CSE 332 Autumn 2023
Lecture 23: Ahmdal’s Law, Parallel Prefix

Nathan Brunelle

http://www.cs.uw.edu/332

Work and Span

- Let $T_P(n)$ be the running time if there are P processors available
- Two key measures of run time:
  - Work: How long it would take 1 processor, so $T_1(n)$
    - Just suppose all forks are done sequentially
    - Cumulative work all processors must complete
    - For array sum: $\Theta(n)$
  - Span: How long it would take an infinite number of processors, so $T_\infty(n)$
    - Theoretical ideal for parallelization
    - Longest "dependence chain" in the algorithm
    - Also called "critical path length" or "computation depth"
    - For array sum: $\Theta(\log n)$

Asymptotically Optimal T_P

- We know how to compute T_1 and T_∞, but what about T_P?
  - T_P cannot be better than $\frac{T_1}{P}$
  - T_P cannot be better than T_∞
- An asymptotically optimal execution would be
  - $T_P(n) \in O\left(\frac{T_1(n)}{P} + T_\infty(n)\right)$
  - $T_1(n)/P$ dominates for small P, $T_\infty(n)$ dominates for large P
- ForkJoin Frameworks gives an expected time guarantee of asymptotically optimal!

And now for some bad news…

- In practice it’s common for your program to have:
  - Parts that parallelize well
    - Maps/reduces over arrays and other data structures
  - And parts that don’t parallelize at all
    - Reading a linked list, getting input, or computations where each step needs the results of previous step
- These unparallelized parts can turn out to be a big bottleneck

Amdahl’s Law (mostly bad news)

- Suppose $T_1 = 1$
  - Work for the entire program is 1
- Let S be the proportion of the program that cannot be parallelized
  - $T_1 = S + (1 - S) = 1$
- Suppose we get perfect linear speedup on the parallel portion
  - $T_P = S + \frac{1-S}{P}$
- For the entire program, the speed is:
  - $\frac{T_1}{T_P} = \frac{1}{S+\frac{1-S}{P}}$
- And so the parallelism (infinite processors) is:
  - $\frac{T_1}{T\_\infty} = \frac{1}{S}$

Ahmdal’s Law Example

- Suppose 2/3 of your program is parallelizable, but 1/3 is not.
  - $S = \frac{2}{3}$
  - $T_1 = \frac{2}{3} + \frac{1}{3} = 1$
- $T_P = S + \frac{1-S}{P}$
- So if T_1 is 100 seconds:
  - $T_P = 33 + \frac{67}{P}$
  - $T_3 = 33 + \frac{67}{3} = 33 + 22 = 55$

Conclusion

- Even with many many processors the sequential part of your program becomes a bottleneck
- Parallelizable code requires skill and insight from the developer to recognize where parallelism is possible, and how to do it well.

Which Data Structures are “Suitable” for Parallelism?

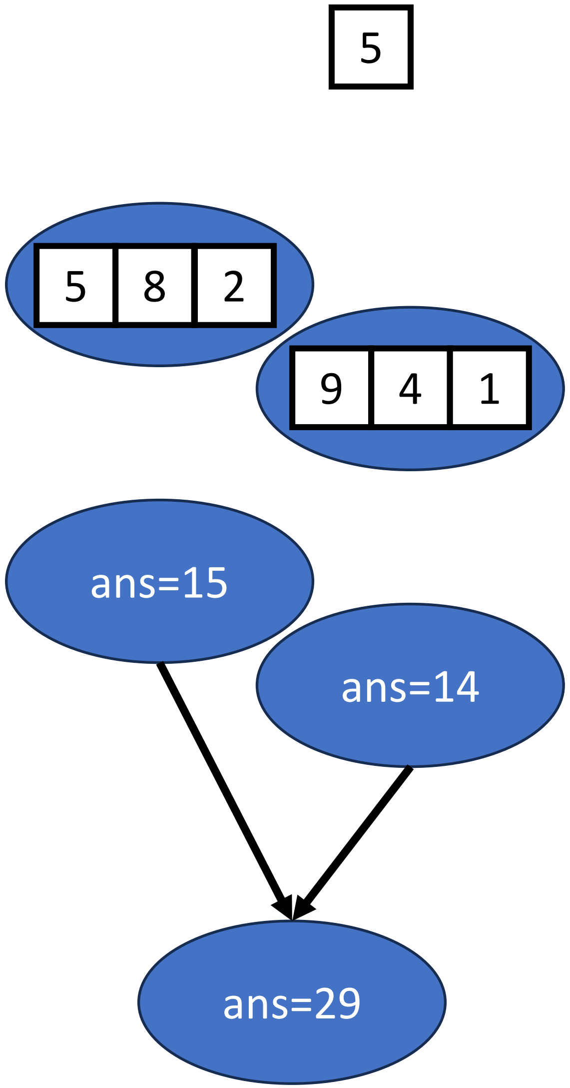
- For each data structure, can we write a parallel algorithm to some all of its values such that $T_1 > T_\infty$?
  - Array
  - Linked List
  - Tree

