CSE 332 Autumn 2023
Lecture 25: Minimum Spanning Trees, P & NP

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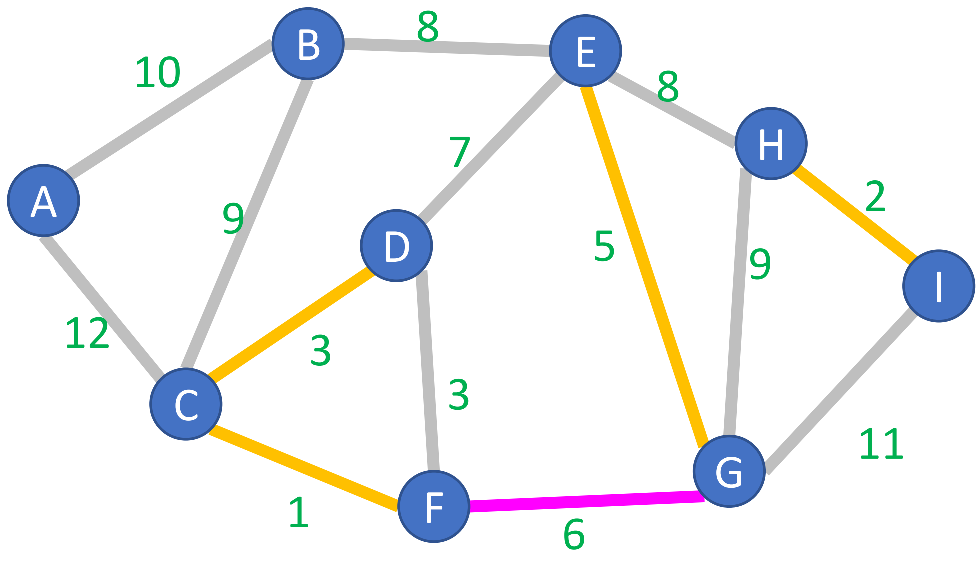
http://www.cs.uw.edu/332

Kruskal's Algorithm

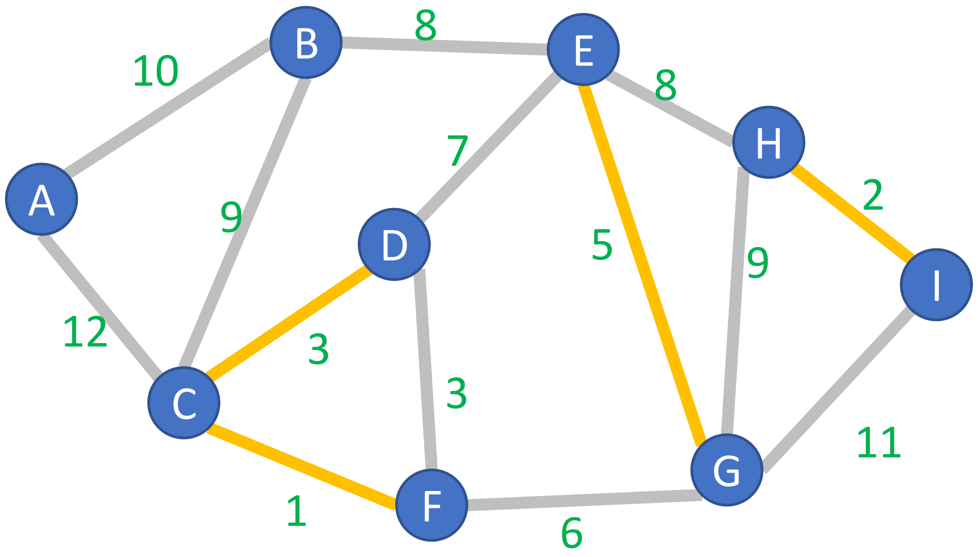
Start with an empty tree A

Add to A the lowest-weight edge that does not create a cycle



Cut Theorem

If a set of edges A is a subset of a minimum spanning tree T, let $(S, V - S)$ be any cut which A respects. Let e be the least-weight edge which crosses $(S, V - S)$. $A \cup \{e\}$ is also a subset of a minimum spanning tree.



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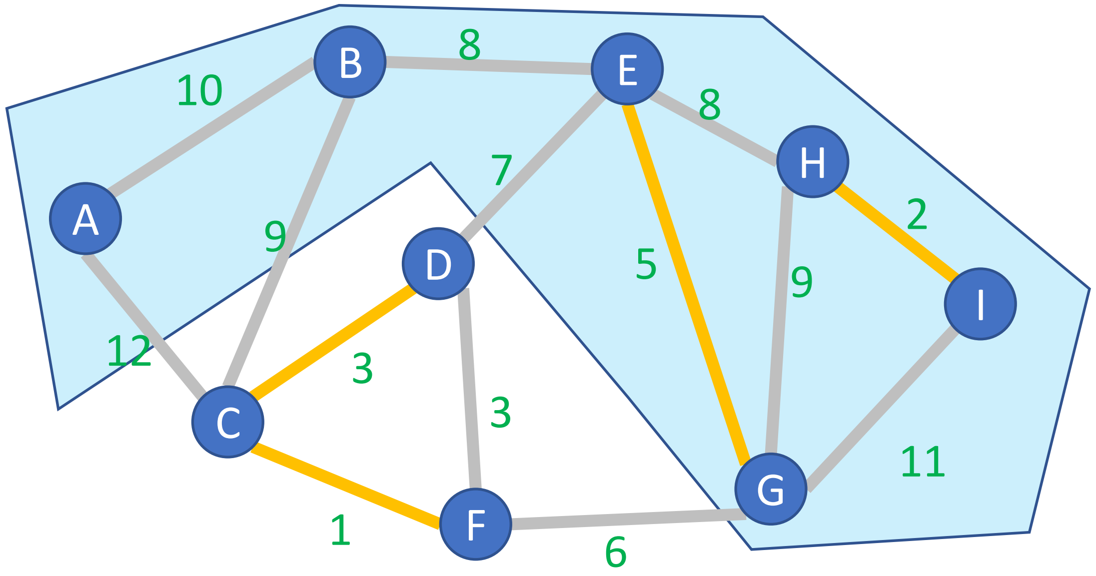

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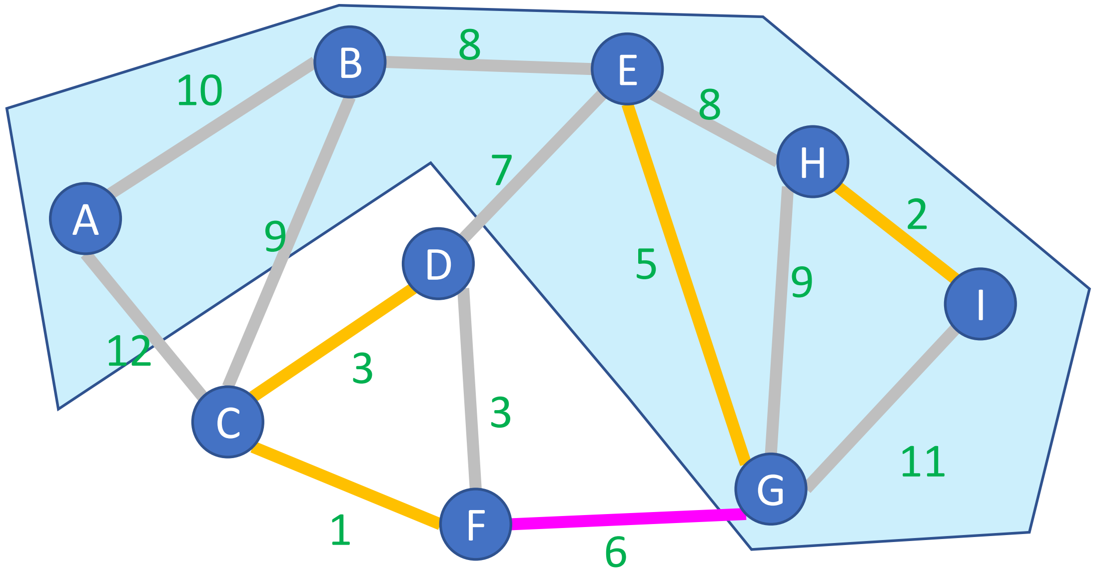

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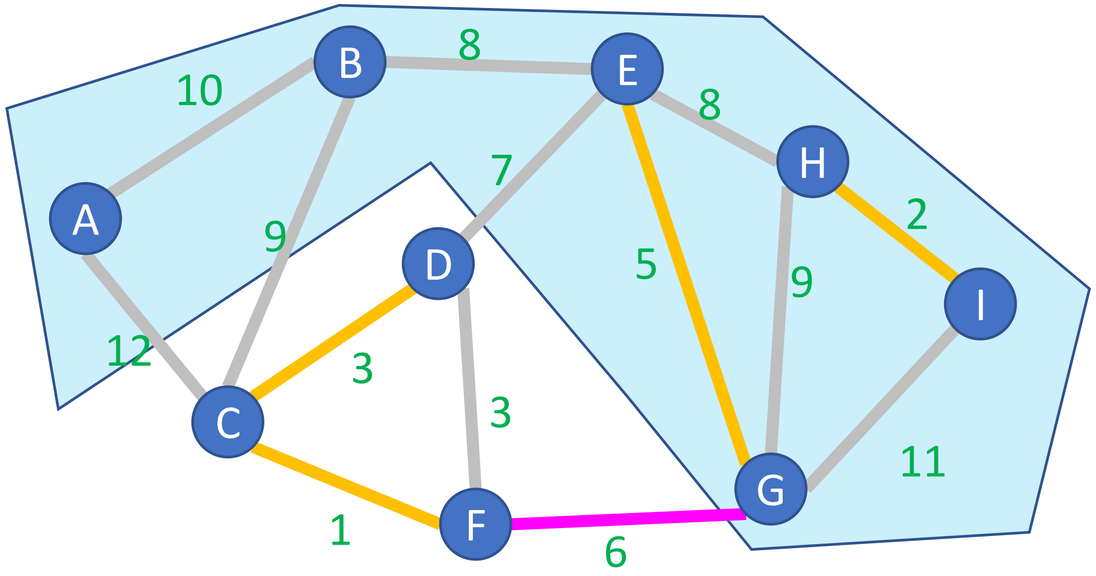
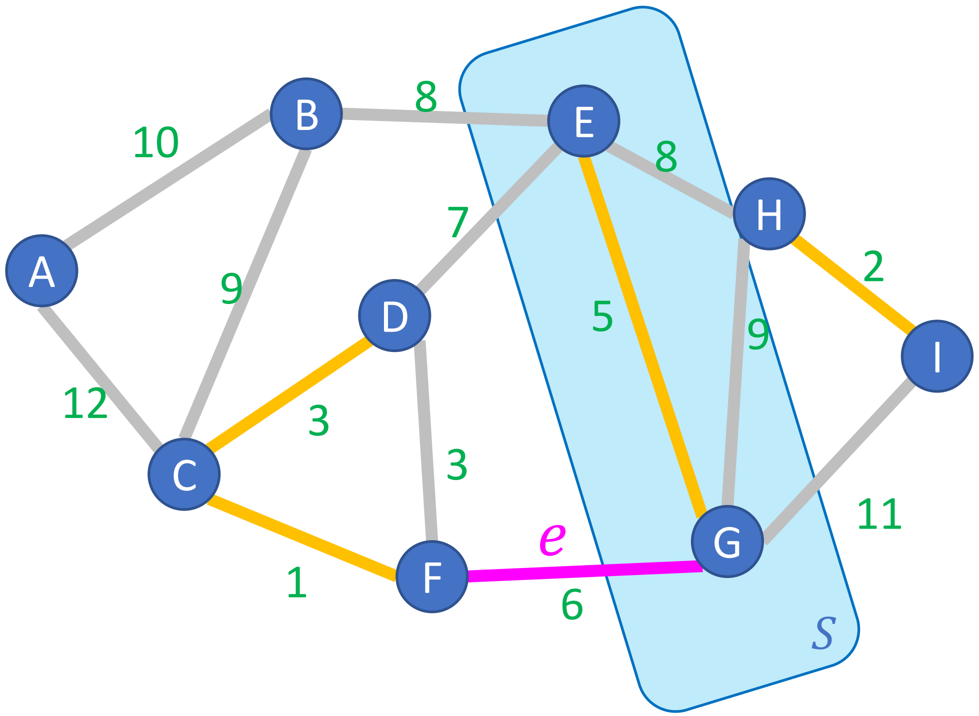


Proof of Kruskal's Algorithm

Start with an empty tree A
Repeat $V-1$ times:
Add the min-weight edge that doesn't cause a cycle



Proof: Suppose we have some arbitrary set of edges A that Kruskal's has already selected to include in the MST. $e=(F,G)$ is the edge Kruskal's selects to add next

We know that there cannot exist a path from F to G using only edges in A because e does not cause a cycle

We can cut the graph therefore into 2 disjoint sets:

- nodes reachable from G using edges in A
- All other nodes

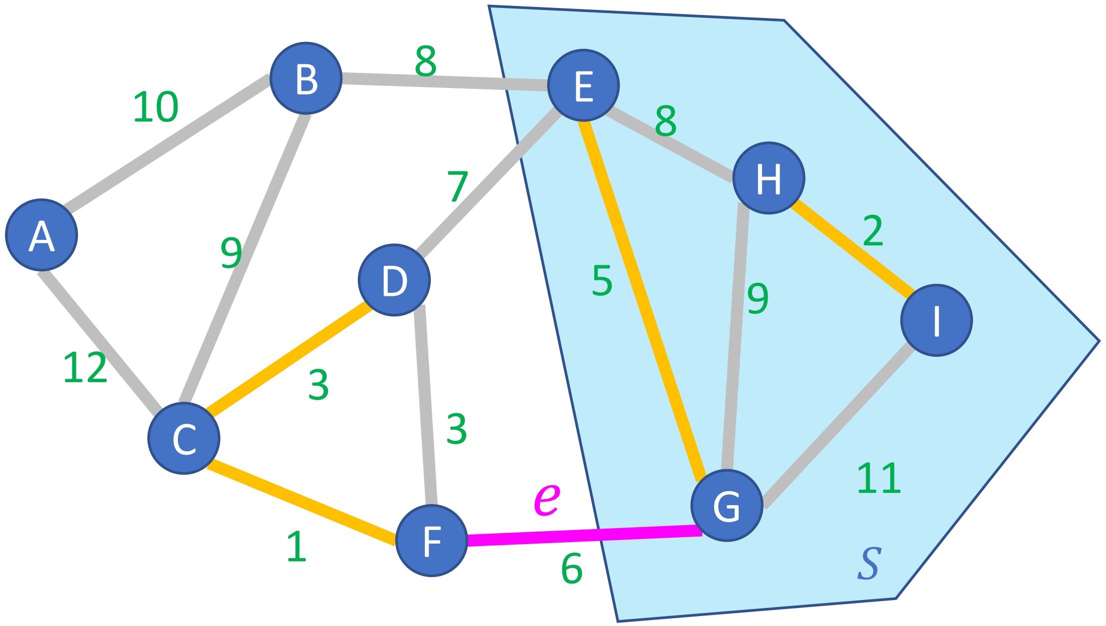
e is the minimum cost edge that crosses this cut, so by the Cut Theorem, Kruskal's is optimal!

