Q1D + Q0D



Vending machine: Moore to synch. Mealy

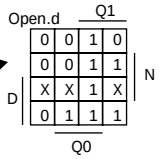
- OPEN = Q1Q0 creates a combinational delay after Q1 and Q0 change in Moore implementation
- This can be corrected by retiming, i.e., move flip-flops and logic through each other to improve delay
- OPEN.d = (Q1 + D + Q0N)(Q0'N + Q0N' + Q1N + Q1D)
  = Q1Q0N' + Q1N + Q1D + Q0'ND + Q0N'D
- Implementation now looks like a synchronous Mealy machine
  - another reason programmable devices have FF at end of logic

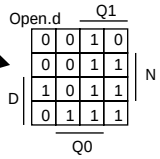


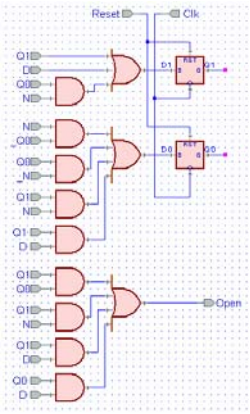


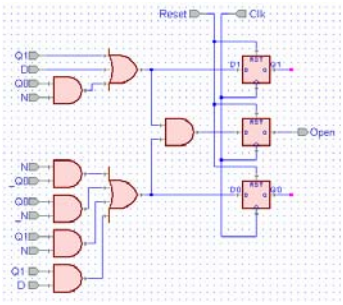
Vending machine: Mealy to synch. Mealy

- OPEN.d = Q1Q0 + Q1N + Q1D + Q0D
- OPEN.d = (Q1 + D + Q0N)(Q0'N + Q0N' + Q1N + Q1D)
  = Q1Q0N' + Q1N + Q1D + Q0'ND + Q0N'D



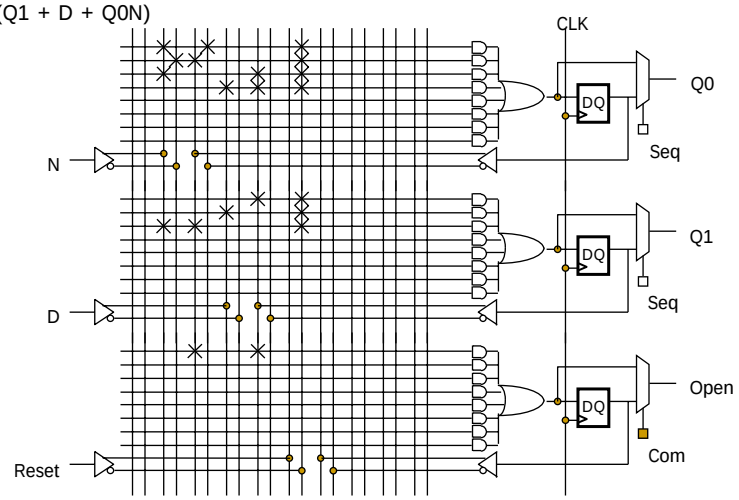






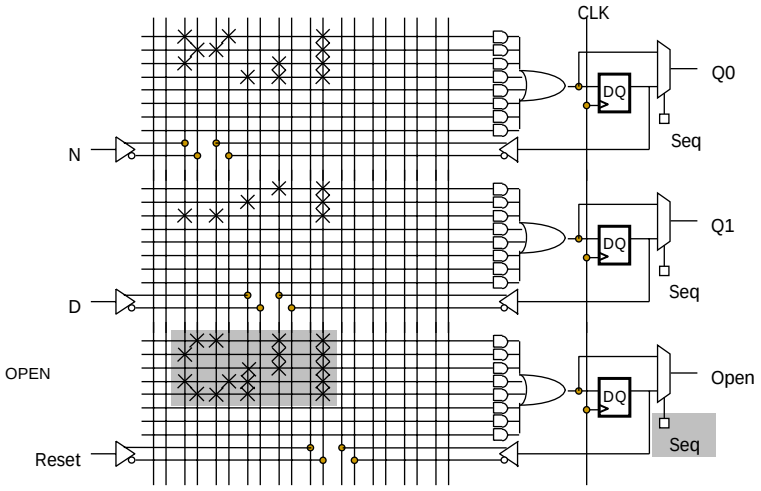
Vending machine example (Moore PLD mapping)

D0 = reset'(Q0'N + Q0N' + Q1N + Q1D)
D1 = reset'(Q1 + D + Q0N)
OPEN = Q1Q0



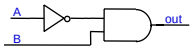
Vending machine (synch. Mealy PLD mapping)

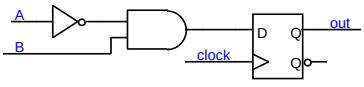
OPEN = reset'(Q1Q0N' + Q1N + Q1D + Q0'ND + Q0N'D)

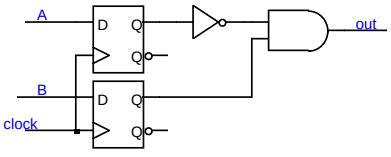


Mealy and Moore examples

- Recognize A,B = 0,1
  - Mealy or Moore?