Reductions

- “Reduce” all elements in an array to a single item
  - Requires operation done among elements is associative
    - $(x + y) + z = x + (y + z)$
  - The “single item” can itself be complex
    - E.g. create a histogram of results from an array of trials

Reduction (sum an array)

5	8	2	9	4	1

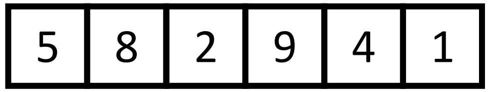


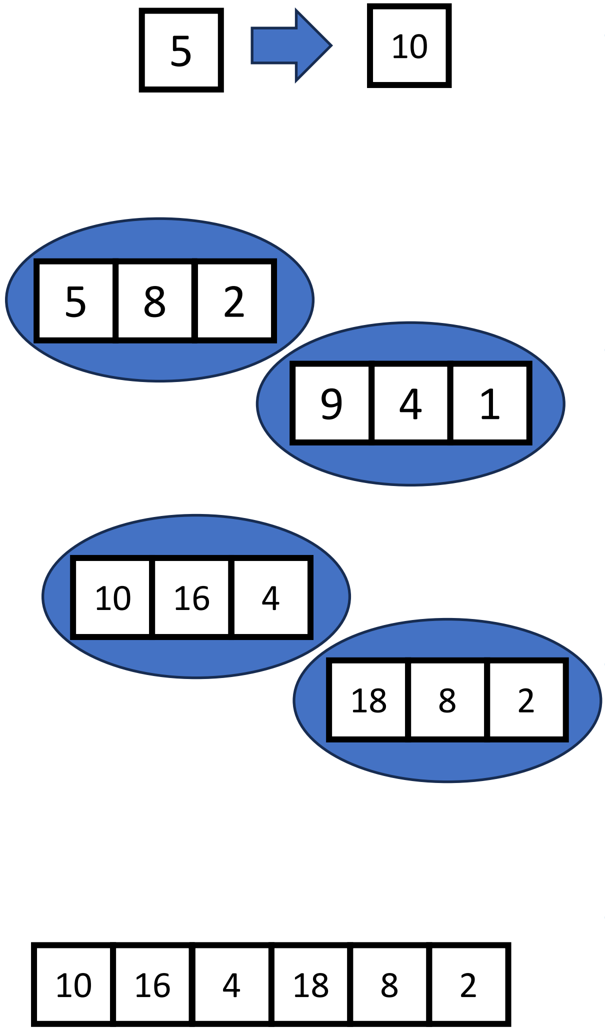
- **Base Case:**
  - If the list’s length is smaller than the Sequential Cutoff, reduce things sequentially
- **Divide:**
  - Split the list into two “sublists” of (roughly) equal length, create a thread to reduce each sublist.
- **Conquer:**
  - Call **start()** for each thread
- **Combine:**
  - Reduce the answers from each thread