Kruskal's Algorithm Runtime

Start with an empty tree A

Repeat $V - 1$ times:

Add the min-weight edge that doesn't cause a cycle



Keep edges in a Disjoint-set data structure (very fancy)

$O(E \log V)$

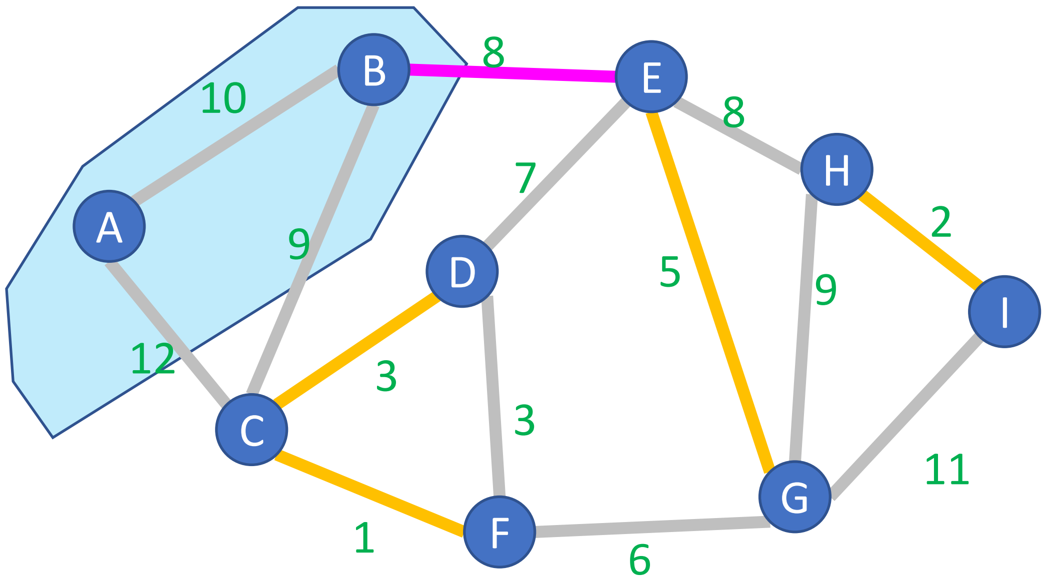
General MST Algorithm

Start with an empty tree A

Repeat $V - 1$ times:

Pick a cut $(S, V - S)$ which A respects (typically implicitly)

Add the min-weight edge which crosses $(S, V - S)$



Prim's Algorithm

Start with an empty tree A

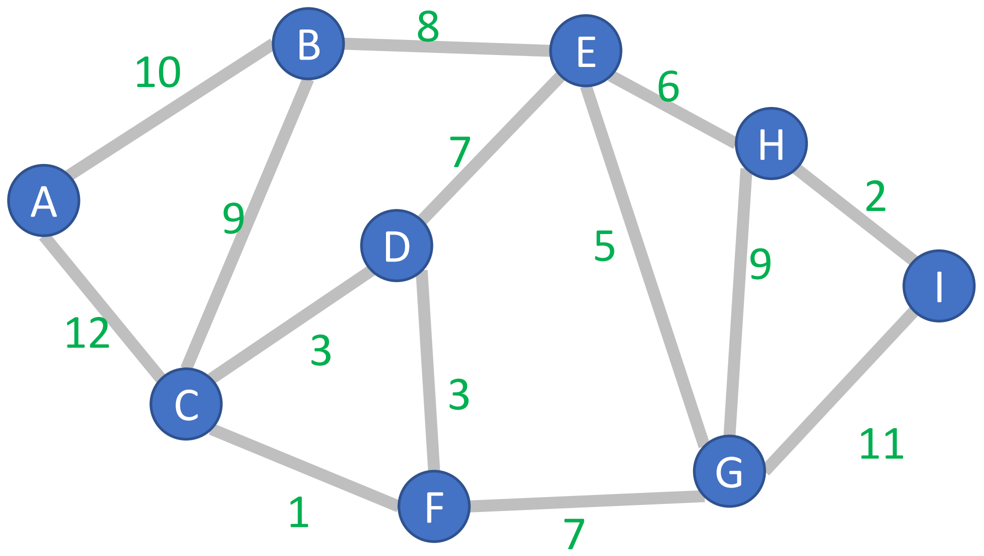
Repeat $V - 1$ times:

  Pick a cut $(S, V - S)$ which A respects

  Add the min-weight edge which crosses $(S, V - S)$

S is all endpoint of edges in A

e is the min-weight edge that grows the tree



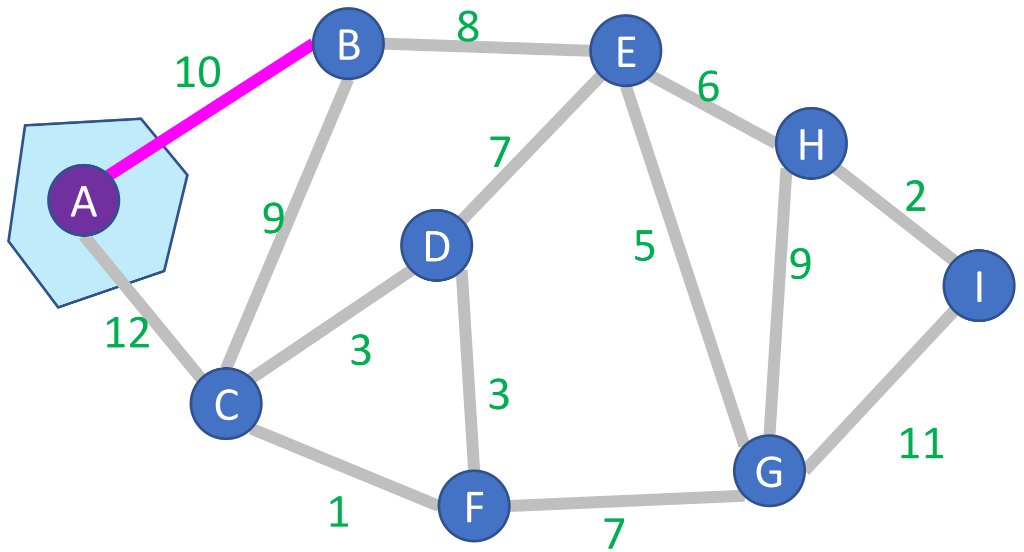
Prim's Algorithm

Start with an empty tree A

Pick a start node

Repeat $V - 1$ times:

Add the min-weight edge which connects to node in A with a node not in A



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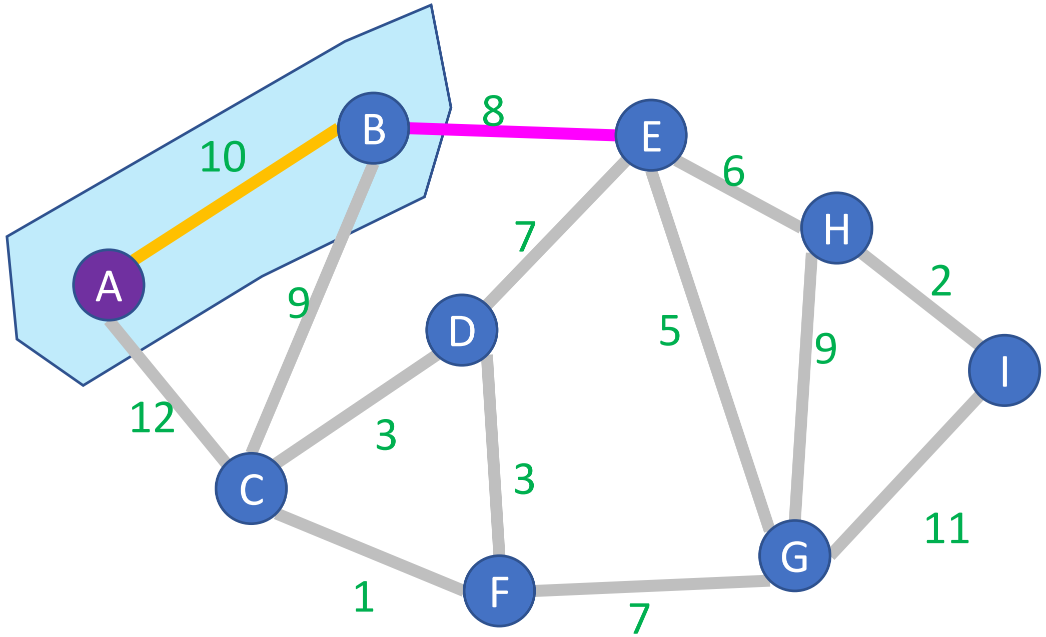

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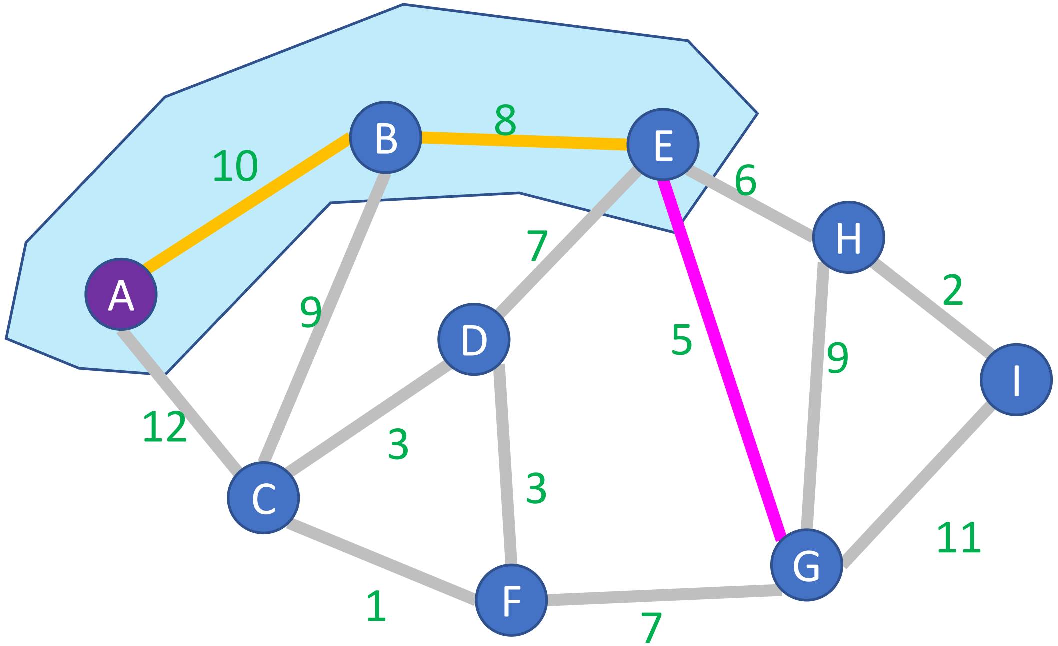

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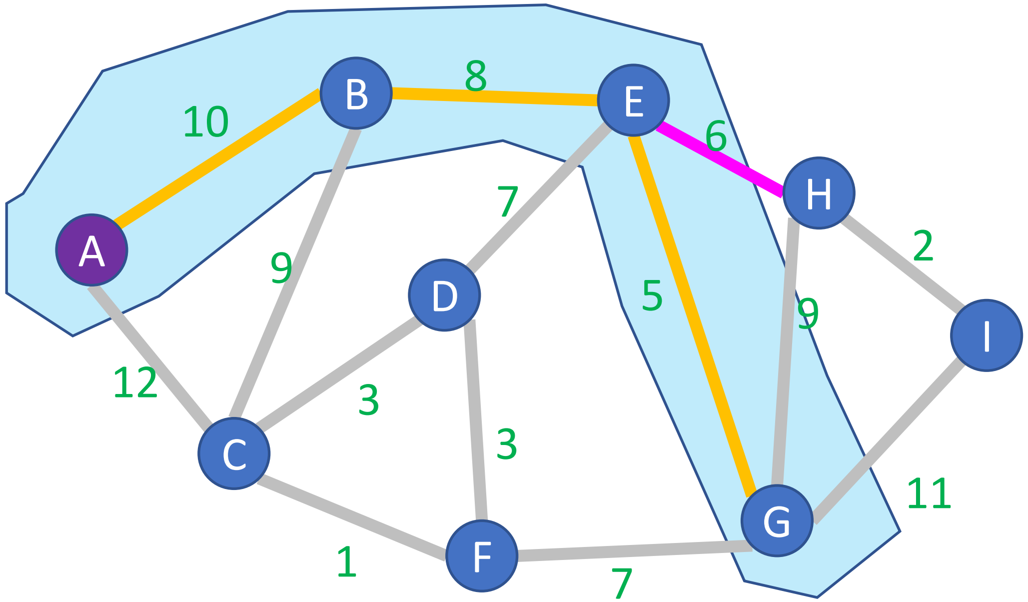


