CSE 332 Autumn 2023
Lecture 18: Graphs

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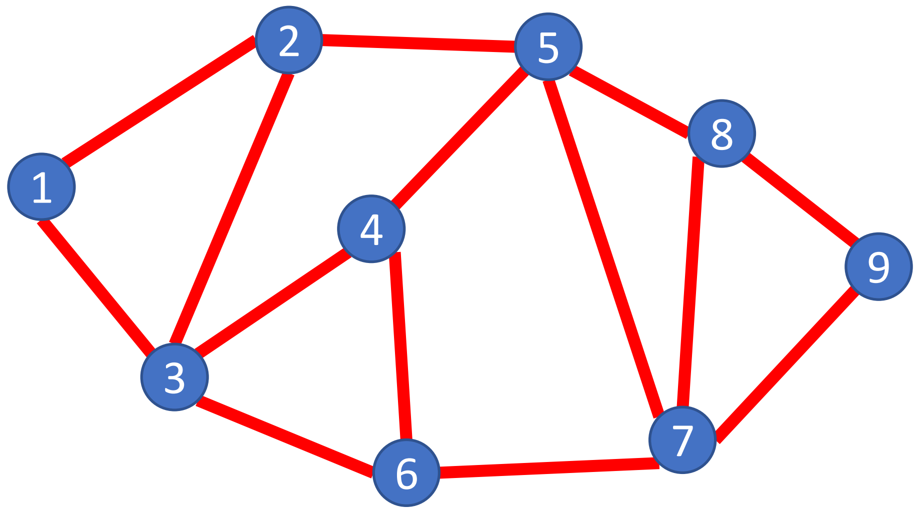
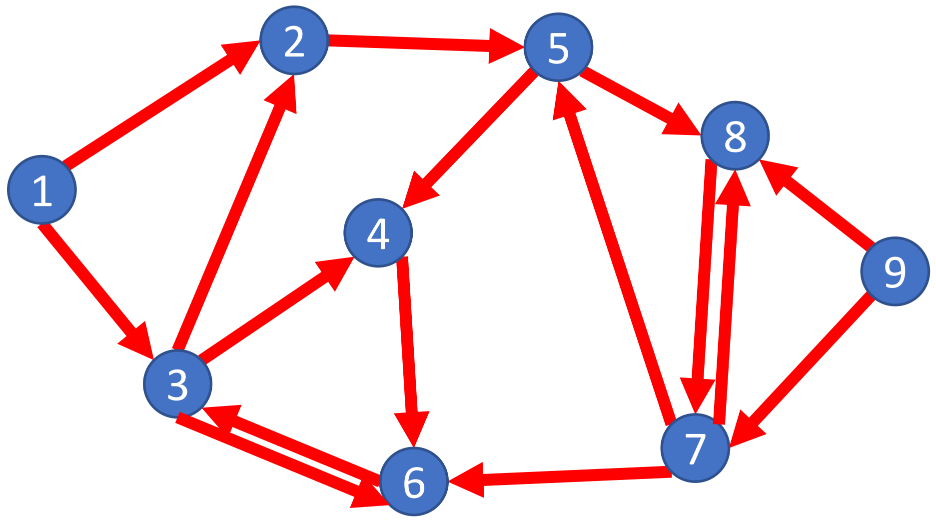
Some Graph Terms

- Adjacent/Neighbors
  - Nodes are adjacent/neighbors if they share an edge
- Degree
  - Number of “neighbors” of a vertex
- Indegree
  - Number of incoming neighbors
- Outdegree
  - Number of outgoing neighbors

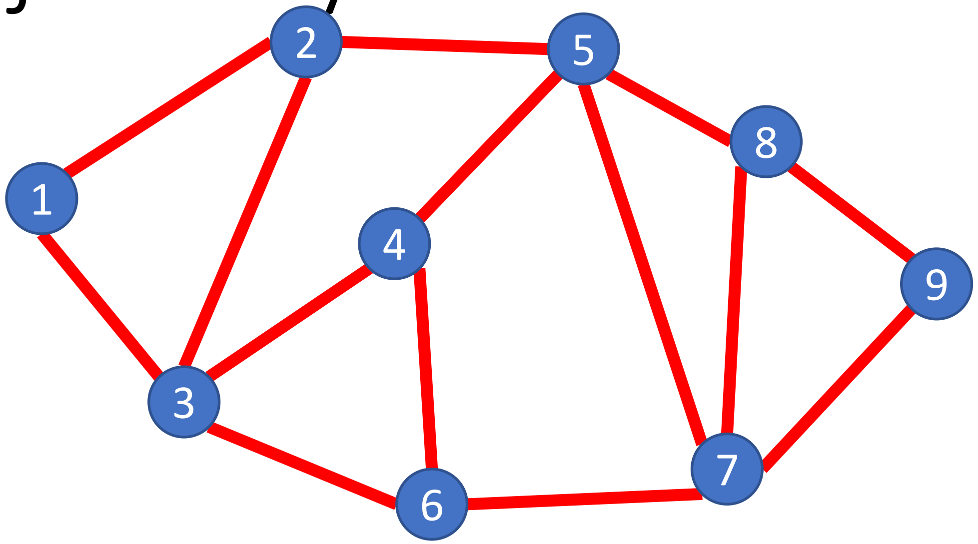


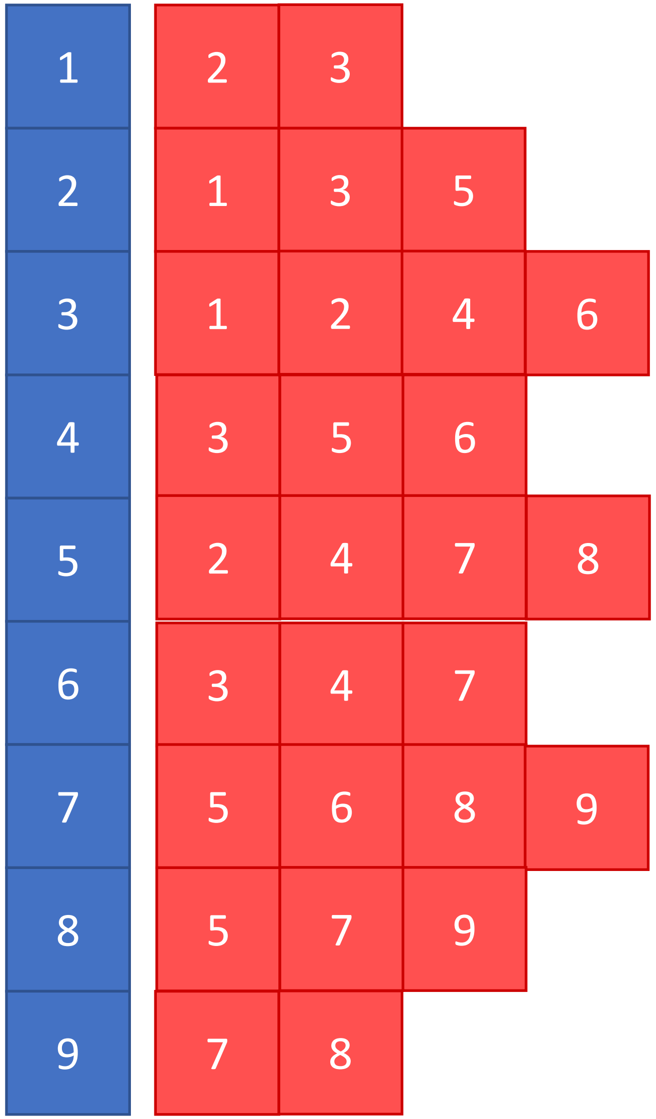


Graph Operations

- To represent a Graph (i.e. build a data structure) we need:
  - Add Edge
  - Remove Edge
  - Check if Edge Exists
  - Get Neighbors (incoming)
  - Get Neighbors (outgoing)

Adjacency List





<u>Time/Space Tradeoffs</u>

Space to represent: $\Theta(n+m)$

Add Edge: $\Theta(1)$

Remove Edge: $\Theta(\deg(v))$

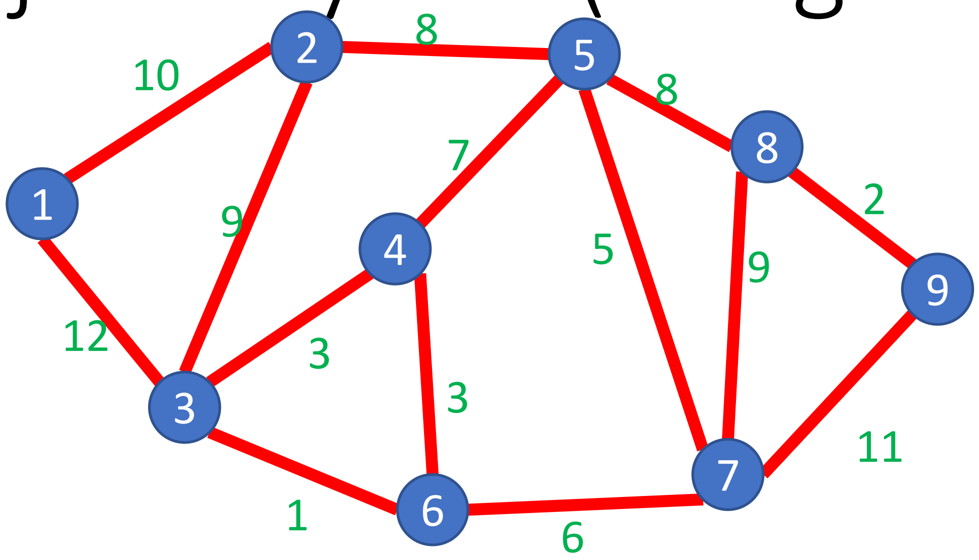
Check if Edge Exists: $\Theta(\deg(v))$

Get Neighbors (incoming): $\Theta(n+m)$

Get Neighbors (outgoing): $\Theta(\deg(v))$

$$|V| = n$$
$$|E| = m$$

Adjacency List (Weighted)



