Number systems

- Last lecture
  - Course overview
  - The Digital Age
- Today's lecture
  - Binary numbers
  - Base conversion
  - Number systems
    - Twos-complement
  - A/D and D/A conversion

Digital

- Digital = discrete
  - Binary codes (example: BCD)
  - Decimal digits 0-9
  - DNA nucleotides
- Binary codes
  - Represent symbols using binary digits (bits)
- Digital computers:
  - I/O is digital
    - ASCII, decimal, etc.
  - Internal representation is binary
    - Process information in bits

Decimal Symbols	BCD Code
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
8	1000
9	1001

The basics: Binary numbers

- Bases we will use
  - Binary: Base 2
  - Octal: Base 8
  - Hexadecimal: Base 16
- Positional number system
  - $101_2 = 1\times2^2 + 0\times2^1 + 1\times2^0$
  - $63_8 = 6\times8^1 + 3\times8^0$
  - $A1_{16} = 10\times16^1 + 1\times16^0$
- Addition and subtraction

$$\begin{array}{r} 1011 \\ +\ 1010 \\ \hline 10101 \end{array} \qquad \begin{array}{r} 1011 \\ -\ 0110 \\ \hline 0101 \end{array}$$

Binary → hex/decimal/octal conversion

- Conversion from binary to octal/hex
  - Binary: 10011110001
  - Octal: 10 | 011 | 110 | 001=2361_8
  - Hex: 100 | 1111 | 0001=$4F1_{16}$
- Conversion from binary to decimal
  - $101_2 = 1\times2^2 + 0\times2^1 + 1\times2^0 = 5_{10}$
  - $63.4_8 = 6\times8^1 + 3\times8^0 + 4\times8^{-1} = 51.5_{10}$
  - $A1_{16} = 10\times16^1 + 1\times16^0 = 161_{10}$

Decimal→ binary/octal/hex conversion

Binary

	Quotient	Remainder
56÷2=	28	0
28÷2=	14	0
14÷2=	7	0
7÷2=	3	1
3÷2=	1	1
1÷2=	0	1

Octal

	Quotient	Remainder
56÷8=	7	0
7÷8=	0	7

$56_{10}=111000_2$
$56_{10}=70_8$

- Why does this work?
  - $N=56_{10}=111000_2$
  - Q=N/2=56/2=111000/2=11100 remainder 0
- Each successive divide liberates an LSB

Number systems

- How do we write negative binary numbers?
- Historically: 3 approaches
  - Sign-and-magnitude
  - Ones-complement
  - Twos-complement
- For all 3, the most-significant bit (msb) is the sign digit
  - 0 ≡ positive
  - 1 ≡ negative
- Learn twos-complement
  - Simplifies arithmetic
  - Used almost universally

Sign-and-magnitude

- The most-significant bit (msb) is the sign digit
  - 0 ≡ positive
  - 1 ≡ negative
- The remaining bits are the number's magnitude
- Problem 1: Two representations for zero
  - 0 = 0000 *and also* –0 = 1000
- Problem 2: Arithmetic is cumbersome

Add		Subtract			Compare and subtract		
4 + 3	0100 + 0011	4 – 3	0100 + 1011	0100 – 0011	– 4 + 3	1100 + 0011	1100 – 0011
= 7	= 0111	= 1	≠ 1111	= 0001	– 1	≠ 1111	= 1001

Ones-complement

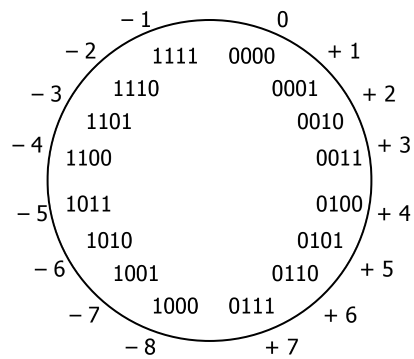
- Negative number: Bitwise complement positive number
  - 0011 ≡ 3_{10}
  - 1100 ≡ -3_{10}
- Solves the arithmetic problem

Add		Invert, add, add carry		Invert and add	
4 + 3	0100 + 0011	4 – 3	0100 + 1100	– 4 + 3	1011 + 0011
= 7	= 0111	= 1 add carry:	1 0000 +1	– 1	1110
			= 0001		

- Remaining problem: Two representations for zero
  - 0 = 0000 *and also* –0 = 1111

Twos-complement

- Negative number: Bitwise complement plus one
  - 0011 ≡ 3_{10}
  - 1101 ≡ -3_{10}
- Number wheel