Prim's Algorithm

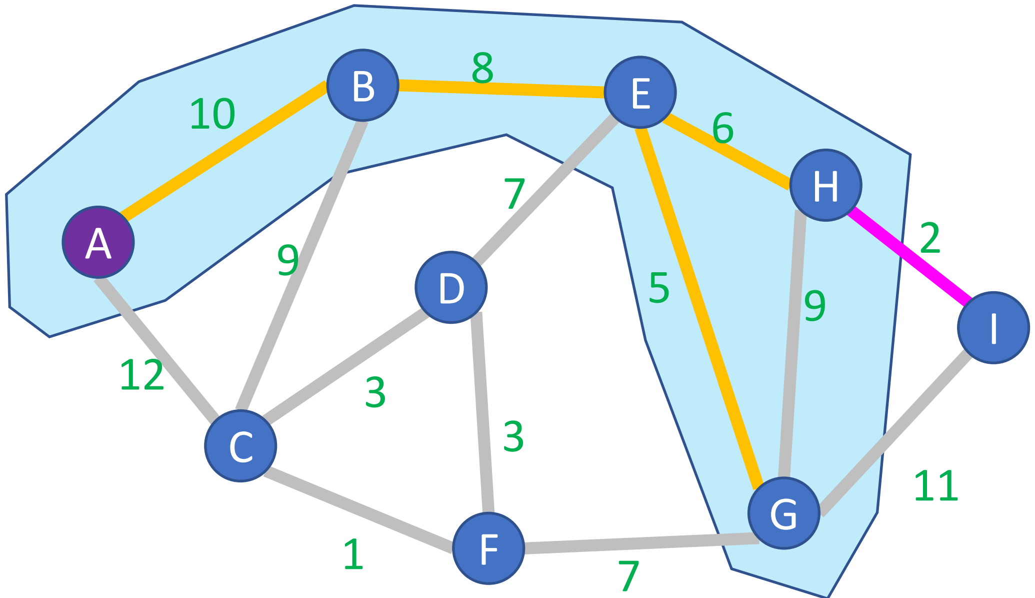
Start with an empty tree A

Pick a start node

Repeat $V - 1$ times:

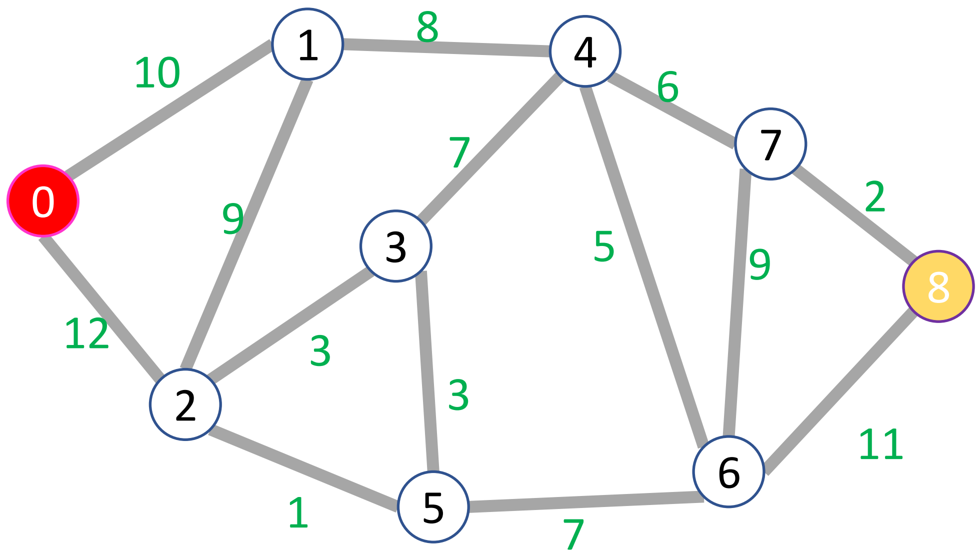
Add the min-weight edge which connects to node in A with a node not in A

Keep edges in a Heap
$O(E \log V)$

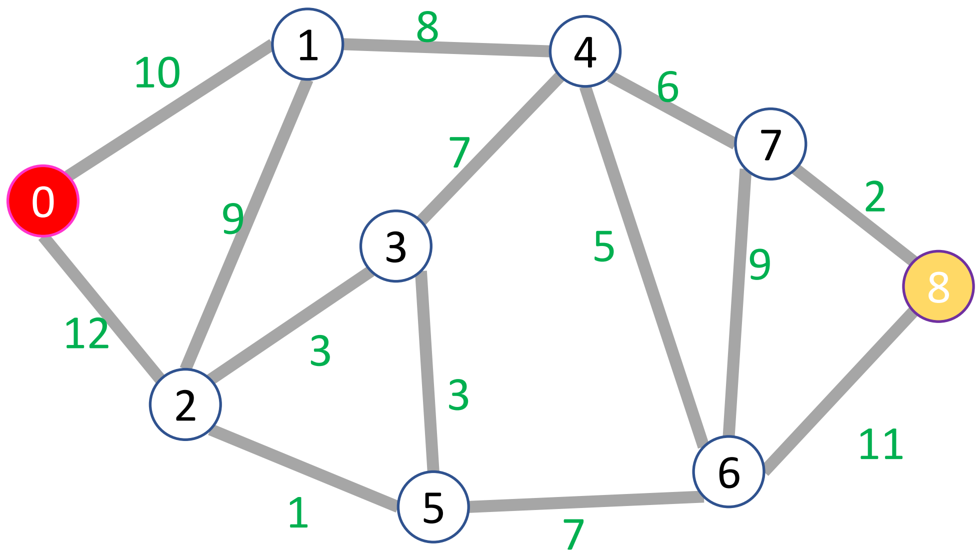


Dijkstra's Algorithm

```
int dijkstras(graph, start, end){
	PQ = new minheap();
	PQ.insert(0, start);  // priority=0, value=start
	start.distance = 0;
	while (!PQ.isEmpty){
		current = PQ.extractmin();
		if (current.known){ continue;}
		current.known = true;
		for (neighbor : current.neighbors){
			if (!neighbor.known){
				new_dist = current.distance + weight(current,neighbor);
				if(neighbor.dist != ∞){ PQ.insert(new_dist, neighbor);}
				else if (new_dist < neighbor. distance){
					neighbor. distance = new_dist;
					PQ.decreaseKey(new_dist,neighbor); }
			}
		}
	}
	return end.distance;
}
```



Prim's Algorithm

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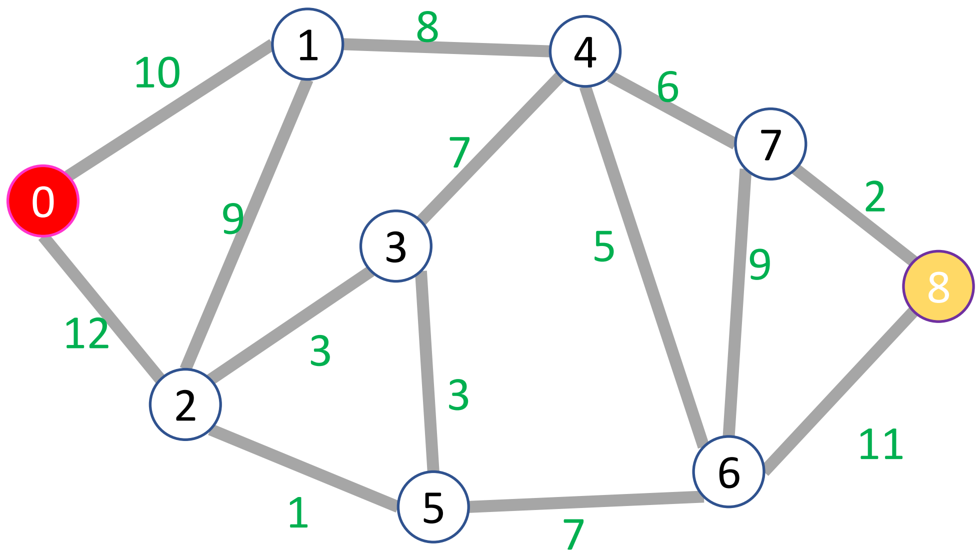

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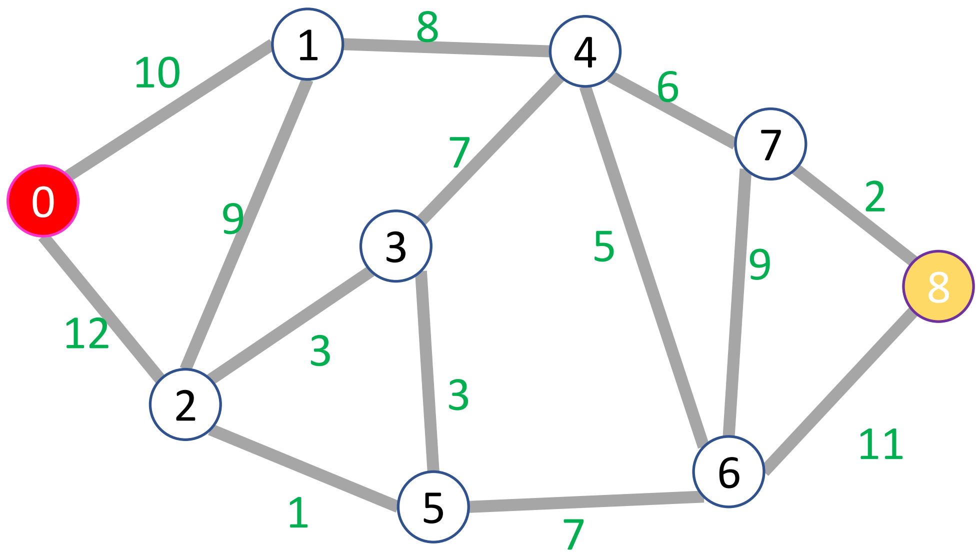
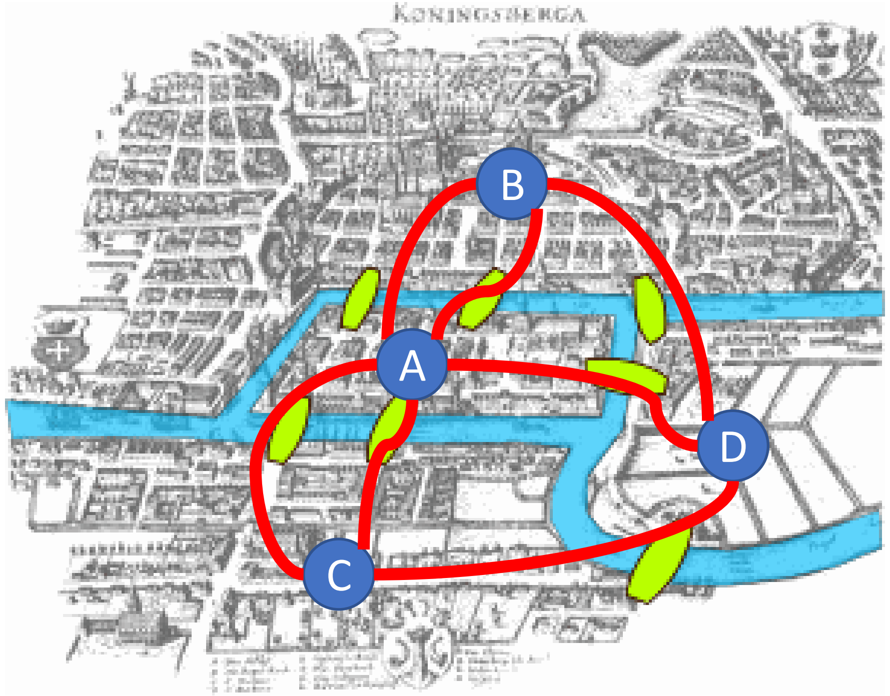
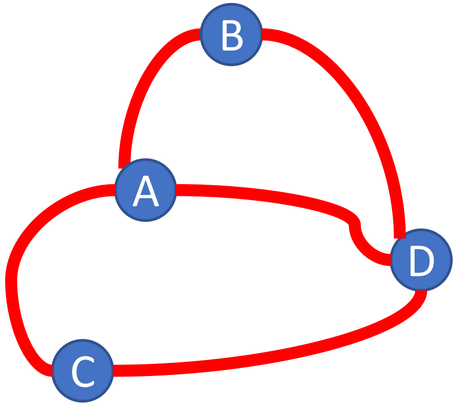


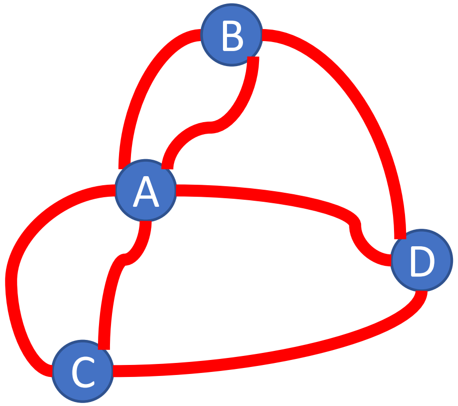
7 Bridges of Königsberg



The Pregel River runs through the city of Koenigsberg, creating 2 islands. Among these 2 islands and the 2 sides of the river, there are 7 bridges. Is there any path starting at one landmass which crosses each bridge exactly once?