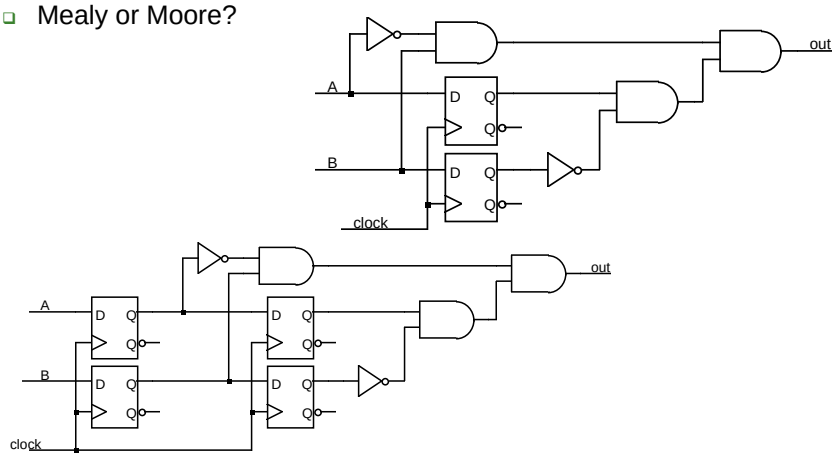






Mealy and Moore examples (cont'd)

- Recognize A,B = 1,0 then 0,1
  - Mealy or Moore?

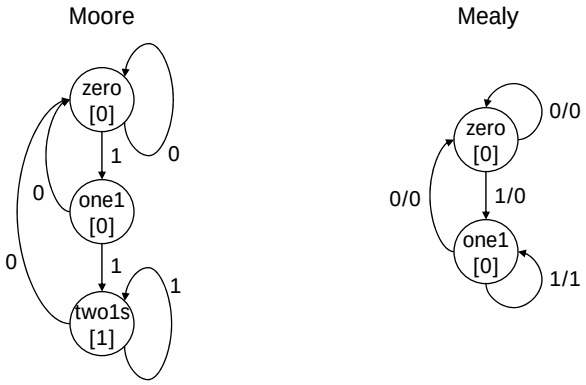


Hardware Description Languages and Sequential Logic

- Flip-flops
  - representation of clocks - timing of state changes
  - asynchronous vs. synchronous
- FSMs
  - structural view (FFs separate from combinational logic)
  - behavioral view (synthesis of sequencers – not in this course)
- Data-paths = data computation (e.g., ALUs, comparators) + registers
  - use of arithmetic/logical operators
  - control of storage elements

Example: reduce-1-string-by-1

- Remove one 1 from every string of 1s on the input

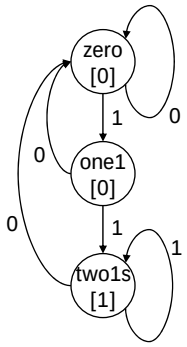


Verilog FSM - Reduce 1s example

- Moore machine

```
module reduce (clk, reset, in, out);
  input clk, reset, in;
  output out;

  parameter zero  = 2'b00;
  parameter one1  = 2'b01;
  parameter two1s = 2'b10;

  reg out;
  reg [2:1] state;       // state variables
  reg [2:1] next_state;

  always @(posedge clk)
    if (reset) state = zero;
    else       state = next_state;
```

state assignment
(easy to change,
if in one place)



Moore Verilog FSM (cont'd)

```
always @(in or state)

  case (state)
    zero:
   // last input was a zero
    begin
      if (in) next_state = one1;
      else    next_state = zero;
    end
     one1:
   // we've seen one 1
    begin
      if (in) next_state = two1s;
      else    next_state = zero;
    end
     two1s:
   // we've seen at least 2 ones
    begin
      if (in) next_state = two1s;
      else    next_state = zero;
    end
   endcase
```

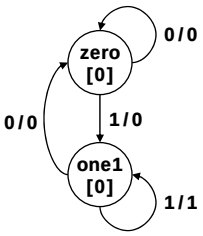
crucial to include all signals that are input to state determination

note that output depends only on state

```
always @(state)
  case (state)
    zero: out = 0;
     one1: out = 0;
    two1s: out = 1;
  endcase

endmodule
```


Mealy Verilog FSM

```
module reduce (clk, reset, in, out);
  input clk, reset, in;
  output out;
  reg out;
  reg state; // state variables
  reg next_state;

  always @(posedge clk)
    if (reset) state = zero;
    else       state = next_state;

  always @(in or state)
    case (state)
      zero:             // last input was a zero
      begin
        out = 0;
        if (in) next_state = one;
        else    next_state = zero;
      end
       one:             // we've seen one 1
      if (in) begin
          next_state = one; out = 1;
      end else begin
          next_state = zero; out = 0;
      end
    endcase
endmodule
```



Synchronous Mealy Machine

```
module reduce (clk, reset, in, out);
  input clk, reset, in;
  output out;
  reg out;
  reg state; // state variables

  always @(posedge clk)
    if (reset) state = zero;
    else
     case (state)
      zero:      // last input was a zero
     begin
       out = 0;
       if (in) state = one;
       else    state = zero;
     end
      one:       // we've seen one 1
     if (in) begin
        state = one; out = 1;
     end else begin
        state = zero; out = 0;
     end
    endcase
endmodule
```


Finite state machines summary

- Models for representing sequential circuits
  - abstraction of sequential elements
  - finite state machines and their state diagrams
  - inputs/outputs
  - Mealy, Moore, and synchronous Mealy machines
- Finite state machine design procedure
  - deriving state diagram
  - deriving state transition table
  - determining next state and output functions
  - implementing combinational logic
- Hardware description languages