1	2	3		
2	1	3	5	
3	1	2	4	6
4	3	5	6	
5	2	4	7	8
6	3	4	7	
7	5	6	8	9
8	5	7	9	
9	7	8		

<u>Time/Space Tradeoffs</u>

Space to represent: $\Theta(n+m)$

Add Edge: $\Theta(1)$

Remove Edge: $\Theta(\deg(v))$

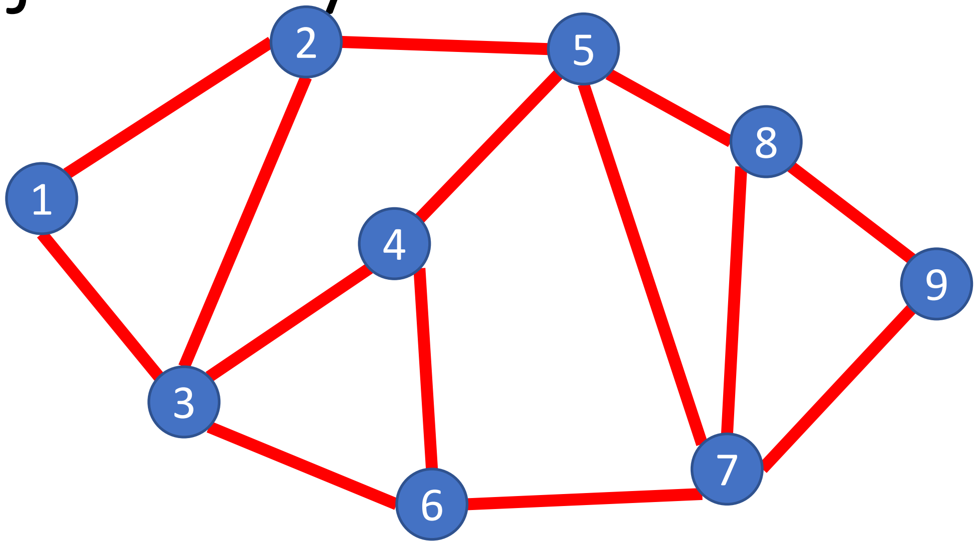
Check if Edge Exists: $\Theta(\deg(v))$

Get Neighbors (incoming): $\Theta(n+m)$

Get Neighbors (outgoing): $\Theta(\deg(v))$

$$|V| = n$$
$$|E| = m$$

Adjacency Matrix



	A	B	C	D	E	F	G	H	I
A		1	1						
B	1		1		1				
C	1	1		1		1			
D			1		1	1			
E		1		1			1	1	
F			1	1			1		
G					1	1		1	1
H					1		1		1
I							1	1	

Time/Space Tradeoffs

Space to represent: $\Theta(?)$

Add Edge: $\Theta(?)$

Remove Edge: $\Theta(?)$

Check if Edge Exists: $\Theta(?)$

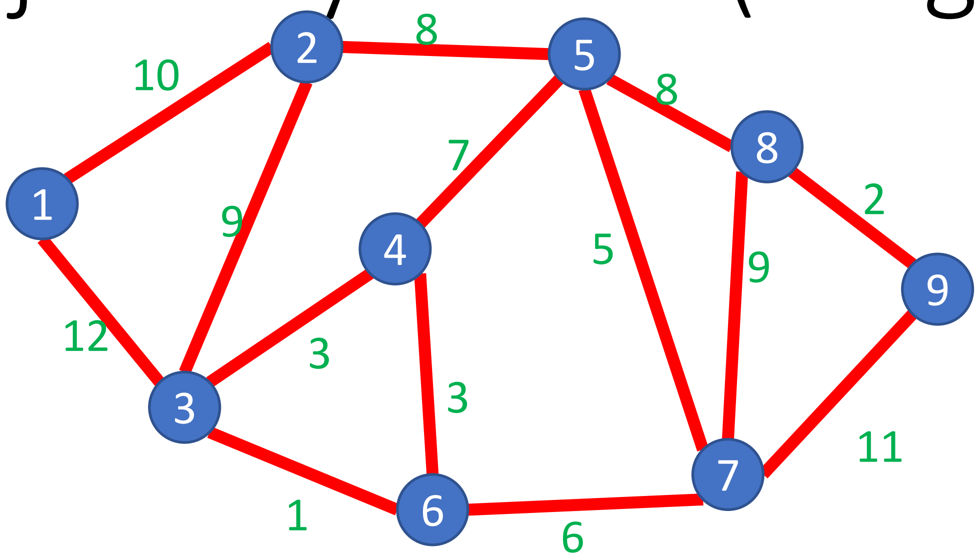
Get Neighbors (incoming): $\Theta(?)$

Get Neighbors (outgoing): $\Theta(?)$

$|V| = n$

$|E| = m$

Adjacency Matrix (weighted)



	A	B	C	D	E	F	G	H	I
A		1	1						
B	1		1		1				
C	1	1		1		1			
D			1		1	1			
E		1		1			1	1	
F			1	1			1		
G					1	1		1	1
H					1		1		1
I							1	1	

<u>Time/Space Tradeoffs</u>

Space to represent: $\Theta(n^2)$

Add Edge: $\Theta(1)$

Remove Edge: $\Theta(1)$

Check if Edge Exists: $\Theta(1)$

Get Neighbors (incoming): $\Theta(n)$

Get Neighbors (outgoing): $\Theta(n)$

$|V| = n$

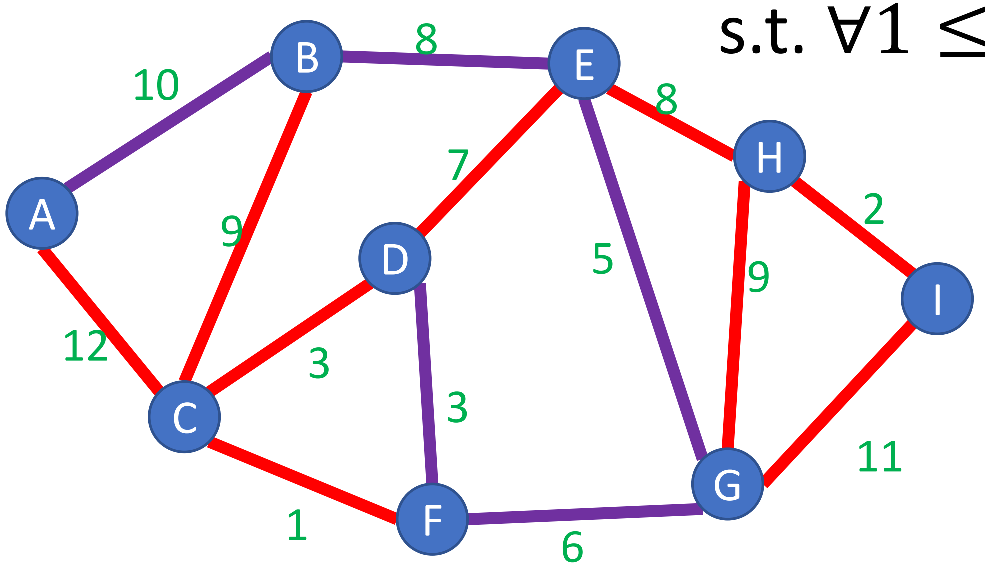
$|E| = m$

Aside

- Almost always, adjacency lists are the better choice
- Most graphs are missing most of their edges, so the adjacency list is much more space efficient and the slower operations aren’t that bad

Definition: Path

A sequence of nodes $(v_1, v_2, \dots, v_k)$
s.t. $\forall 1 \leq i \leq k-1,\ (v_i, v_{i+1}) \in E$

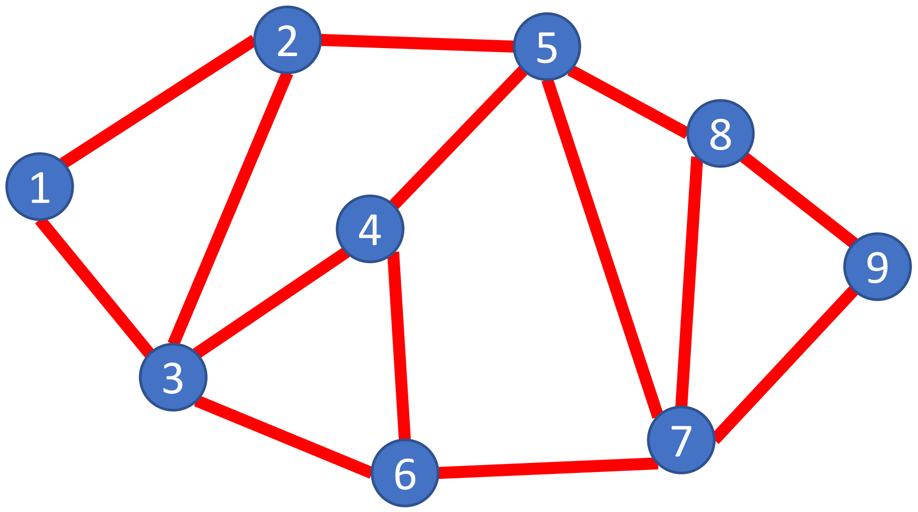


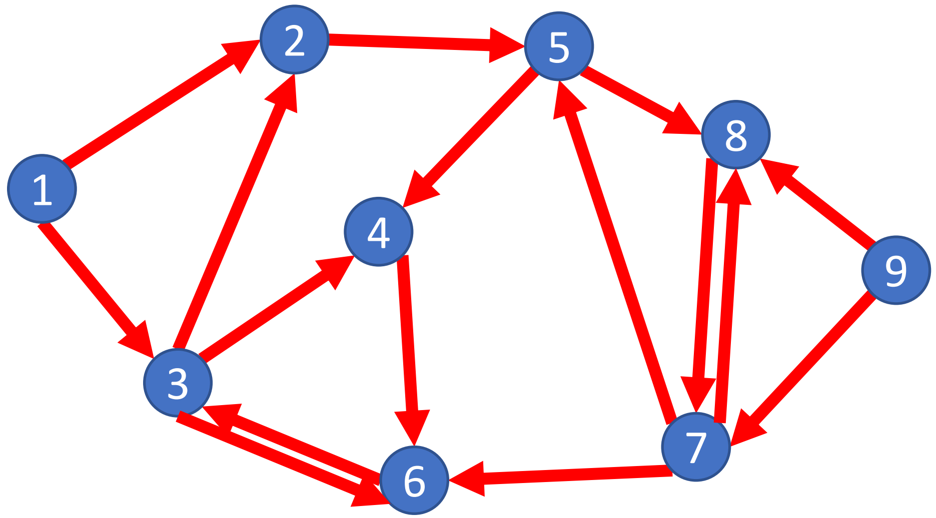
Simple Path:
A path in which each node appears at most once