Map

- Perform an operation on each item in an array to create a new array of the same size
- Examples:
  - Vector addition:
    - sum[i] = arr1[i] + arr2[i]
  - Function application:
    - out[i] = f(arr[i]);

Map (double each value)





- **Base Case:**
  - If the list’s length is smaller than the Sequential Cutoff, convert each thing sequentially
- **Divide:**
  - Split the list into two “sublists” of (roughly) equal length, create a thread to map each sublist.
- **Conquer:**
  - Call **start()** for each thread
- **Combine:**
  - No additional work necessary

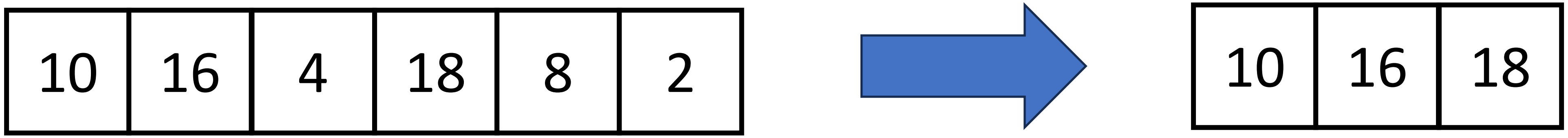
Maps and Reductions

- “Workhorse” constructs in parallel programming
- Many problems can be written in terms of maps and reductions
- With practice, writing them will become second nature
  - Like how over time for loops and if statements have gotten easier

- Today:
  - Filter/Pack to complete the trio!

Pack/Filter

- Given an array of values and a Boolean function, return a new array which contains only elements that were “true



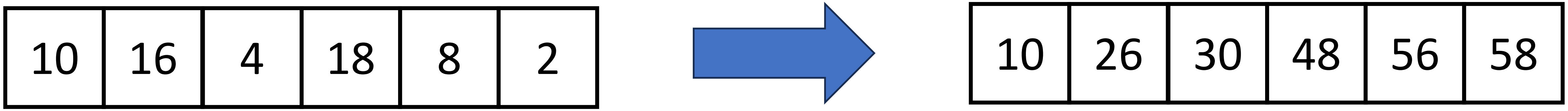
10	16	4	18	8	2

10	16	18

$f(x) = x > 9$

Prefix Sum

- Given an array, compute a new array where each index i is the sum of all values up to i



10	16	4	18	8	2

10	26	30	48	56	58

```
int[] prefixSum(int[] arr){
        int[] output = new int[arr.length];
        output[0] = arr[0];
        for (int i = 1; I < arr.length, i++)
                output[i] = output[i-1] + arr[i];
        return output;
}
```