- Only one zero!
- msb is the sign digit
  - 0 ≡ positive
  - 1 ≡ negative

Twos-complement (con't)

- Complementing a complement ➲ the original number
- Arithmetic is easy
  - Subtraction = negation and addition
    - Easy to implement in hardware

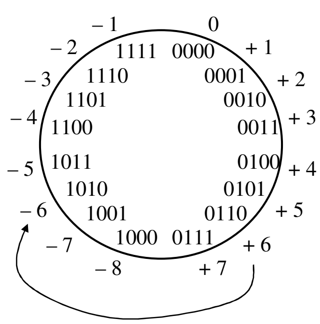
Add		Invert and add		Invert and add	
4 + 3	0100 + 0011	4 – 3	0100 + 1101	– 4 + 3	1100 + 0011
= 7	= 0111	= 1 drop carry	1 0001 = 0001	– 1	1111

Miscellaneous

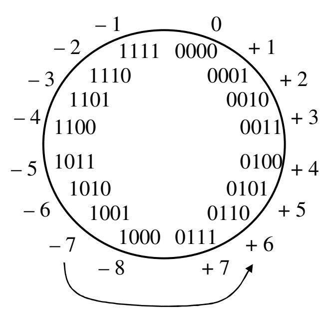
- Twos-complement of non-integers
  - $1.6875_{10} = 01.1011_2$
  - $-1.6875_{10} = 10.0101_2$
- Sign extension
  - Write +6 and –6 as twos complement
    - 0110 and 1010
  - Sign extend to 8-bit bytes
    - 00000110 and 11111010
- Can't infer a representation from a number
  - 11001 is 25 (unsigned)
  - 11001 is –9 (sign magnitude)
  - 11001 is –6 (ones complement)
  - 11001 is –7 (twos complement)

Twos-complement overflow

- Summing two positive numbers gives a negative result
- Summing two negative numbers gives a positive result



$6 + 4 \Rightarrow -6$



$-7 - 3 \Rightarrow +6$

Twos-complement overflow (cont'd)

- Correct results

1111	−1	0011	+3
+ 1010	−6	+ 0010	+2
~~1~~ 1001	−7	0101	+5

- Incorrect results

0110	+6	1001	−7
+ 0100	+4	+ 1010	−6
1010	−6	~~1~~ 0011	+3

- Overflow condition
  - Carry from 2sb-msb and carry from msb-Cout are different

2sb-msb	msb-Cout	Overflow
0	0	0
0	1	1
1	0	1
1	1	0

Gray and BCD codes

Decimal Symbols	Gray Code
0	0000
1	0001
2	0011
3	0010
4	0110
5	0111
6	0101
7	0100
8	1100
9	1101

Decimal Symbols	BCD Code
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
8	1000
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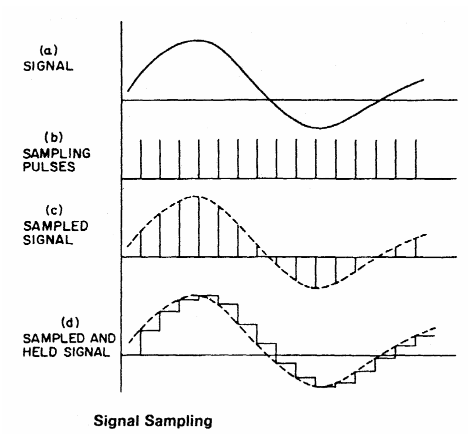
The physical world is analog

- Digital systems need to
  - Measure analog quantities
    - Speech waveforms, etc
  - Control analog systems
    - Drive motors, etc
- How do we connect the analog and digital domains?
  - Analog-to-digital converter (ADC or A/D)
    - Example: CD recording
  - Digital-to-analog converter (DAC or D/A)
    - Example: CD playback

Sampling

- **Quantization**
  - Conversion from analog to discrete values
- Quantizing a signal
  - We sample it



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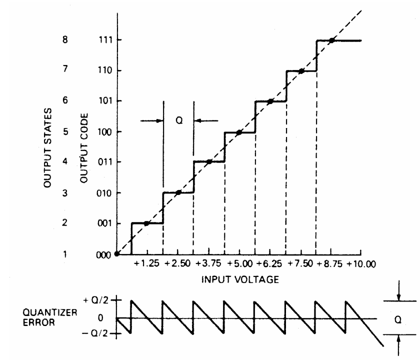
Conversion

- **Encoding**
  - Assigning a digital word to each discrete value
- Encoding a quantized signal
  - Encode the samples
  - Typically Gray or binary codes



Transfer Function of Ideal 3-Bit Quantizer

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