Cycle:
A path which starts and ends in the same place

Definition: (Strongly) Connected Graph

A Graph $G = (V, E)$ s.t. for any pair of nodes $v_1, v_2 \in V$ there is a path from v_1 to v_2





Definition: (Strongly) Connected Graph

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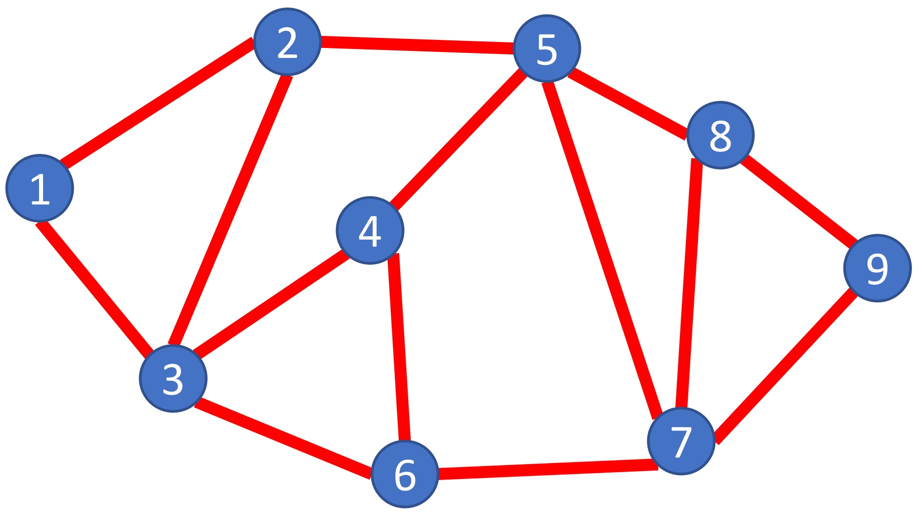
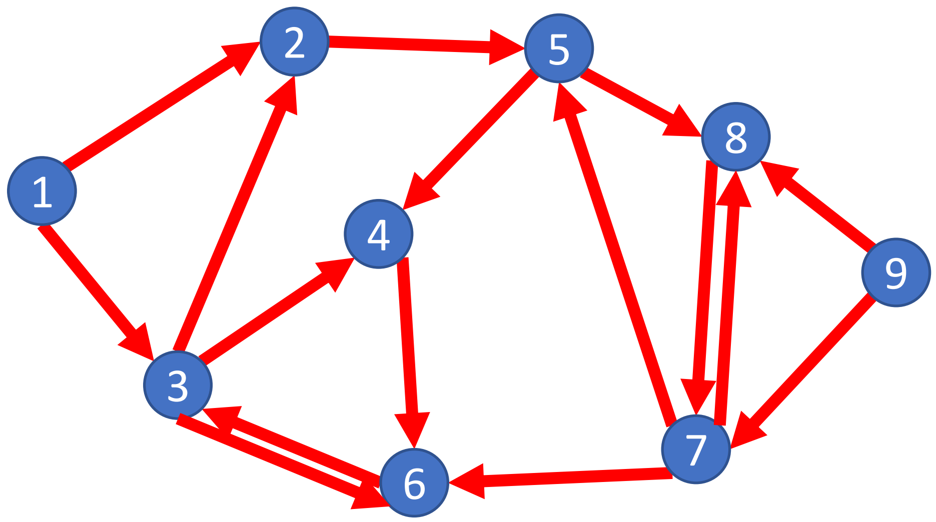


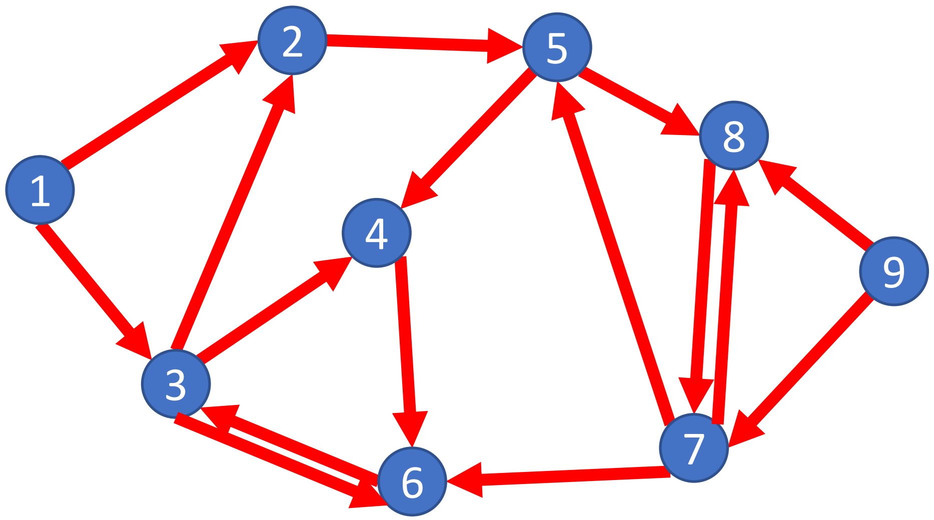
Connected



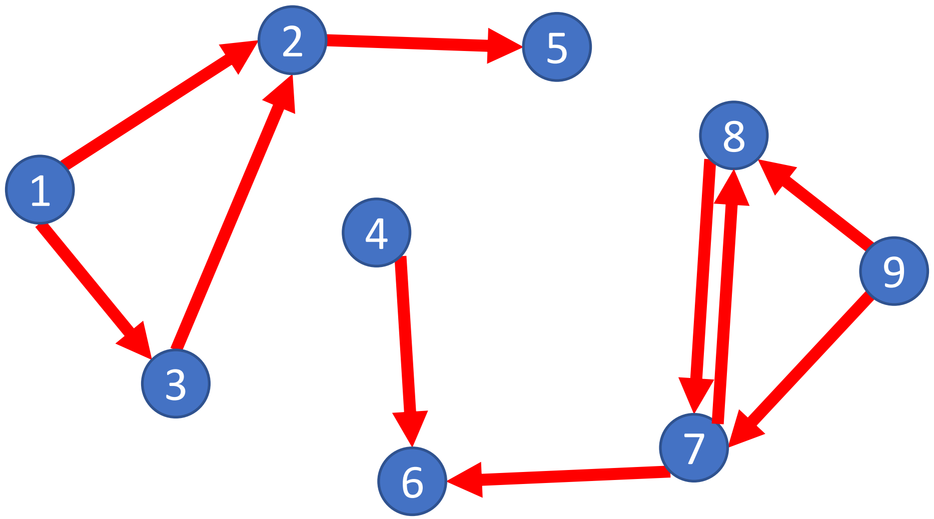
Not (strongly) Connected

Definition: Weakly Connected Graph

A Graph $G = (V, E)$ s.t. for any pair of nodes $v_1, v_2 \in V$ there is a path from v_1 to v_2 ignoring direction of edges



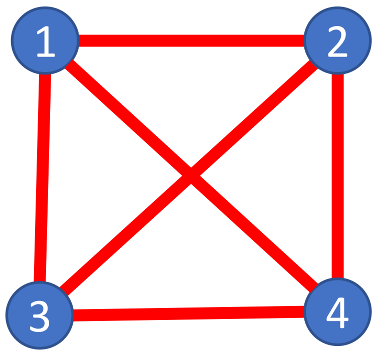
Weakly Connected



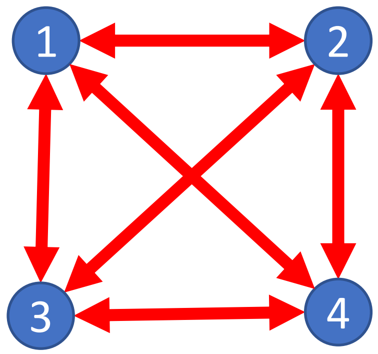
Weakly Connected

Definition: Complete Graph

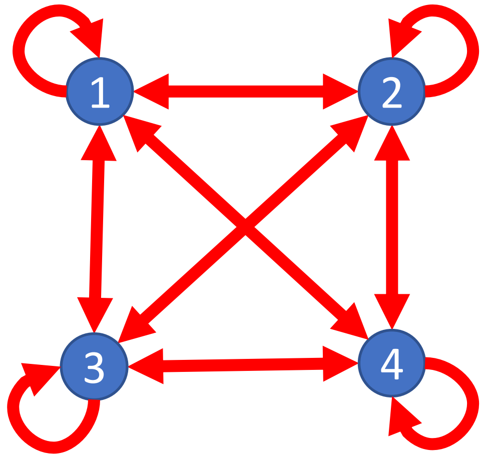
A Graph $G = (V, E)$ s.t. for any pair of nodes $v_1, v_2 \in V$ there is an edge from v_1 to v_2