Euler Path Problem

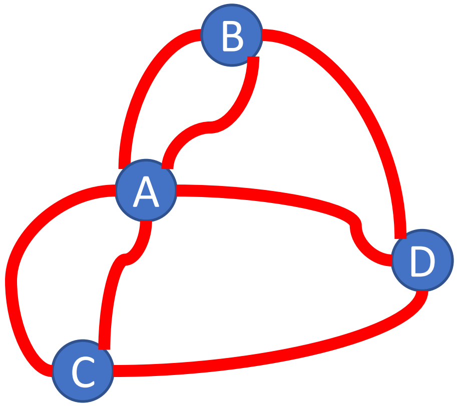




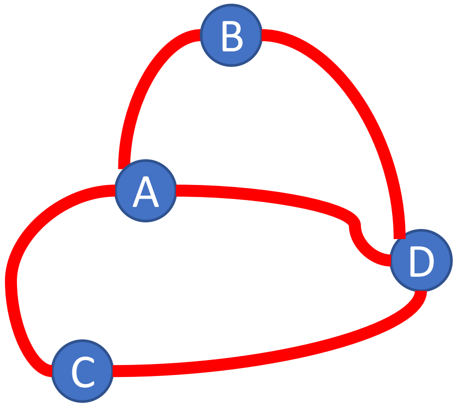
- Path:
  - A sequence of nodes $v_1, v_2, \ldots$ such that for every consecutive pair are connected by an edge (i.e. (v_i, v_{i+1}) is an edge for each i in the path)
- Euler Path:
  - A path such that every edge in the graph appears exactly once
    - If the graph is not simple then some pairs need to appear multiple times!
- Euler path problem:
  - Given an undirected graph $G = (V, E)$, does there exist an Euler path for G?

Examples

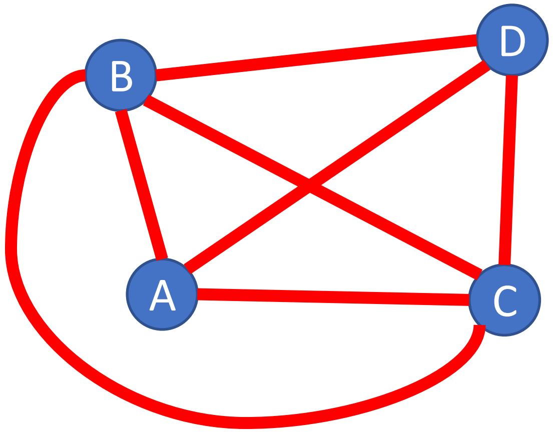
- Which of the graphs below have an Euler path?



No Euler path exists!



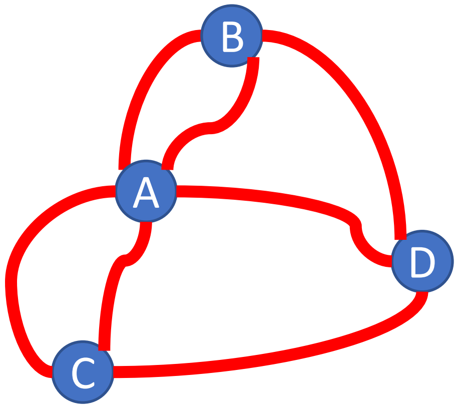
Euler path exists!
A, B, D, A, C, D

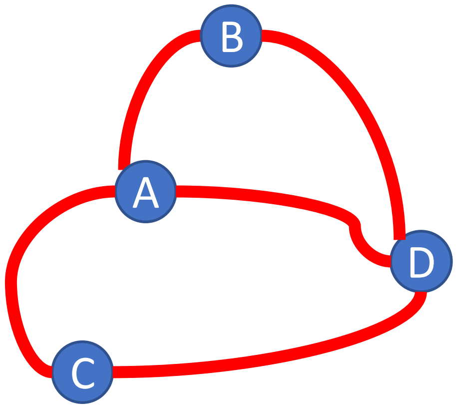


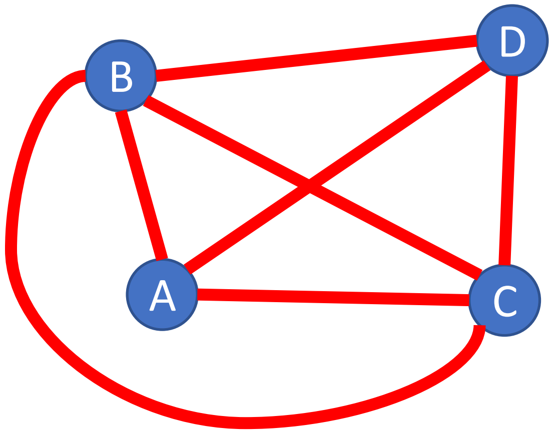
Euler path exists!
A, B, C, D, A, C, B, D

Euler's Theorem

- A graph has an Euler Path if and only if it is connected and has exactly 0 or 2 nodes with odd degree.





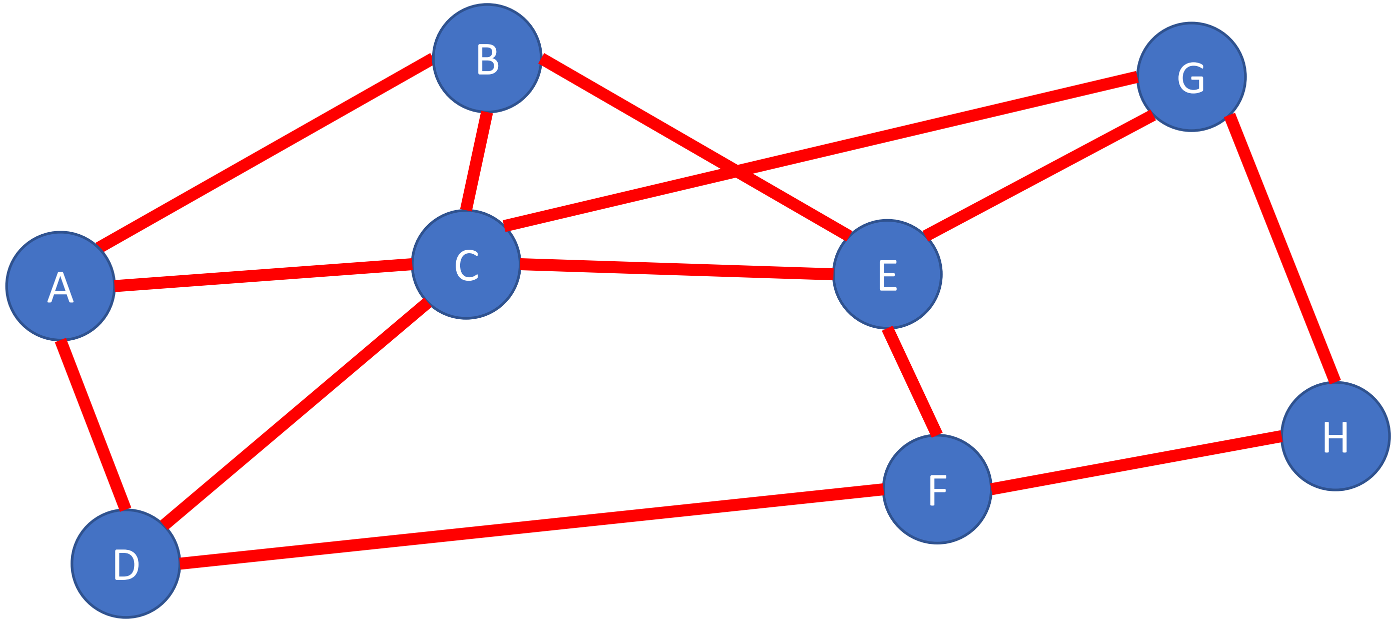


Algorithm for the Euler Path Problem

- Given an undirected graph $G = (V, E)$, does there exist an Euler path for G?
- Algorithm:
  - Check if the graph is connected
  - Check the degree of each node
  - If the number of nodes with odd degree is 0 or 2, return true
  - Otherwise return false
- Running time?

A Seemingly Similar Problem

- Hamiltonian Path:
  - A path that includes every node in the graph exactly once
- Hamiltonian Path Problem:
  - Given a graph $G = (V, E)$, does that graph have a Hamiltonian Path?



True!
A, B, C, E, G, H, F, D

Algorithms for the Hamiltonian Path Problem

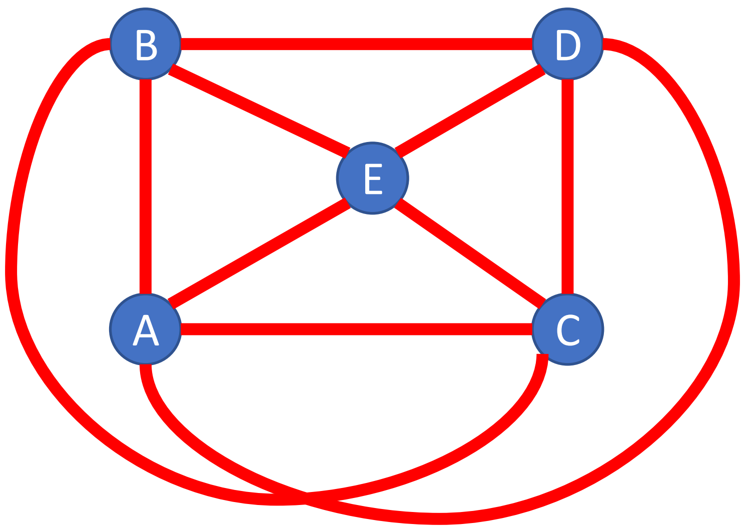
- Option 1:
  - Explore all possible simple paths through the graph
  - Check to see if any of those are length V
- Option 2:
  - Write down every sequence of nodes
  - Check to see if any of those are a path
- Both options are examples of an **Exhaustive Search (“Brute Force”) algorithm**

Option 2: List all sequences, look for a path

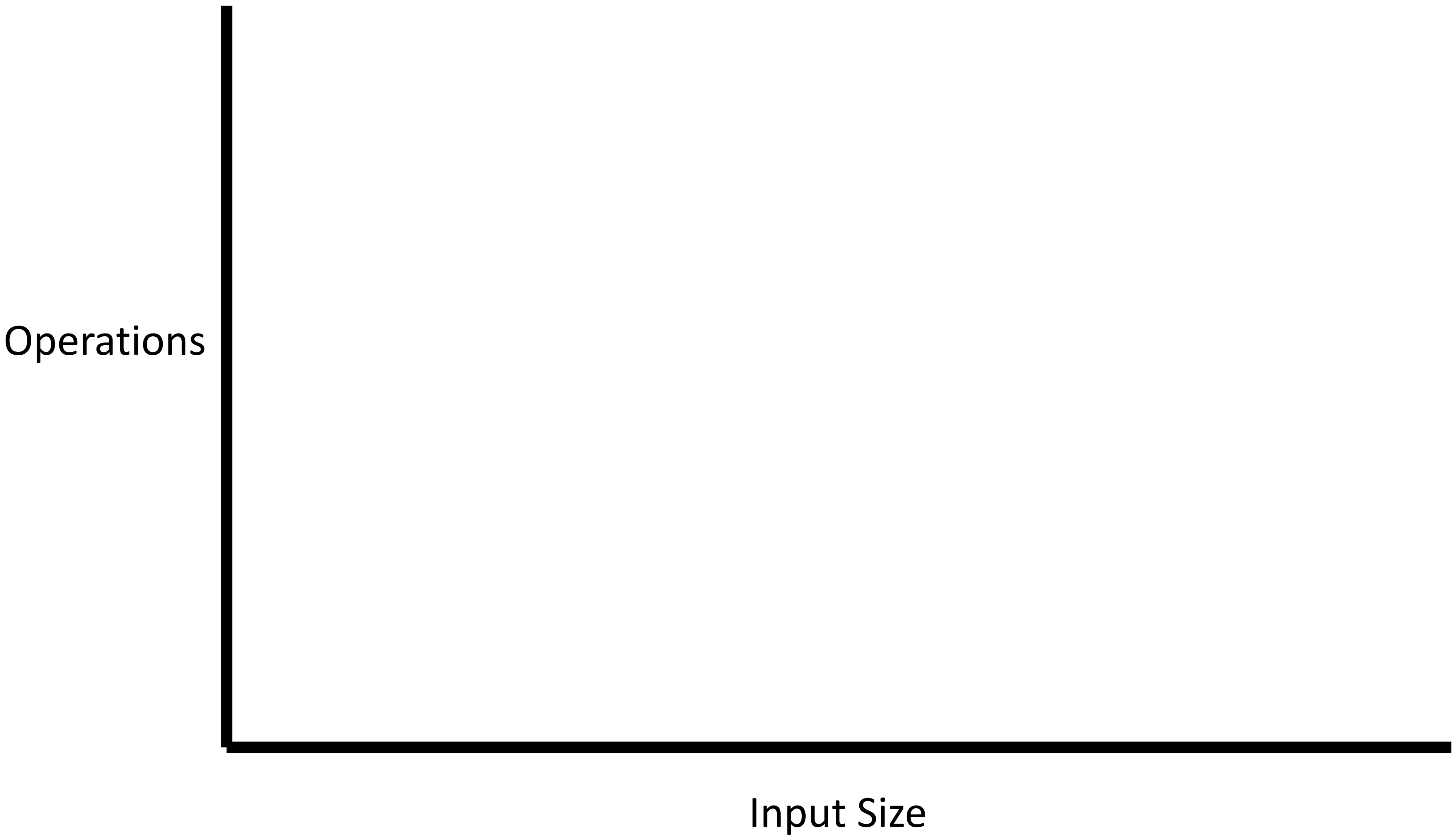
- Running time:
  - $G = (V, E)$
  - Number of permutations of V is $|V|!$
    - $n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1$
  - How does $n!$ compare with 2^n?
    - $n! \in \Omega(2^n)$
  - Exponential running time!