Parallel Prefix Sum

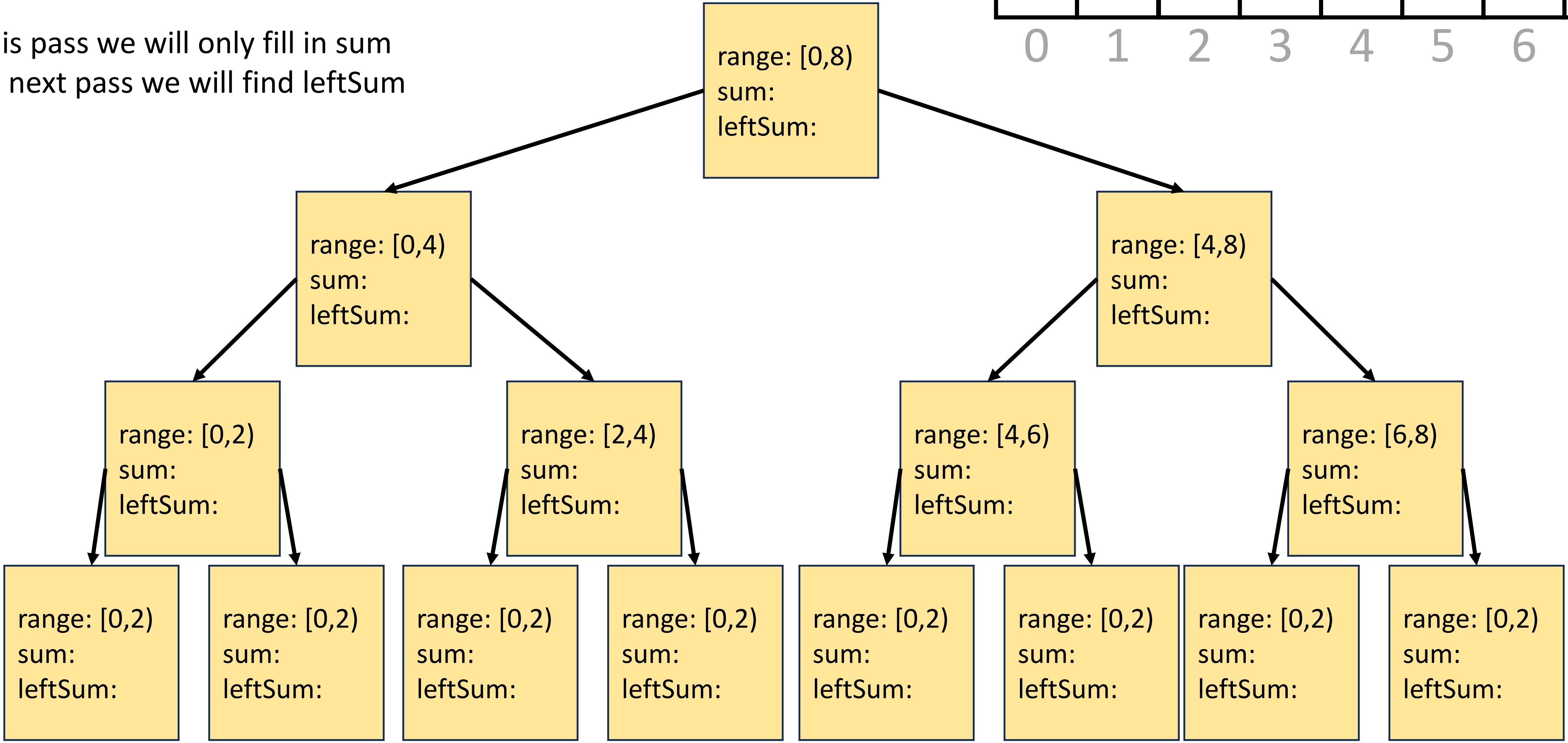
- Algorithm will have two major parallel steps
  - Called a “two pass” parallel algorithm
- First step:
  - Create a tree data structure
- Second Step:
  - Use the tree to fill in the output array

Richard Ladner
Allen School Faculty

Step 1: Create a Tree, Fill in sum

Input:	10	16	4	18	8	2	14	9
Output:								
	0	1	2	3	4	5	6	7

For this pass we will only fill in sum
In the next pass we will find leftSum



range: [0,8)
sum:
leftSum:

range: [0,4)
sum:
leftSum:

range: [4,8)
sum:
leftSum:

range: [0,2)
sum:
leftSum:

range: [2,4)
sum:
leftSum:

range: [4,6)
sum:
leftSum:

range: [6,8)
sum:
leftSum:

range: [0,2)
sum:
leftSum:

range: [0,2)
sum:
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sum:
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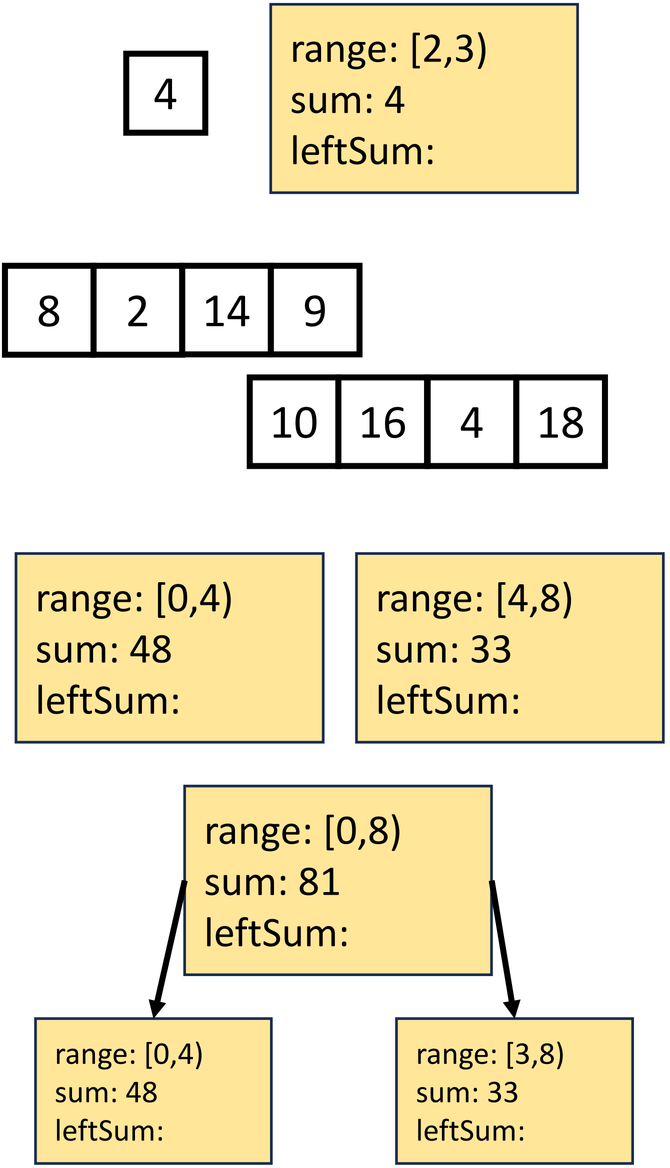
range: [0,2)
sum:
leftSum:

range: [0,2)
sum:
leftSum:

range: [0,2)
sum:
leftSum:

Step 1: Create a Tree, Fill in sum

10	16	4	18	8	2	14	9



- **Base Case**:
  - If the rand is smaller than the Sequential Cutoff, create a node for that range and find the sum sequentially
- **Divide**:
  - Split the list into two “sublists” of (roughly) equal length, create a thread for each sublist.
- **Conquer**:
  - Call **start()** for each thread to compute the left and right subtrees
- **Combine**:
  - Create parent node, connect to children, fill in sum