Complete
Undirected Graph



Complete
Directed Graph



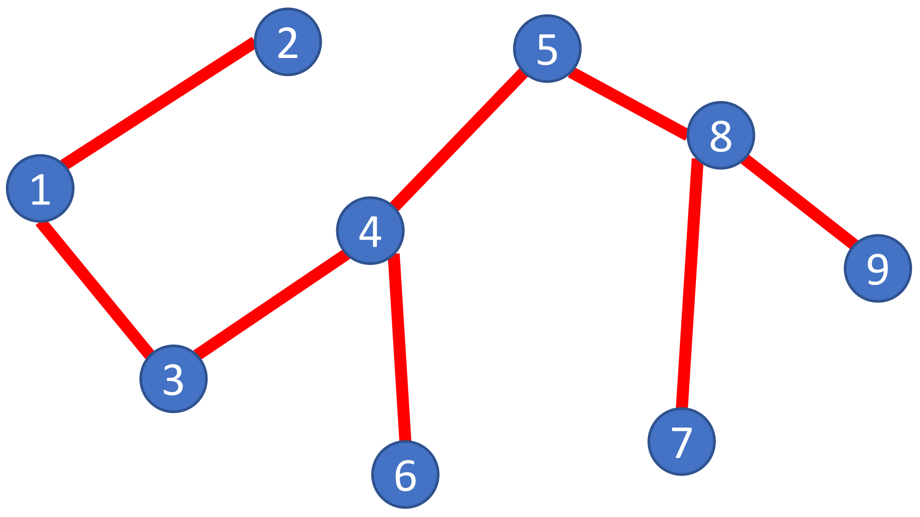
Complete Directed
Non-simple Graph

Graph Density, Data Structures, Efficiency

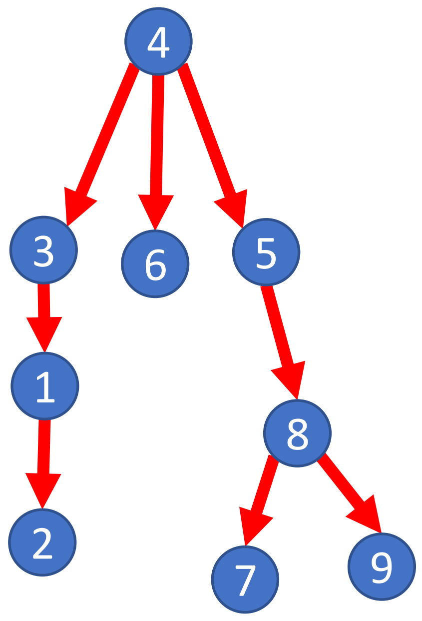
- The maximum number of edges in a graph is $\Theta(|V|^2)$:
  - Undirected and simple: $\frac{|V|(|V|-1)}{2}$
  - Directed and simple: $|V|(|V|-1)$
  - Direct and non-simple (but no duplicates): $|V|^2$
- If the graph is connected, the minimum number of edges is $|V|-1$
- If $|E| \in \Theta(|V|^2)$ we say the graph is **dense**
- If $|E| \in \Theta(|V|)$ we say the graph is **sparse**
- Because $|E|$ is not always near to $|V|^2$ we do not typically substitute $|V|^2$ for $|E|$ in running times, but leave it as a separate variable

Definition: Tree

A Graph $G = (V, E)$ is a tree if it is undirect, connected, and has no cycles (i.e. is acyclic). Often one node is identified as the “root”



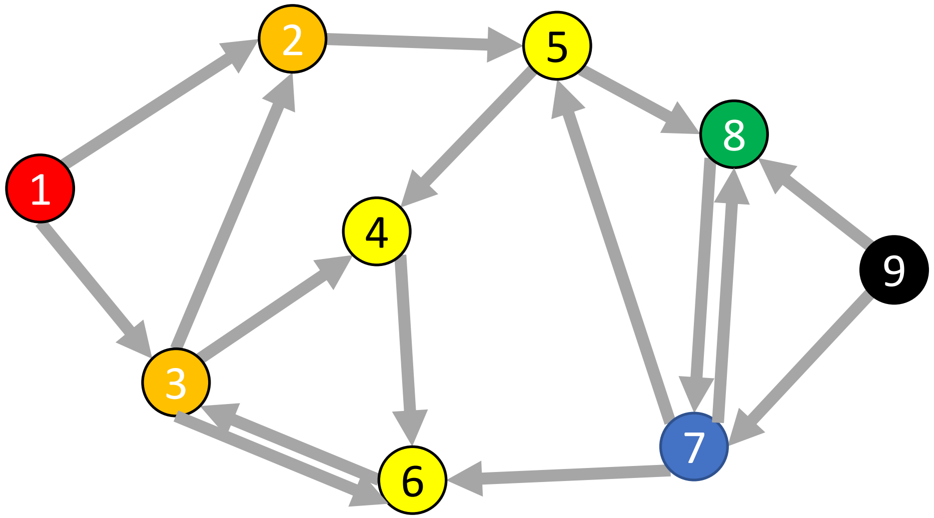
A Tree



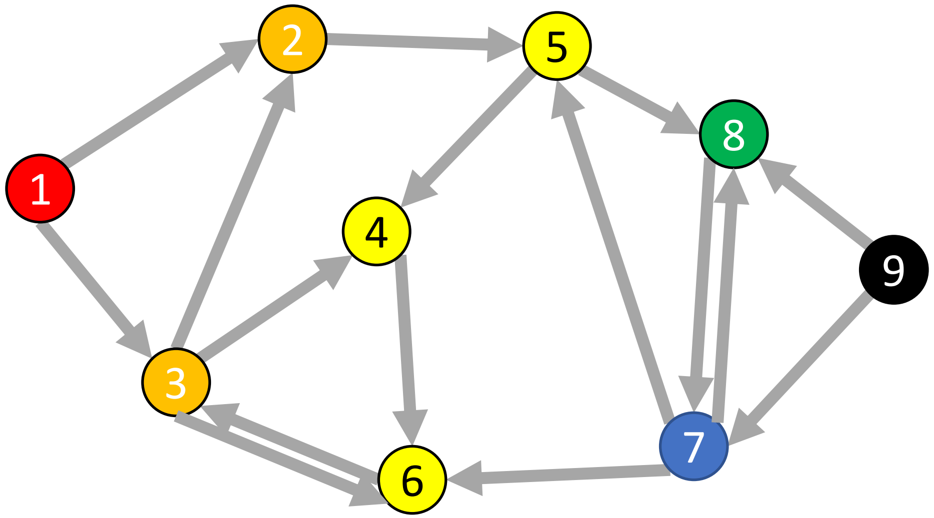
A Rooted Tree

Breadth-First Search

- Input: a node s
- Behavior: Start with node s, visit all neighbors of s, then all neighbors of neighbors of s, …
- Output:
  - How long is the shortest path?
  - Is the graph connected?



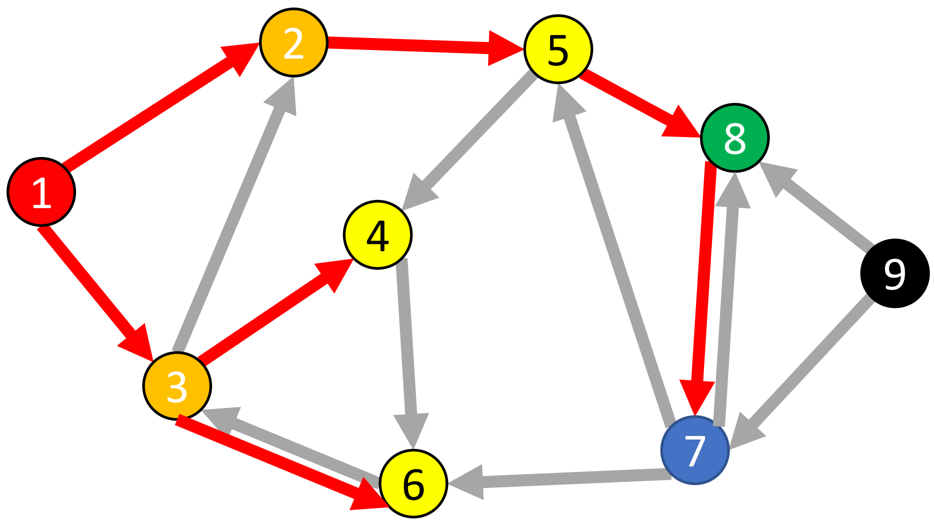
BFS



Running time: $\Theta(|V| + |E|)$

```
void bfs(graph, s){
        found = new Queue();
        found.enqueue(s);
        mark s as "visited";
        While (!found.isEmpty()){
                current = found.dequeue();
                for (v : neighbors(current)){
                        if (! v marked "visited"){
                                mark v as "visited";
                                found.enqueue(v);
                        }
                }
        }
}
```

Shortest Path (unweighted)



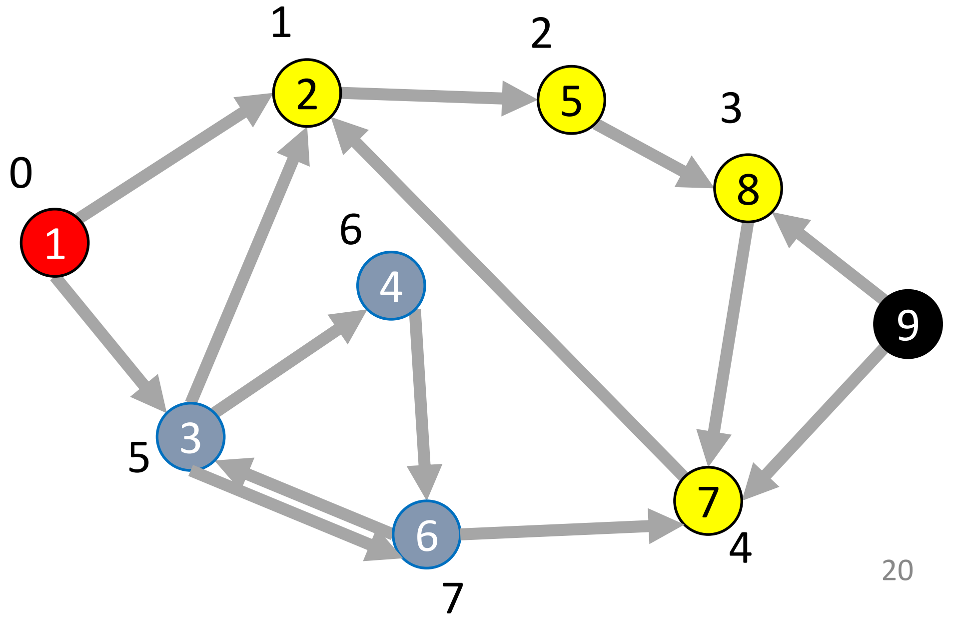
Idea: when it's seen, remember its "layer" depth!

```
int shortestPath(graph, s, t){
        found = new Queue();
        layer = 0;
        found.enqueue(s);
        mark s as "visited";
        While (!found.isEmpty()){
                current = found.dequeue();
                layer = depth of current;
                for (v : neighbors(current)){
                        if (! v marked "visited"){
                                mark v as "visited";
                                depth of v = layer + 1;
                                found.enqueue(v);
                        }
                }
        }
        return depth of t;
}
```