Option 1: Explore all simple paths, check for one of length V

- Running time:
  - $G = (V, E)$
  - Number of paths
    - Pick a first node ($|V|$ choices)
    - Pick a neighbor (up to $|V| - 1$ choices)
    - Pick a neighbor (up to $|V| - 2$ choices)
    - …. Repeat $|V| - 1$ total times
    - Overall: $|V|!$ paths
  - Exponential running time



Running Times



Running times we've seen:

- $\Theta(1)$
- $\Theta(\log n)$
- $\Theta(n)$
- $\Theta(n \log n)$
- $\Theta(n^2)$
- $\Theta(2^n)$

Running Times

Table 2.1 The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds 10^{25} years, we simply record the algorithm as taking a very long time.

	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	18 min	10^{25} years
$n = 50$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	11 min	36 years	very long
$n = 100$	< 1 sec	< 1 sec	< 1 sec	1 sec	12,892 years	10^{17} years	very long
$n = 1{,}000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long	very long
$n = 10{,}000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long	very long
$n = 100{,}000$	< 1 sec	2 sec	3 hours	32 years	very long	very long	very long
$n = 1{,}000{,}000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

Tractability

- Tractable:
  - Feasible to solve in the “real world”
- Intractable:
  - Infeasible to solve in the “real world”
- Whether a problem is considered “tractable” or “intractable” depends on the use case
  - For machine learning, big data, etc. tractable might mean $\mathrm{O}(n)$ or even $O(\log n)$
  - For most applications it’s more like $O(n^3)$ or $O(n^2)$
- A strange pattern:
  - Most “natural” problems are either done in small-degree polynomial (e.g. n^2) or else exponential time (e.g. 2^n)
  - It’s rare to have problems which require a running time of n^5, for example

Complexity Classes

- A Complexity Class is a set of problems (e.g. sorting, Euler path, Hamiltonian path)
  - The problems included in a complexity class are those whose most efficient algorithm has a specific upper bound on its running time (or memory use, or…)
- Examples:
  - The set of all problems that can be solved by an algorithm with running time $O(n)$
    - Contains: Finding the minimum of a list, finding the maximum of a list, buildheap, summing a list, etc.
  - The set of all problems that can be solved by an algorithm with running time $O(n^2)$
    - Contains: everything above as well as sorting, Euler path
  - The set of all problems that can be solved by an algorithm with running time $O(n!)$
    - Contains: everything we've seen in this class so far

Complexity Classes and Tractability

- To explore what problems are and are not tractable, we give some complexity classes special names:
- Complexity Class P:
  - Stands for "Polynomial"
  - The set of problems which have an algorithm whose running time is $O(n^p)$ for some choice of $p \in \mathbb{R}$.
  - We say all problems belonging to P are "Tractable"
- Complexity Class EXP:
  - Stands for "Exponential"
  - The set of problems which have an algorithm whose running time is $O\left(2^{n^p}\right)$ for some choice of $p \in \mathbb{R}$
  - We say all problems belonging to $EXP - P$ are "Intractable"
    - Disclaimer: Really it's all problems outside of P, and there are problems which do not belong to EXP, but we're not going to worry about those in this class

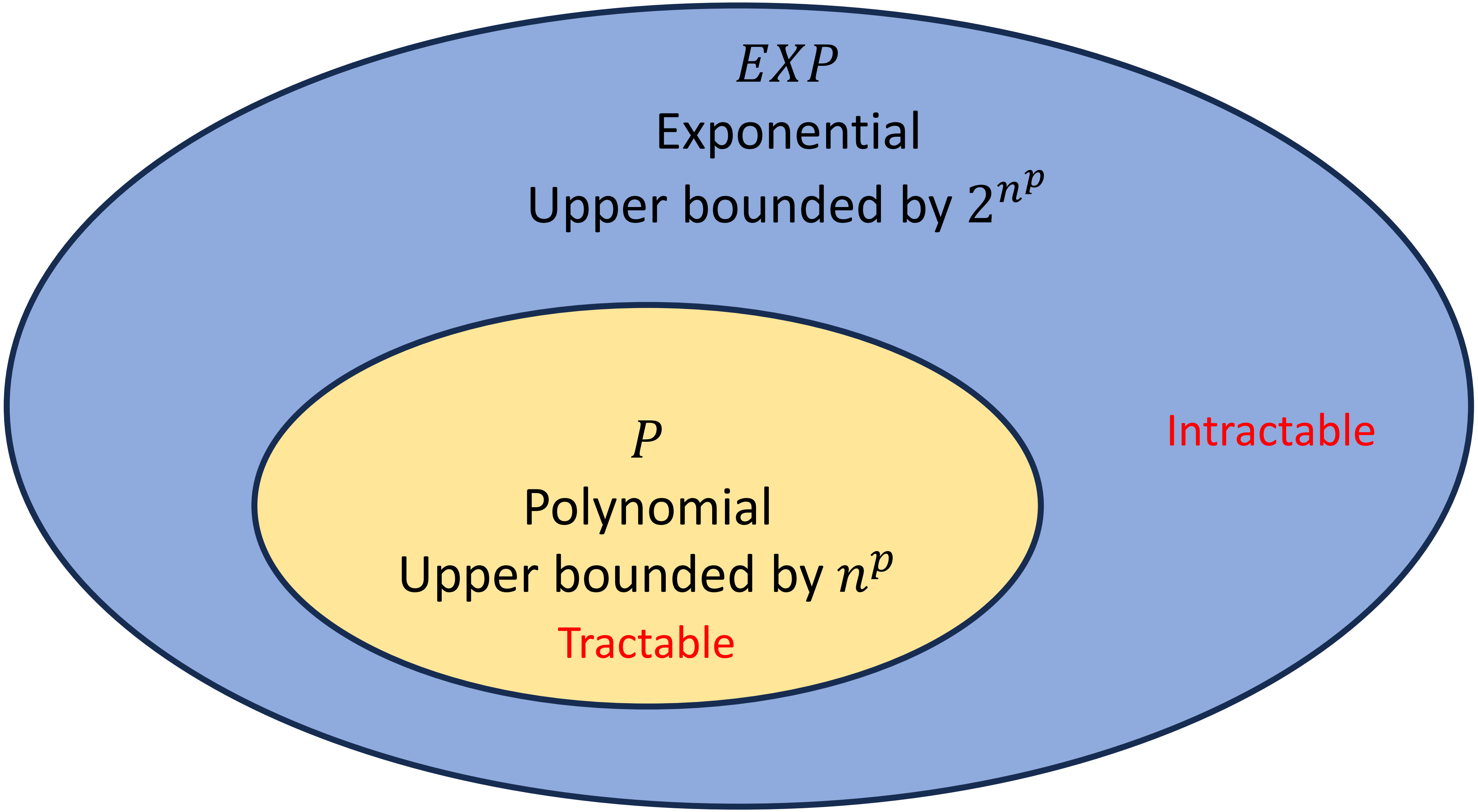
EXP and P

Important!

$P \subset EXP$

Every problem within P is also within EXP

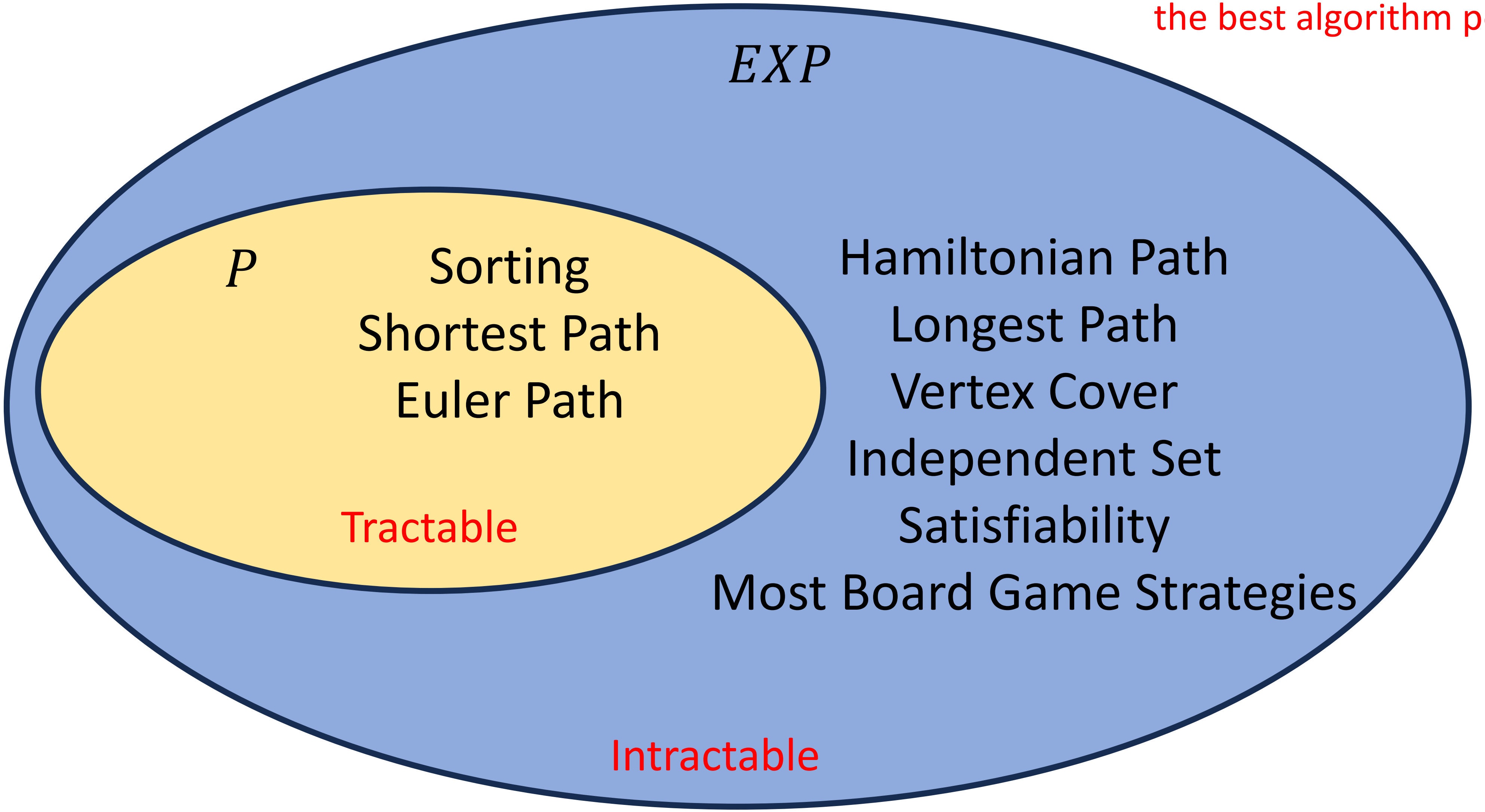
The intractable ones are the problems within EXP but NOT P



Members

Important!

Some of the problems listed in EXP could also be members of P
Since membership is determined by a problems most efficient algorithm, knowing if a problem belongs to P requires knowing the best algorithm possible!



Studying Complexity and Tractability

- Organizing problems into complexity classes helps us to reason more carefully and flexibly about tractability
- The goal for each problem is to either
  - Find an efficient algorithm if it exists
    - i.e. show it belongs to P
  - Prove that no efficient algorithm exists
    - i.e. show it does not belong to P
- Complexity classes allow us to reason about sets of problems at a time, rather than each problem individually
  - If we can find more precise classes to organize problems into, we might be able to draw conclusions about the entire class
  - It may be easier to show a problem belongs to class C than to P, so it may help to show that $C \subseteq P$

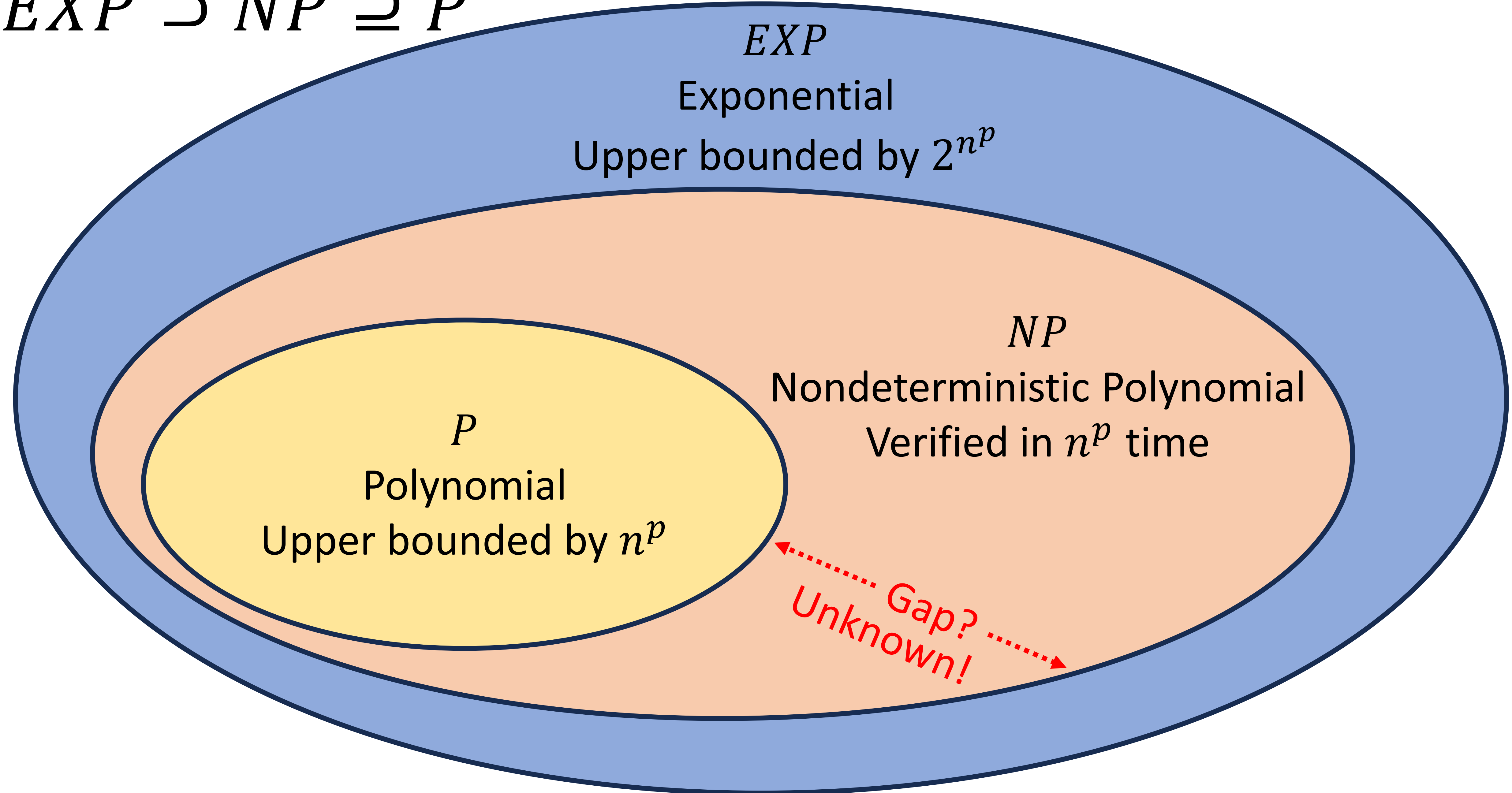
Some problems in EXP seem “easier”

- There are some problems that we do not have polynomial time algorithms to solve, but provided answers are easy to check
- Hamiltonian Path:
  - It’s “hard” to look at a graph and determine whether it has a Hamiltonian Path
  - It’s “easy” to look at a graph and a candidate path together and determine whether THAT path is a Hamiltonian Path
    - It’s easy to **verify** whether a given path is a Hamiltonian path

Class NP

- NP
  - The set of problems for which a candidate solution can be verified in polynomial time
  - Stands for “Non-deterministic Polynomial”
    - Corresponds to algorithms that can guess a solution (if it exists), that solution is then verified to be correct in polynomial time
    - Can also think of as allowing a special operation that allows the algorithm to magically guess the right choice at each step of an exhaustive search
- $P \subseteq NP$
  - Why?

$EXP \supset NP \supseteq P$

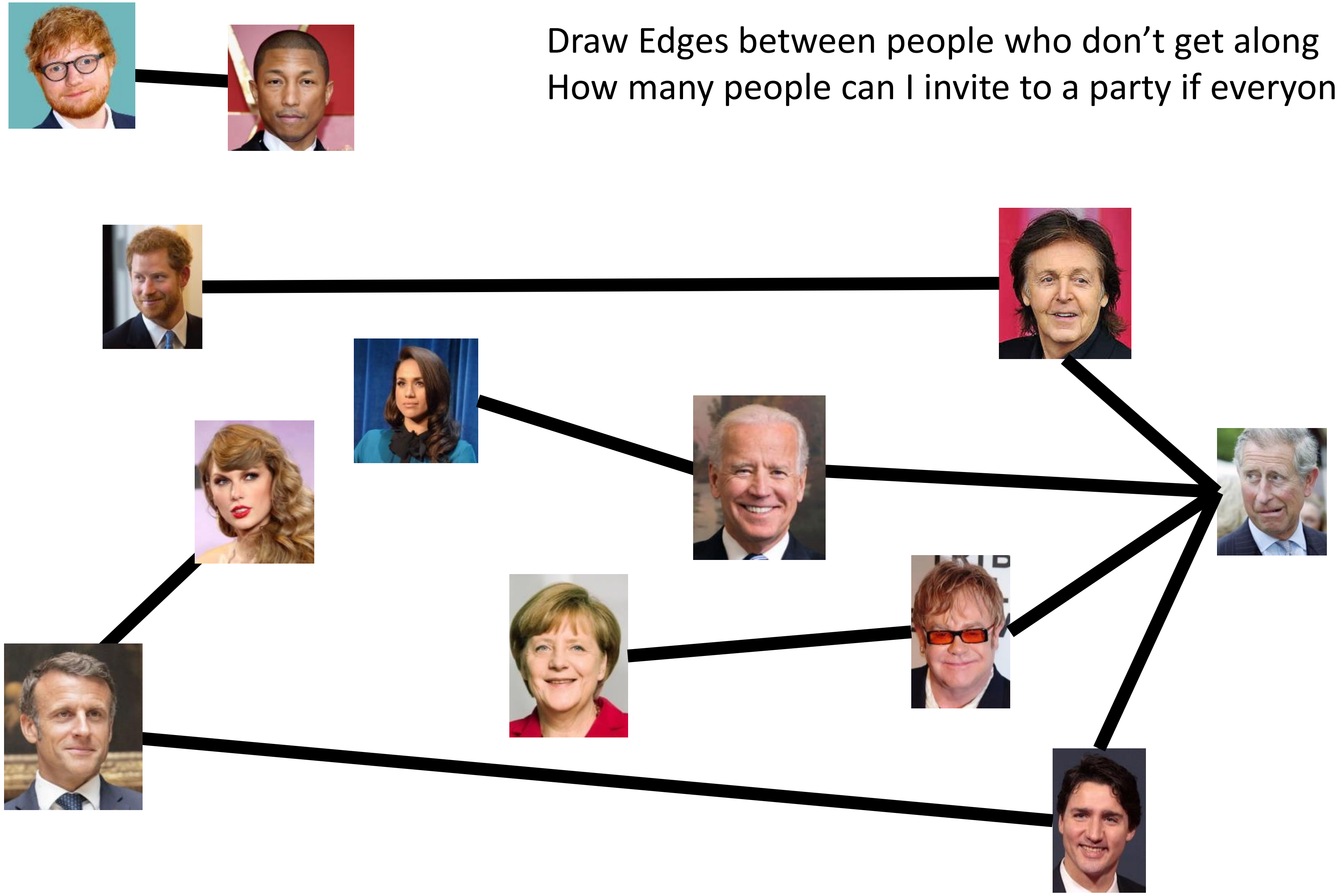


Solving and Verifying Hamiltonian Path

- Give an algorithm to solve Hamiltonian Path
  - Input: $G = (V, E)$
  - Output: True if G has a Hamiltonian Path
  - Algorithm: Check whether each permutation of V is a path.
    - Running time: $|V|!$, so does not show whether it belongs to P
- Give an algorithm to verify Hamiltonian Path
  - Input: $G = (V, E)$ and a sequence of nodes
  - Output: True if that sequence of nodes is a Hamiltonian Path
  - Algorithm:
    - Check that each node appears in the sequence exactly once
    - Check that the sequence is a path
    - Running time: $O(V \cdot E)$, so it belongs to NP

Party Problem

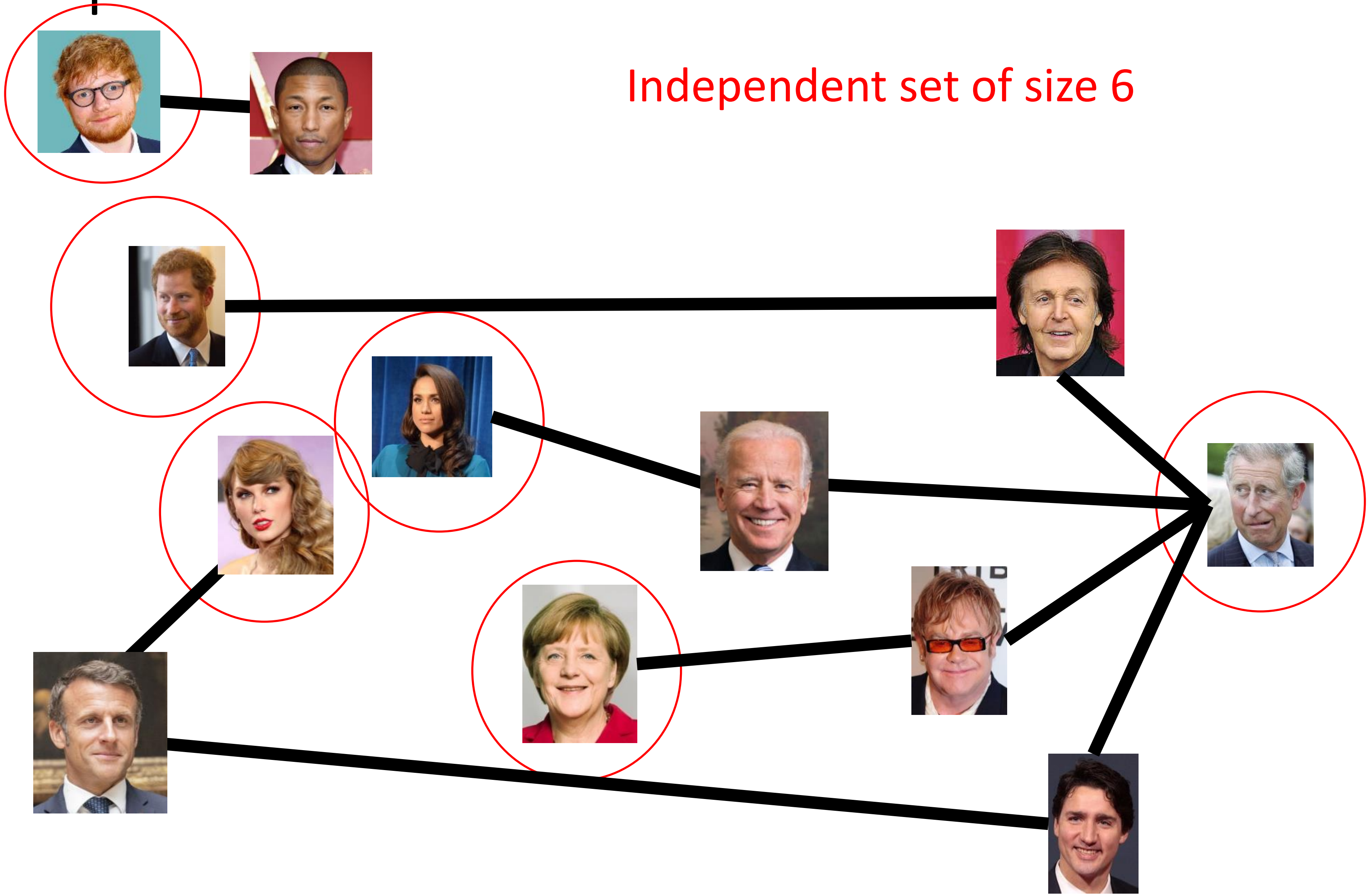
Draw Edges between people who don't get along
How many people can I invite to a party if everyone must get along?