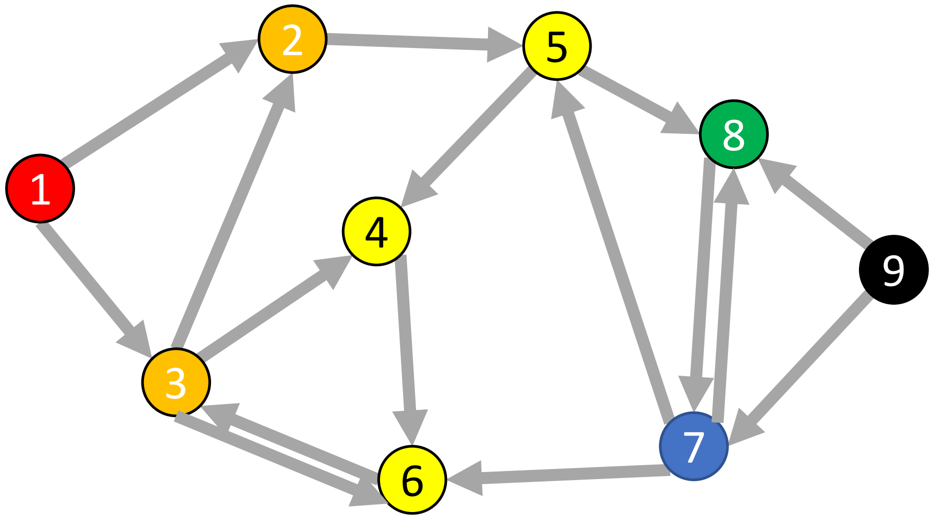
Depth-First Search

Depth-First Search

- Input: a node s
- Behavior: Start with node s, visit one neighbor of s, then all nodes reachable from that neighbor of s, then another neighbor of s,…
  - Before moving on to the second neighbor of s, visit everything reachable from the first neighbor of s
- Output:
  - Does the graph have a cycle?
  - A **topological sort** of the graph.



DFS (non-recursive)

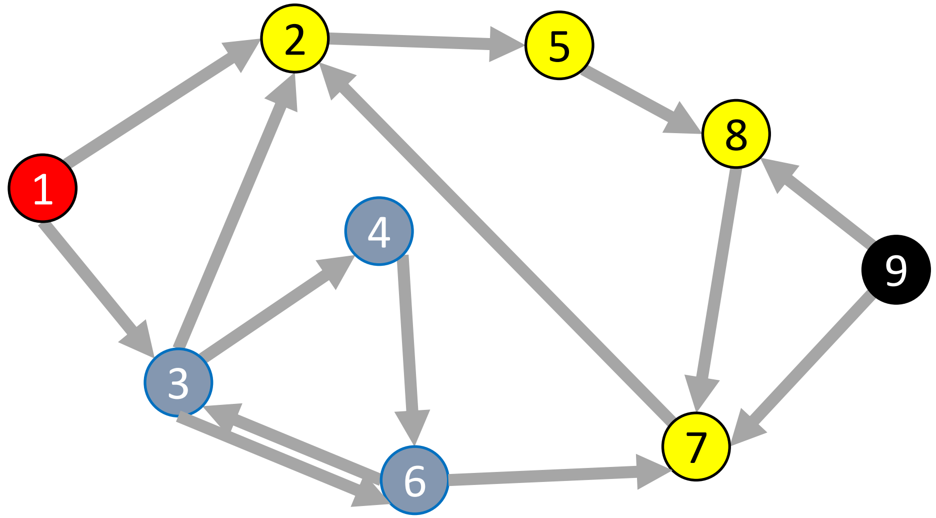


Running time: $\Theta(|V| + |E|)$

```
void dfs(graph, s){
        found = new Stack();
        found.pop(s);
        mark s as "visited";
        While (!found.isEmpty()){
                current = found.pop();
                for (v : neighbors(current)){
                        if (! v marked "visited"){
                                mark v as "visited";
                                found.push(v);
                        }
                }
        }
}
```

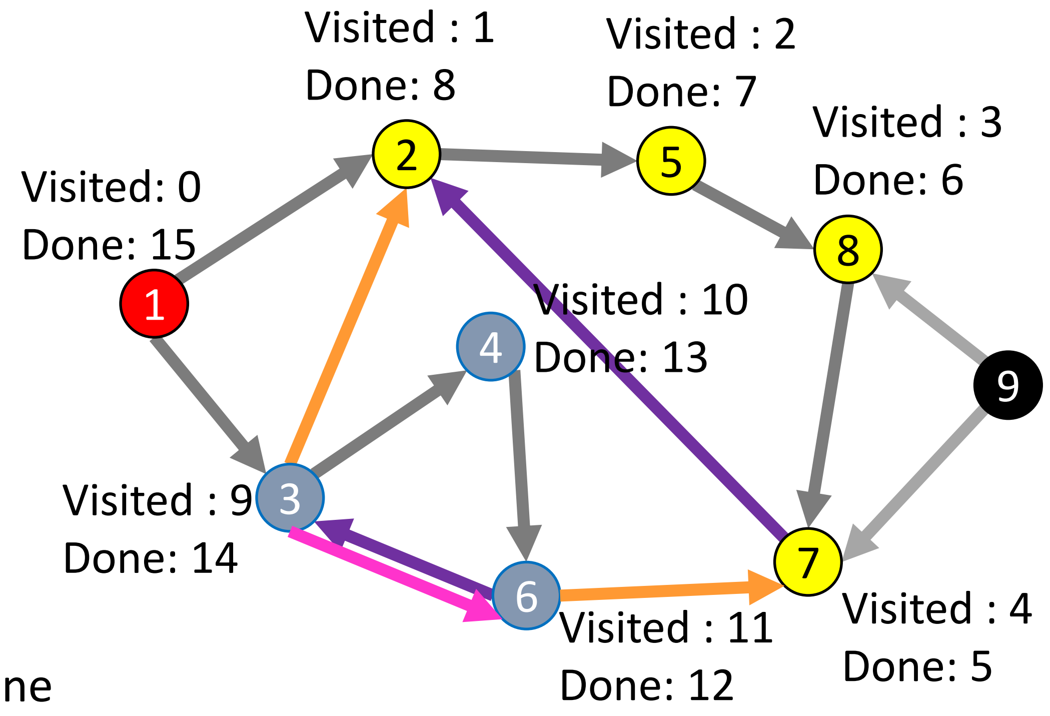
DFS Recursively (more common)

```
void dfs(graph, curr){
        mark curr as “visited”;
        for (v : neighbors(current)){
                if (! v marked “visited”){
                        dfs(graph, v);
                }
        }
        mark curr as “done”;
}
```



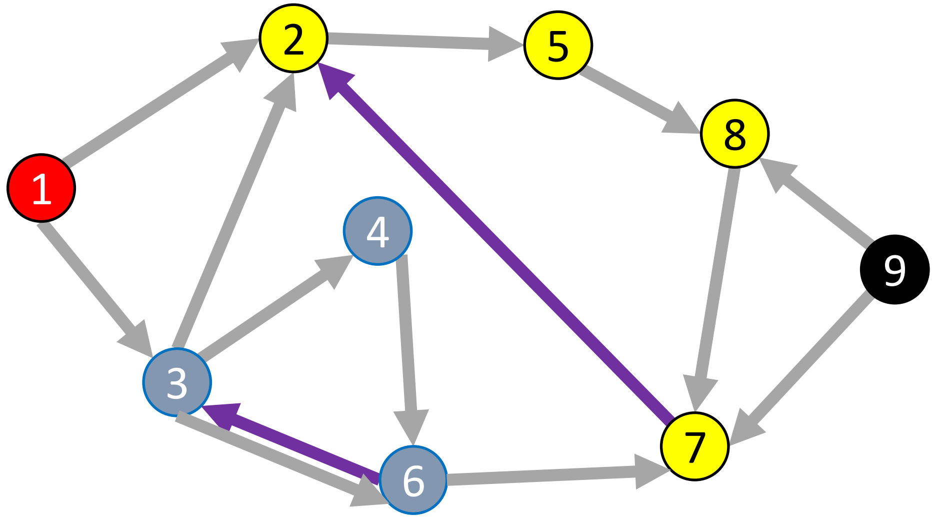
Using DFS

- Consider the "visited times" and "done times"
- Edges can be categorized:
  - Tree Edge
    - (a, b) was followed when pushing
    - (a, b) when b was unvisited when we were at a
  - Back Edge
    - (a, b) goes to an "ancestor"
    - a and b visited but not done when we saw (a, b)
    - $t_{visited}(b) < t_{visited}(a) < t_{done}(a) < t_{done}(b)$
  - Forward Edge
    - (a, b) goes to a "descendent"
    - b was visited and done between when a was visited and done
    - $t_{visited}(a) < t_{visited}(b) < t_{done}(b) < t_{done}(a)$
  - Cross Edge
    - (a, b) goes to a node that doesn't connect to a
    - b was seen and done before a was ever visited
    - $t_{done}(b) < t_{visited}(a)$



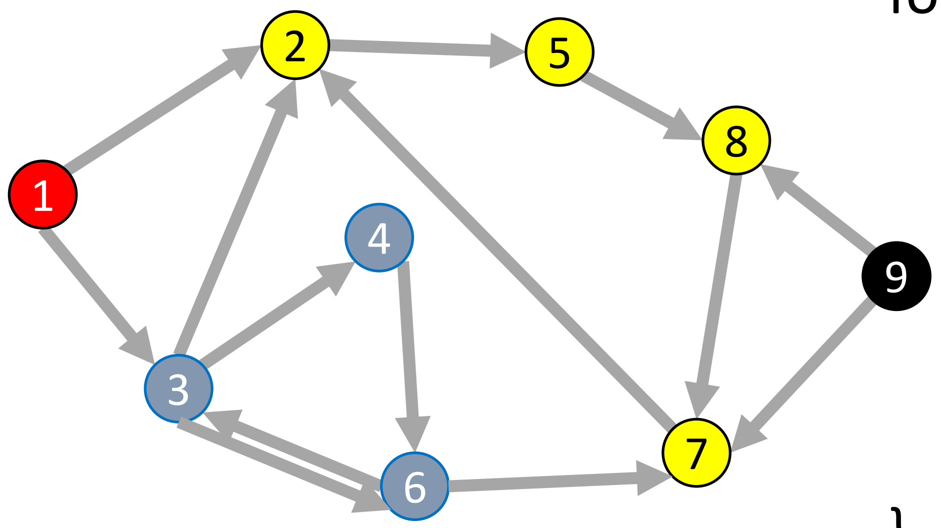
Back Edges

- Behavior of DFS:
  - “Visit everything reachable from the current node before going back”
- Back Edge:
  - The current node’s neighbor is an “in progress” node
  - Since that other node is “in progress”, the current node is reachable from it
  - The back edge is a path to that other node
  - **Cycle!**



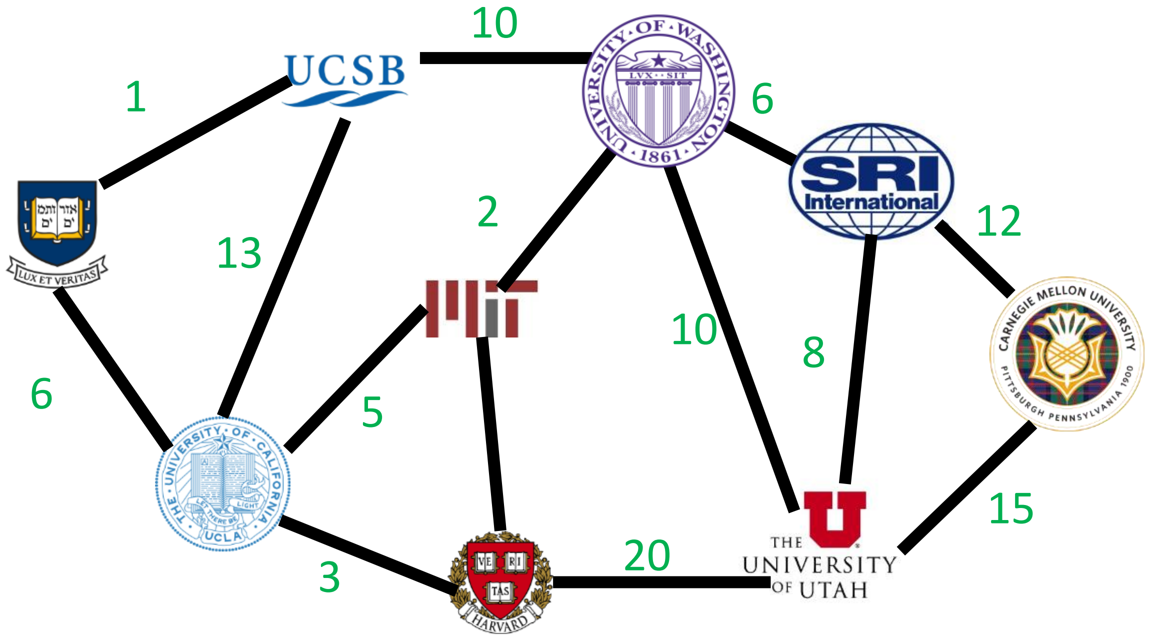
Cycle Detection

Idea: Look for a back edge!




```
boolean hasCycle(graph, curr){
        mark curr as "visited";
        cycleFound = false;
        for (v : neighbors(current)){
                if (v marked "visited" && ! v marked "done"){
                        cycleFound=true;
                }
                if (! v marked "visited" && !cycleFound){
                        cycleFound = hasCycle(graph, v);
                }
        }
        mark curr as "done";
        return cycleFound;
}
```

Single-Source Shortest Path



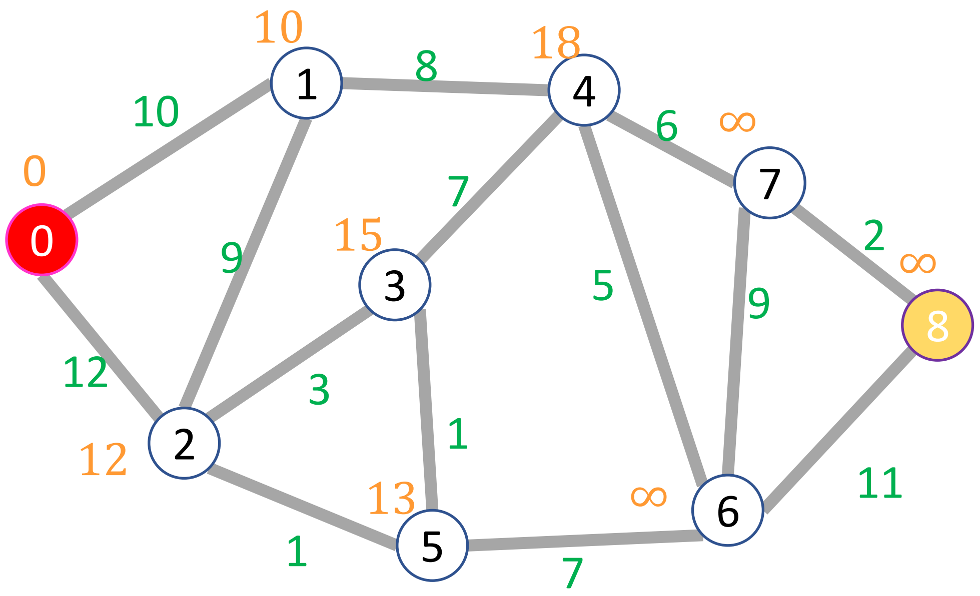
Find the quickest way to get from UVA to each of these other places

Given a graph $G = (V, E)$ and a start node $s \in V$, for each $v \in V$ find the least-weight path from $s \to v$ (call this weight $\delta(s, v)$)

(assumption: all edge weights are positive)

Dijkstra's Algorithm

- Input: graph with **no negative edge weights**, start node s, end node t
- Behavior: Start with node s, repeatedly go to the incomplete node "nearest" to s, stop when
- Output:
  - Distance from start to end
  - Distance from start to every node



Dijkstra’s Algorithm

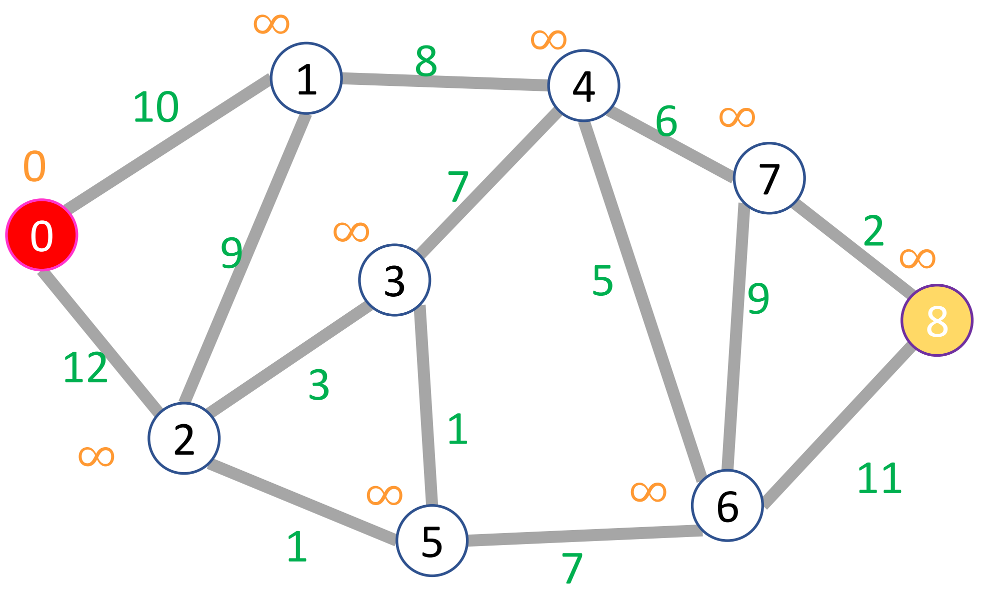
Start: 0

End: 8

Idea: When a node is the closest undiscovered thing to the start, we have found its shortest path