Independent Set

- Independent set:
  - $S \subseteq V$ is an independent set if no two nodes in S share an edge
- Independent Set Problem:
  - Given a graph $G = (V, E)$ and a number k, determine whether there is an independent set S of size k

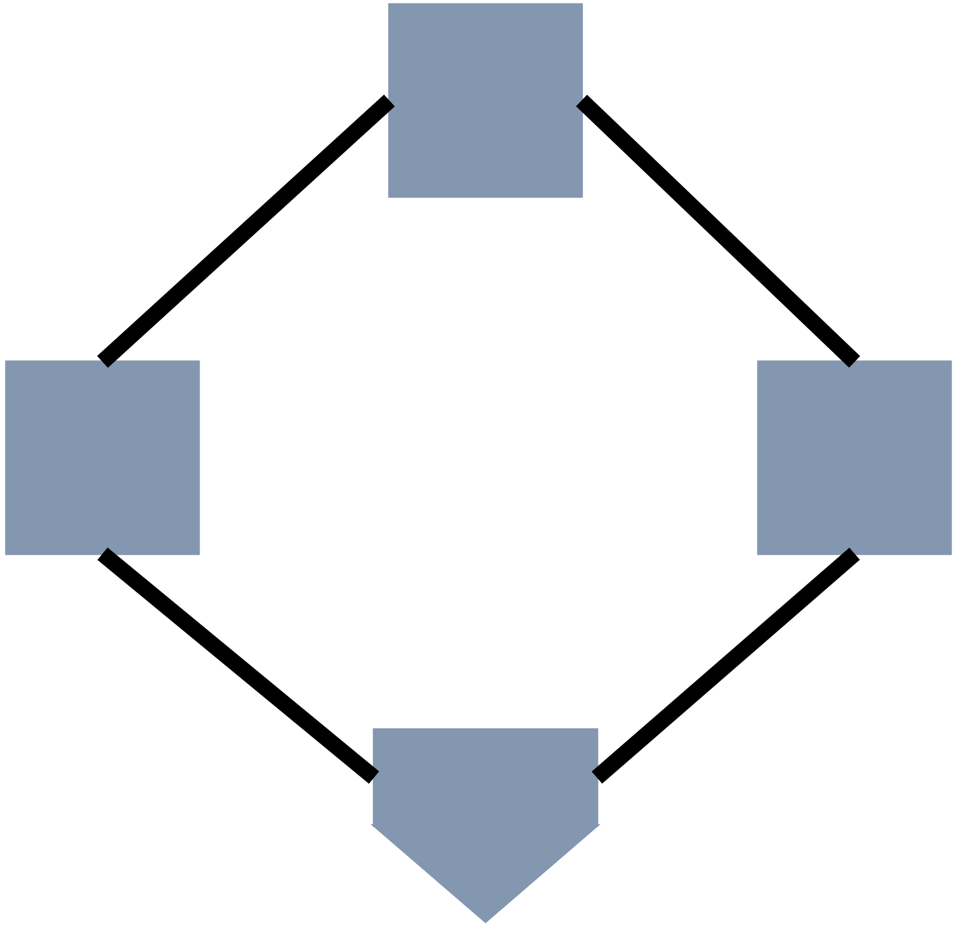
Example



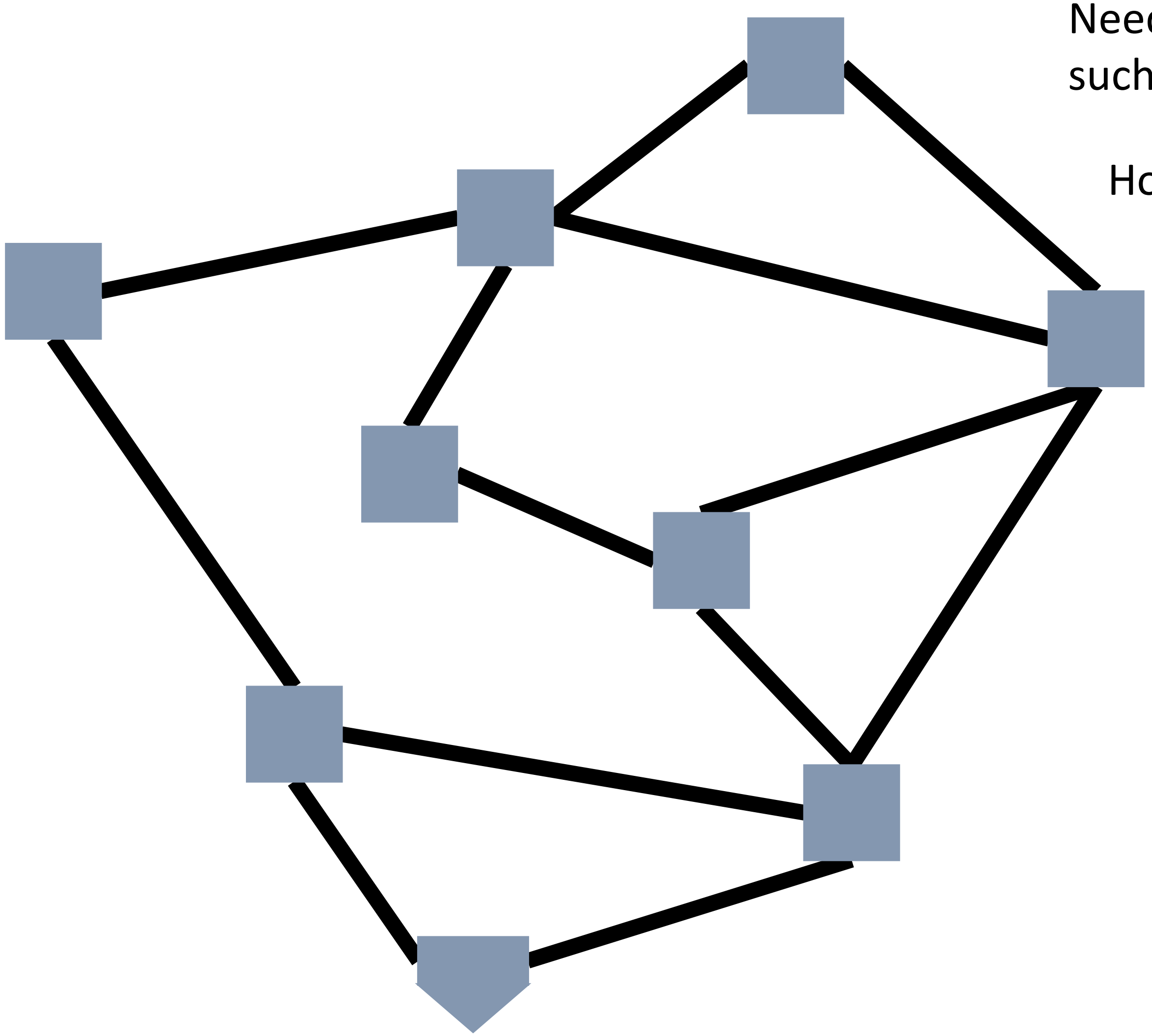
Solving and Verifying Independent Set

- Give an algorithm to solve independent set
  - Input: $G = (V, E)$ and a number k
  - Output: True if G has an independent set of size k
- Give an algorithm to verify independent set
  - Input: $G = (V, E)$,a number k, and a set $S \subseteq V$
  - Output: True if S is an independent set of size k

Generalized Baseball



Generalized Baseball



Need to place defenders on bases such that every edge is defended

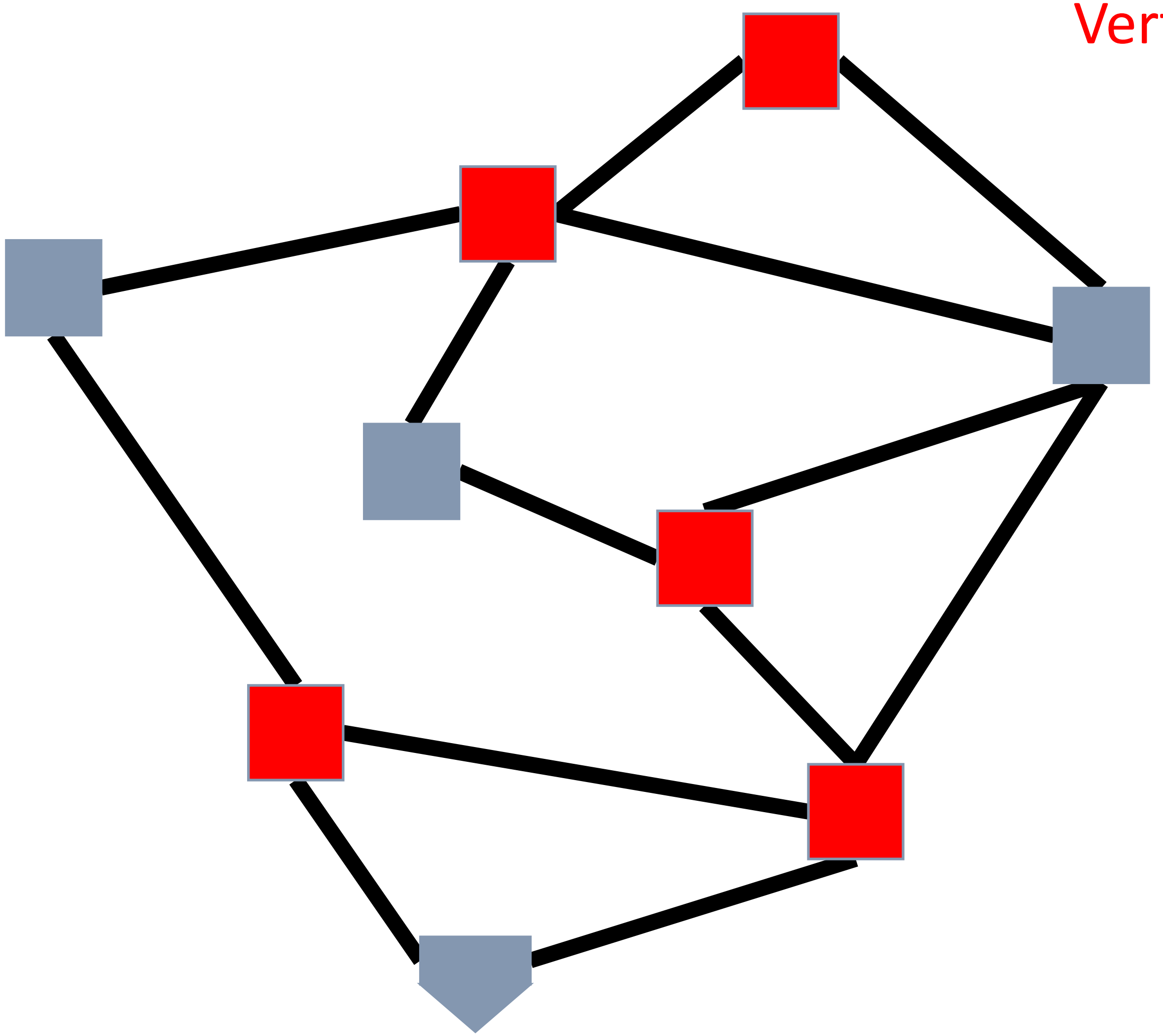
How many defenders would suffice?

Vertex Cover

- Vertex Cover:
  - $C \subseteq V$ is a vertex cover if every edge in E has one of its endpoints in C
- Vertex Cover Problem:
  - Given a graph $G = (V, E)$ and a number k, determine if there is a vertex cover C of size k

Example

Vertex cover of size 5

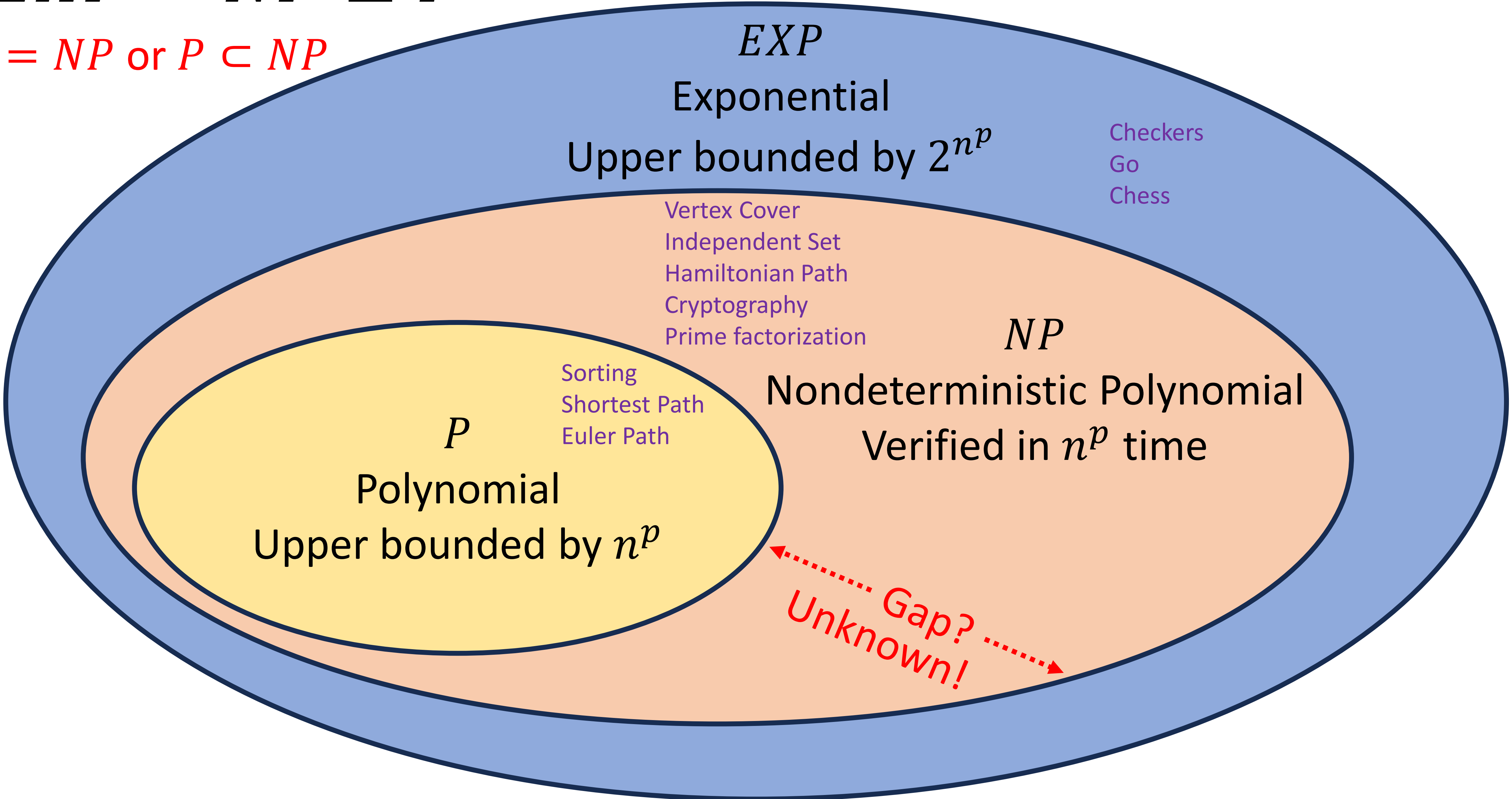


Solving and Verifying Vertex Cover

- Give an algorithm to solve vertex cover
  - Input: $G = (V, E)$ and a number k
  - Output: True if G has a vertex cover of size k
- Give an algorithm to verify vertex cover
  - Input: $G = (V, E)$,a number k, and a set $S \subseteq E$
  - Output: True if S is a vertex cover of size k

$EXP \supset NP \supseteq P$

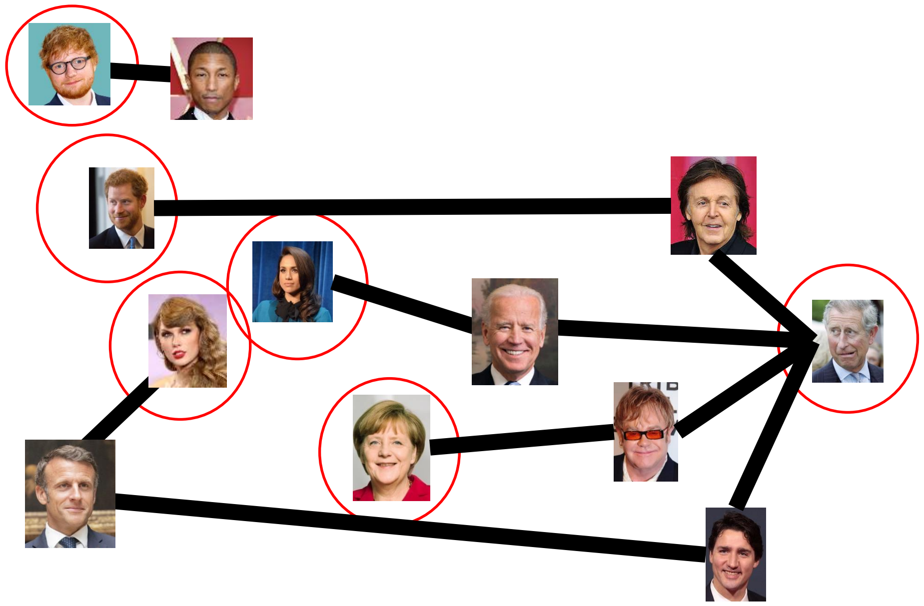
$P = NP$ or $P \subset NP$



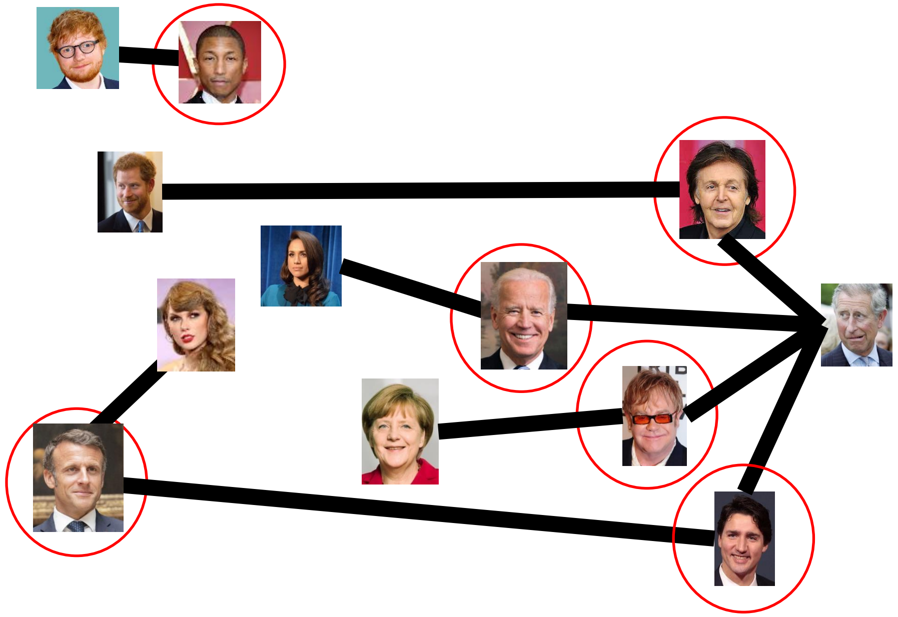
Way Cool!

S is an independent set of G iff $V - S$ is a vertex cover of G

Independent Set



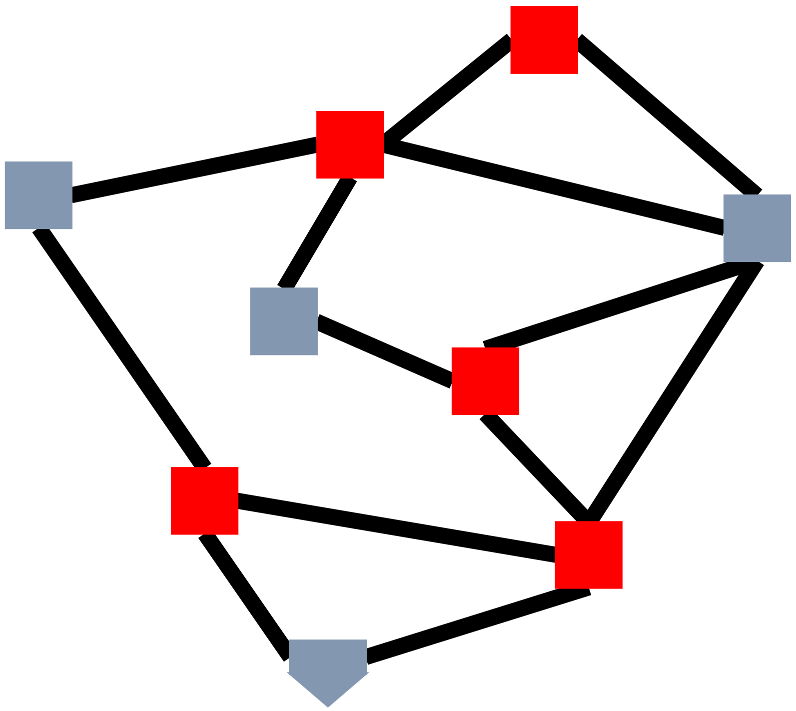
Vertex Cover



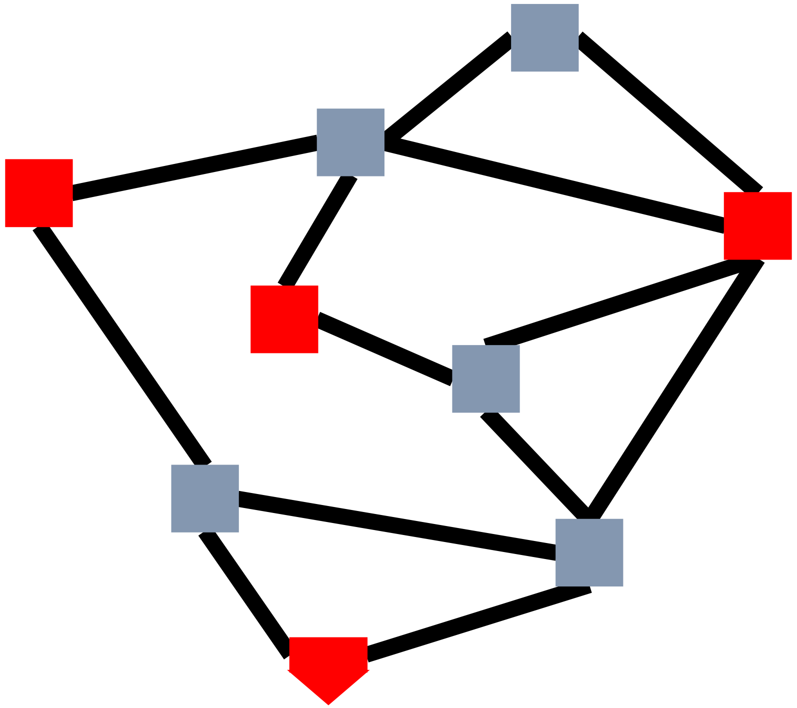
Way Cool!

S is an independent set of G iff $V - S$ is a vertex cover of G

Vertex Cover



Independent Set



Solving Vertex Cover and Independent Set

- Algorithm to solve vertex cover
  - Input: $G = (V, E)$ and a number k
  - Output: True if G has a vertex cover of size k
    - Check if there is an Independent Set of G of size $|V| - k$
- Algorithm to solve independent set
  - Input: $G = (V, E)$ and a number k
  - Output: True if G has an independent set of size k
    - Check if there is a Vertex Cover of G of size $|V| - k$

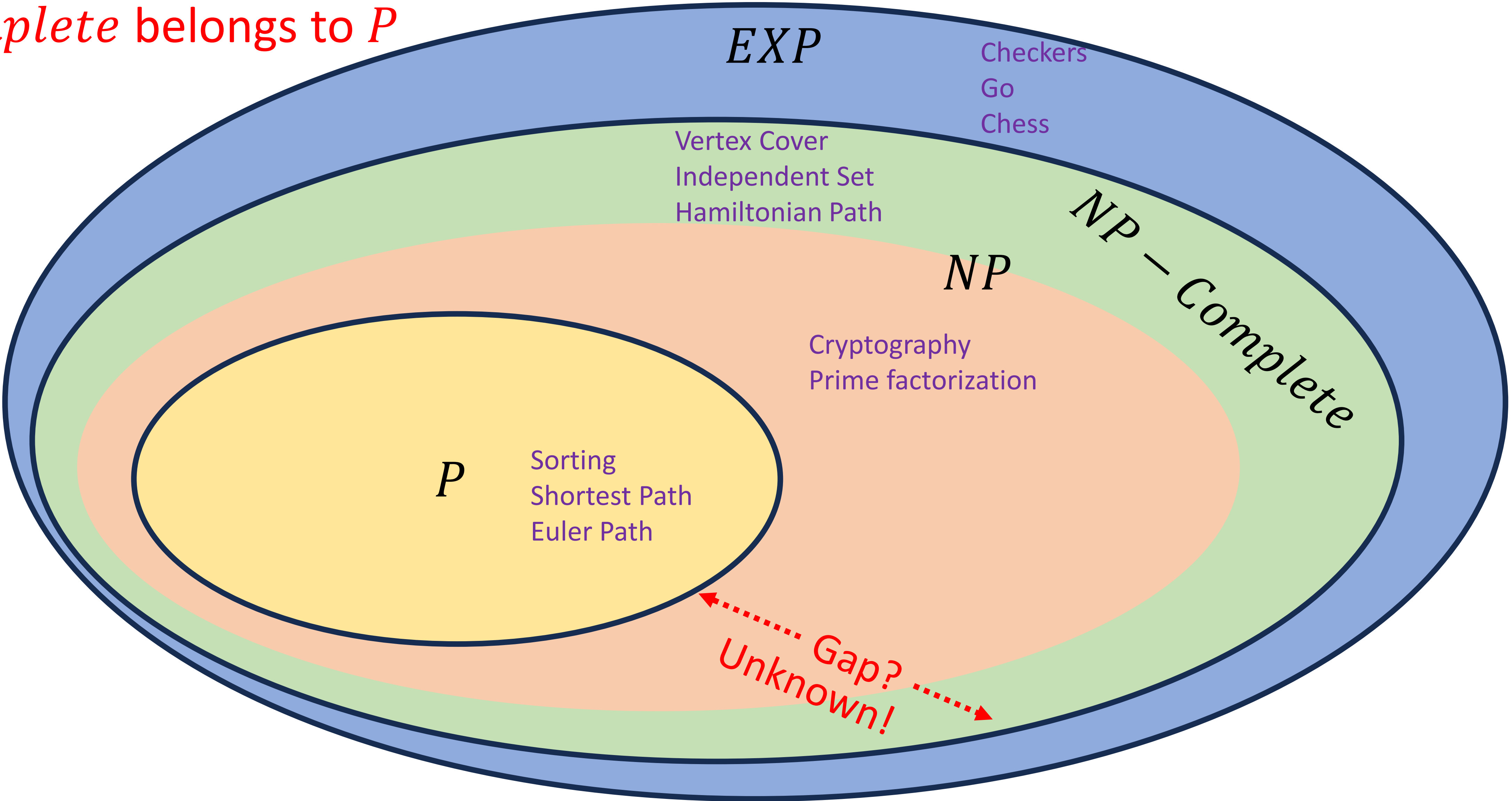
Either both problems belong to P, or else neither does!

NP-Complete

- A set of “together they stand, together they fall” problems
- The problems in this set either all belong to P, or none of them do
- Intuitively, the “hardest” problems in NP
- Collection of problems from NP that can all be “transformed” into each other in polynomial time
  - Like we could transform independent set to vertex cover, and vice-versa
  - We can also transform vertex cover into Hamiltonian path, and Hamiltonian path into independent set, and …

$EXP \supset NP - Complete \supseteq NP \supseteq P$

$P = NP$ iff some problem from $NP - Complete$ belongs to P



Overview

- Problems not belonging to P are considered intractable
- The problems within NP have some properties that make them seem like they might be tractable, but we've been unsuccessful with finding polynomial time algorithms for many
- The class $NP - Complete$ contains problems with the properties:
  - All members are also members of NP
  - All members of NP can be transformed into every member of $NP - Complete$
  - Therefore if any one member of $NP - Complete$ belongs to P, then $P = NP$

Why should YOU care?

- If you can find a polynomial time algorithm for any $NP - Complete$ problem then:
  - You will win $1million
  - You will win a Turing Award
  - You will be world famous
  - You will have done something that no one else on Earth has been able to do in spite of the above!
- If you are told to write an algorithm a problem that is $NP - Complete$
  - You can tell that person everything above to set expectations
  - Change the requirements!
  - **Approximate the solution**: Instead of finding a path that visits every node, find a path that visits at least 75% of the nodes
  - **Add Assumptions**: problem might be tractable if we can assume the graph is acyclic, a tree
  - **Use Heuristics**: Write an algorithm that's "good enough" for small inputs, ignore edge cases