Node	Done?
0	F
1	F
2	F
3	F
4	F
5	F
6	F
7	F
8	F

Node	Distance
0	0
1	∞
2	∞
3	∞
4	∞
5	∞
6	∞
7	∞
8	∞



Dijkstra's Algorithm

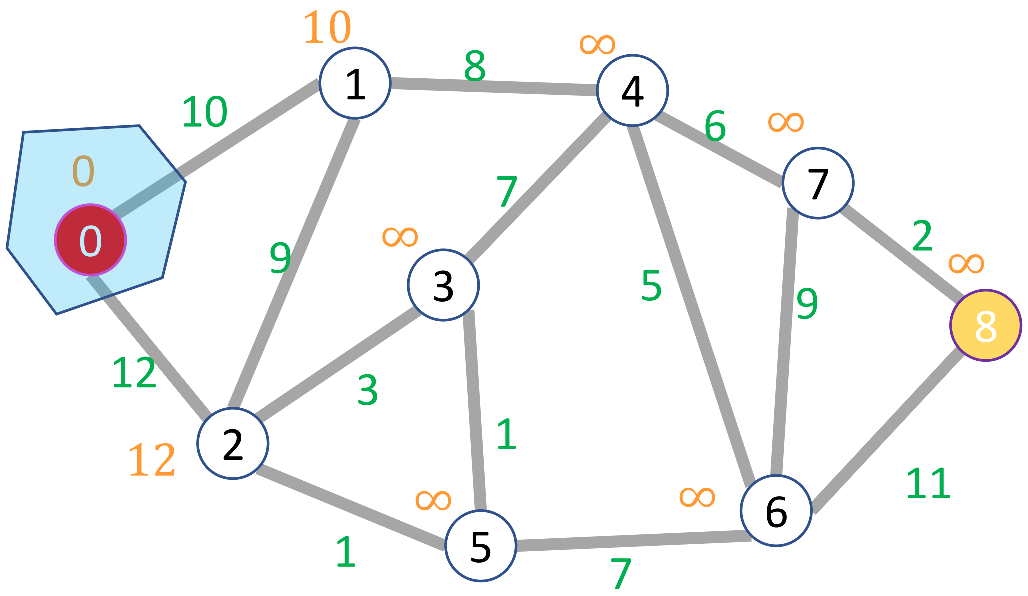
Start: 0

End: 8

Idea: When a node is the closest undiscovered thing to the start, we have found its shortest path

Node	Done?
0	T
1	F
2	F
3	F
4	F
5	F
6	F
7	F
8	F

Node	Distance
0	0
1	10
2	12
3	∞
4	∞
5	∞
6	∞
7	∞
8	∞



Dijkstra's Algorithm

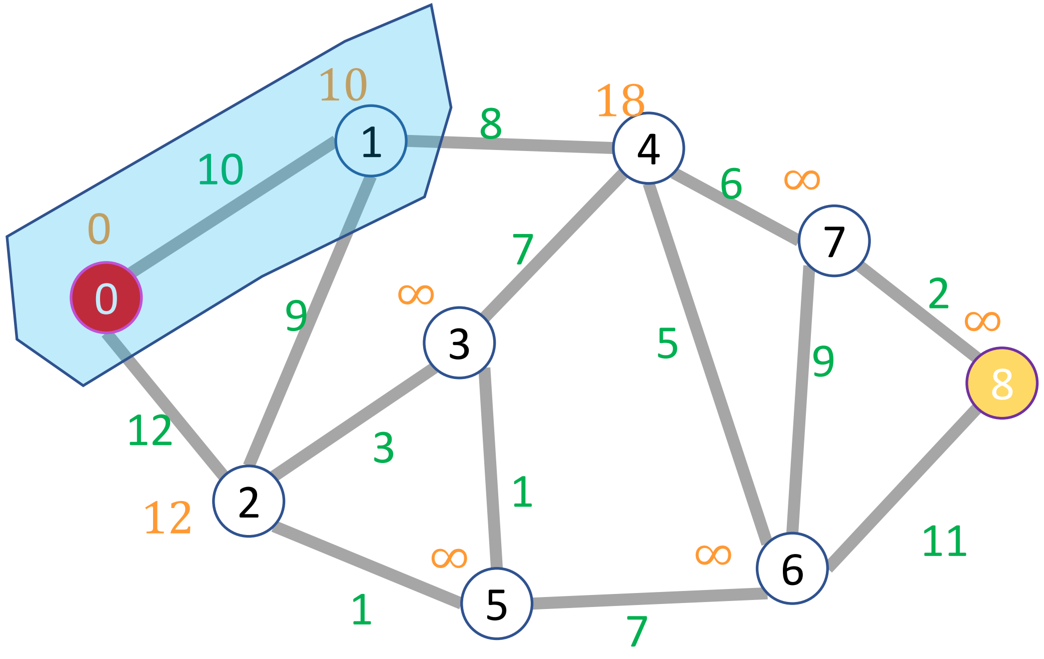
Start: 0

End: 8

Idea: When a node is the closest undiscovered thing to the start, we have found its shortest path

Node	Done?
0	T
1	T
2	F
3	F
4	F
5	F
6	F
7	F
8	F

Node	Distance
0	0
1	10
2	12
3	∞
4	18
5	∞
6	∞
7	∞
8	∞



Dijkstra's Algorithm

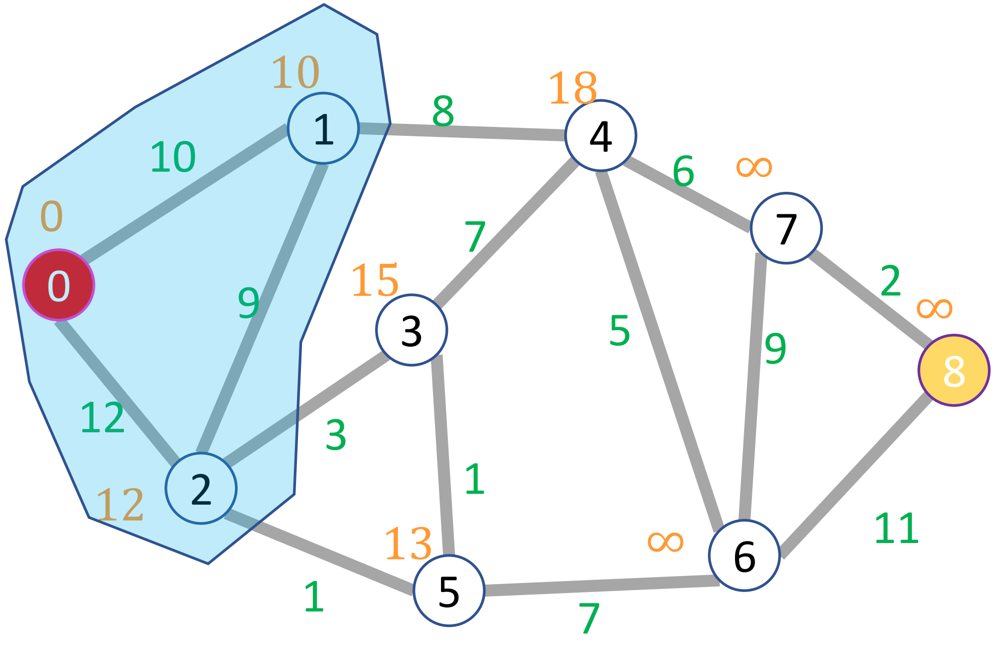
Start: 0

End: 8

Idea: When a node is the closest undiscovered thing to the start, we have found its shortest path

Node	Done?
0	T
1	T
2	T
3	F
4	F
5	F
6	F
7	F
8	F

Node	Distance
0	0
1	10
2	12
3	15
4	18
5	13
6	∞
7	∞
8	∞



Dijkstra's Algorithm

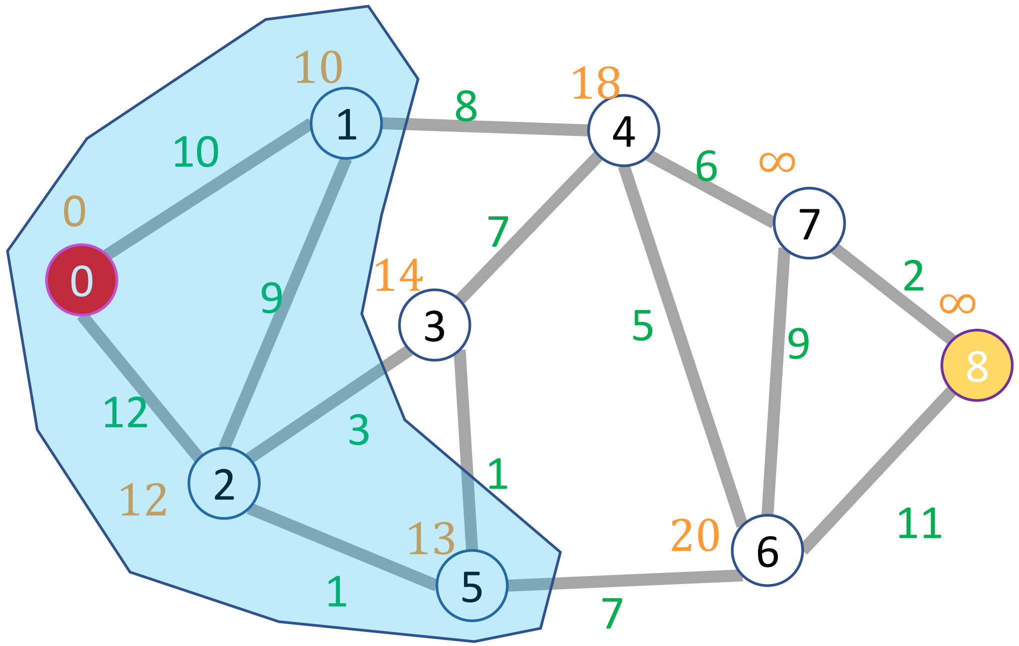
Start: 0

End: 8

Idea: When a node is the closest undiscovered thing to the start, we have found its shortest path

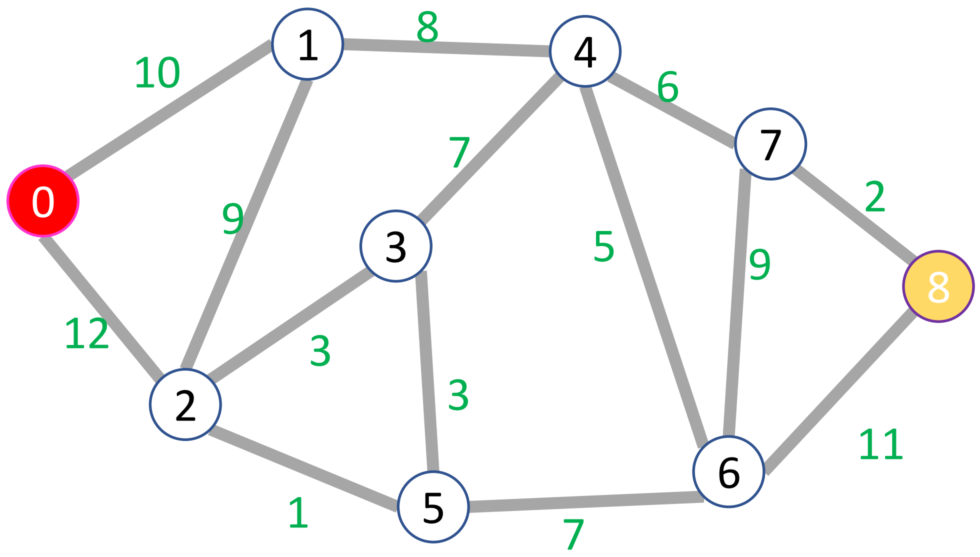
Node	Done?
0	T
1	T
2	T
3	F
4	F
5	T
6	F
7	F
8	F

Node	Distance
0	0
1	10
2	12
3	14
4	18
5	13
6	∞
7	20
8	∞



Dijkstra’s Algorithm

```
int dijkstras(graph, start, end){
        distances = [∞, ∞, ∞,…];  // one index per node
        done = [False,False,False,…];  // one index per node
        PQ = new minheap();
        PQ.insert(0, start);  // priority=0, value=start
        distances[start] = 0;
        while (!PQ.isEmpty){
                current = PQ.deleteMin();
                done[current] = true;
                for (neighbor : current.neighbors){
                        if (!done[neighbor]){
                                new_dist = distances[current]+weight(current,neighbor);
                                if new_dist < distances[neighbor]{
                                        distances[neighbor] = new_dist;
                                        PQ.decreaseKey(new_dist,neighbor); }
                        }
                }
        }
        return distances[end]
}
```

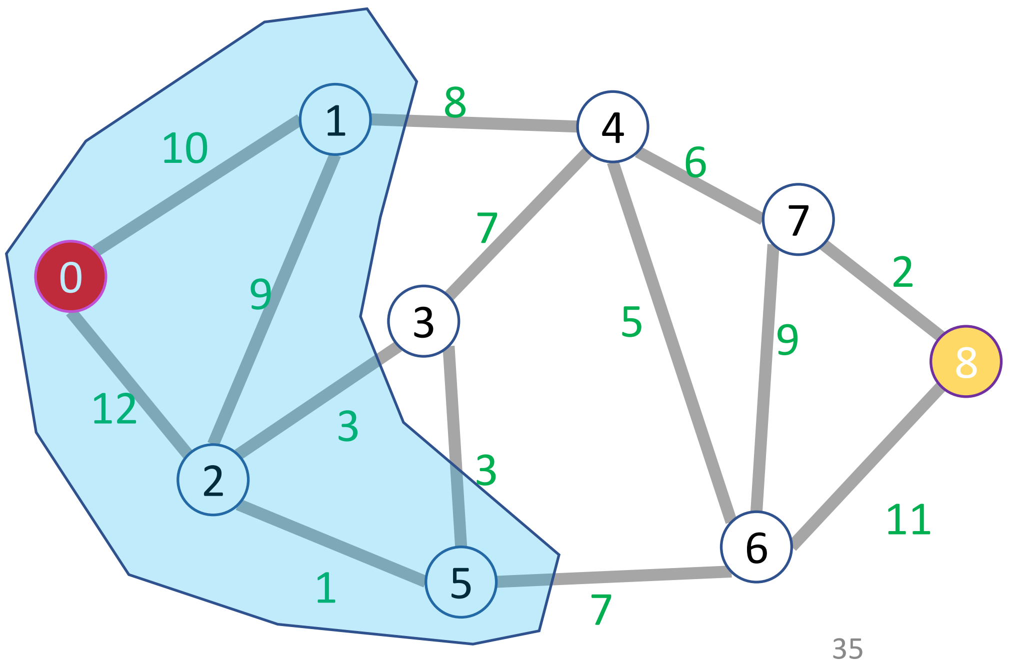


Dijkstra’s Algorithm: Running Time

- How many total priority queue operations are necessary?
  - How many times is each node added to the priority queue?
  - How many times might a node’s priority be changed?
- What’s the running time of each priority queue operation?
- Overall running time:
  - $\Theta(|E|\log|V|)$

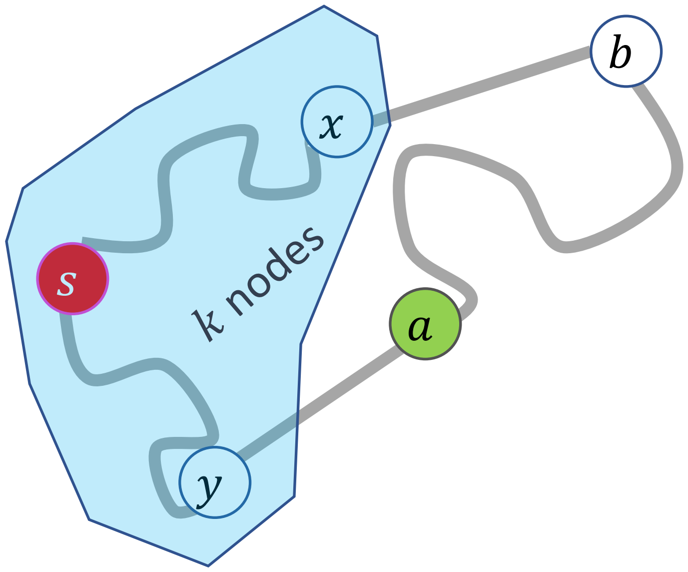
Dijkstra's Algorithm: Correctness

- Claim: when a node is removed from the priority queue, we have found its shortest path
- Induction over number of completed nodes
- Base Case:
- Inductive Step:



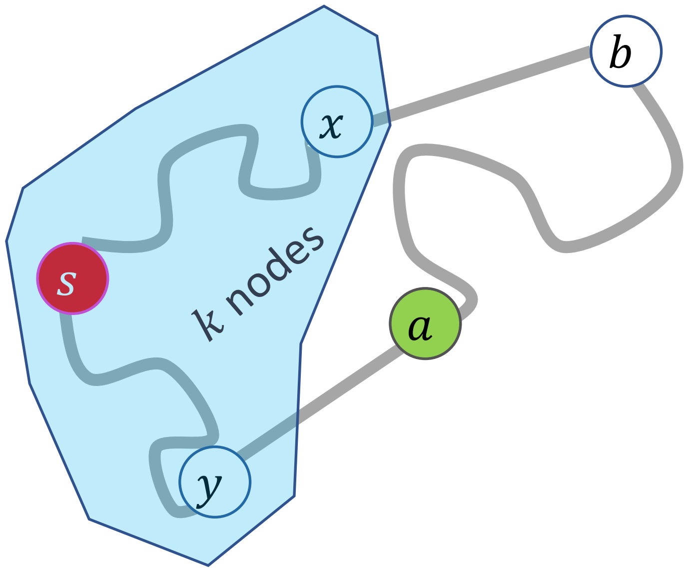
Dijkstra's Algorithm: Correctness

- Claim: when a node is removed from the priority queue, its distance is that of the shortest path
- Induction over number of completed nodes
- Base Case: Only the start node removed
  - It is indeed 0 away from itself
- Inductive Step:
  - If we have correctly found shortest paths for the first k nodes, then when we remove node $k+1$ we have found its shortest path



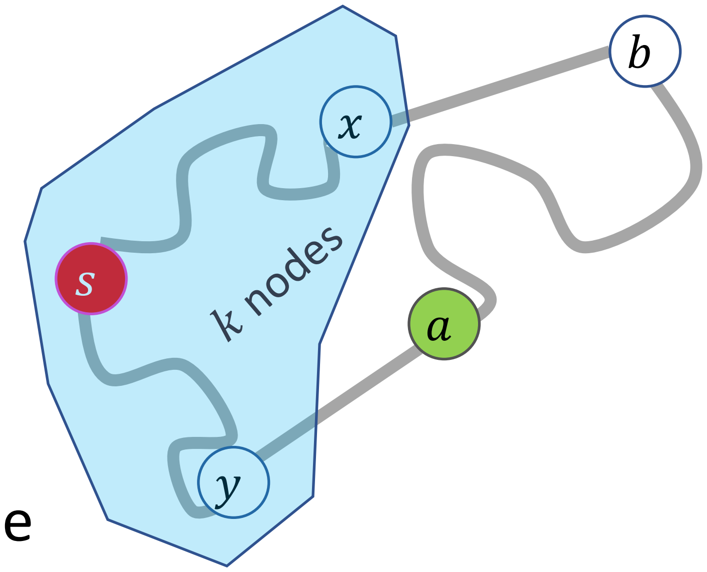
Dijkstra’s Algorithm: Correctness

- Suppose a is the next node removed from the queue. What do we know bout a?



Dijkstra's Algorithm: Correctness

- Suppose a is the next node removed from the queue.
  - No other node incomplete node has a shorter path discovered so far
- Claim: no undiscovered path to a could be shorter
  - Consider any other incomplete node b that is 1 edge away from a complete node
  - a is the closest node that is one away from a complete node
  - Thus no path that includes b can be a shorter path to a
  - Therefore the shortest path to a must use only complete nodes, and therefore we have found it already!



Dijkstra’s Algorithm: Correctness

- Suppose a is the next node removed from the queue.
  - No other node incomplete node has a shorter path discovered so far
- Claim: no undiscovered path to a could be shorter
  - Consider any other incomplete node b that is 1 edge away from a complete node
  - a is the closest node that is one away from a complete node
  - No path from b to a can have negative weight
  - Thus no path that includes b can be a shorter path to a
  - Therefore the shortest path to a must use only complete nodes, and therefore we have found it already!