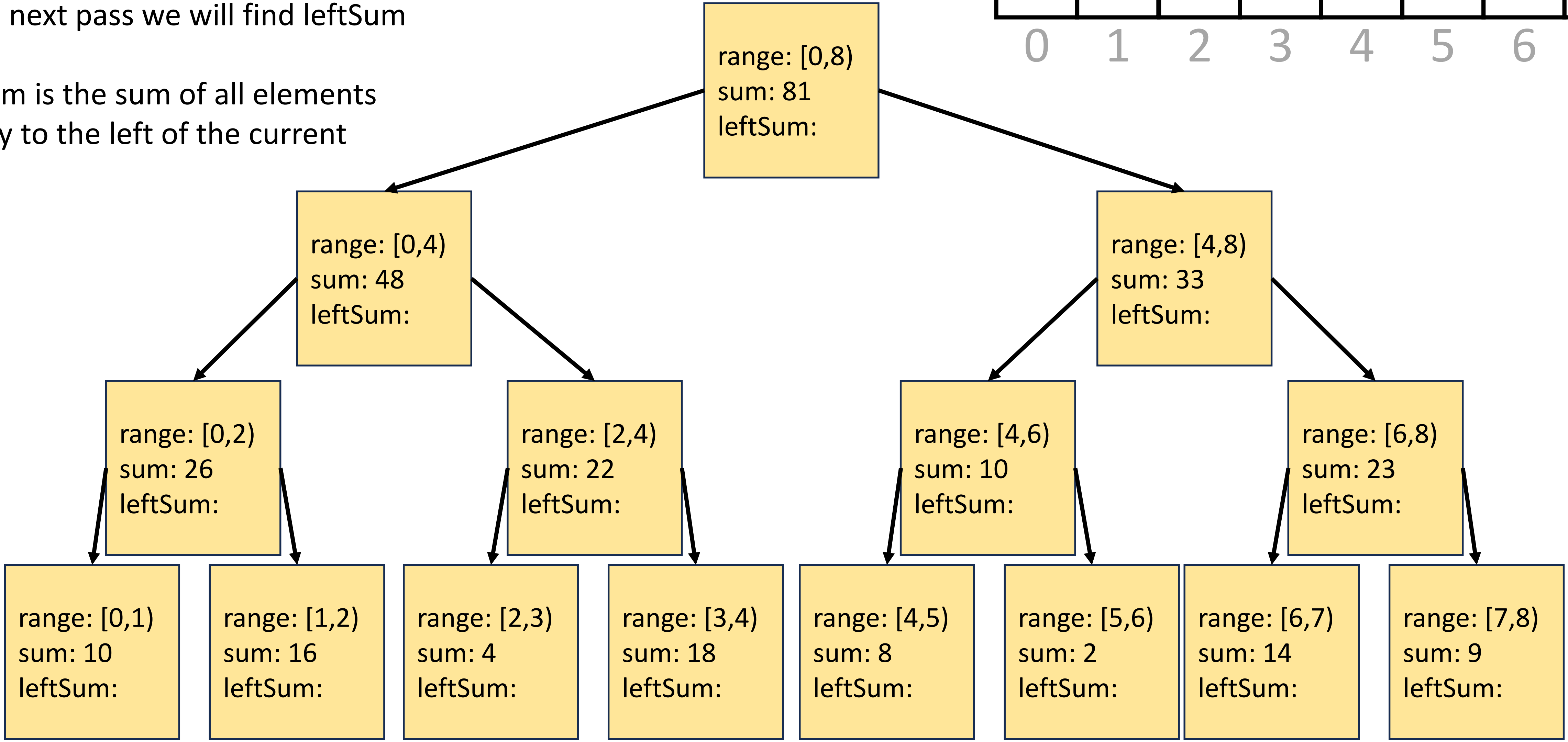
```
class BuildTree extends RecursiveTask<Node> {
        protected Node compute(){
                if(hi – lo < SEQUENTIAL_CUTOFF) {  // base case
                        int ans = 0; // local var, not a field
                        for(int i=lo; i < hi; i++)
                                ans += arr[i];
                        return new Node(lo, hi, ans); }
                else {
                        BuildTree left = new BuildTree(arr,lo,(hi+lo)/2);
                        BuildTree right= new BuildTree(arr,(hi+lo)/2,hi);
                        left.fork();
                        Node rightChild = right.compute();
                        Node leftChild = left.join();
                        int ans = rightChild.sum + leftChild.sum;
                        parent = new Node(lo, hi, ans);
                        parent.left = leftChild;
                        parent.right = rightChild;
                        return parent; }
        }
}
```

After Step 1

All sums filled in per node
In the next pass we will find leftSum

leftSum is the sum of all elements strictly to the left of the current range

Input:	10	16	4	18	8	2	14	9
Output:								
	0	1	2	3	4	5	6	7



range: [0,8)
sum: 81
leftSum:

range: [0,4)
sum: 48
leftSum:

range: [4,8)
sum: 33
leftSum:

range: [0,2)
sum: 26
leftSum:

range: [2,4)
sum: 22
leftSum:

range: [4,6)
sum: 10
leftSum:

range: [6,8)
sum: 23
leftSum:

range: [0,1)
sum: 10
leftSum:

range: [1,2)
sum: 16
leftSum:

range: [2,3)
sum: 4
leftSum:

range: [3,4)
sum: 18
leftSum:

range: [4,5)
sum: 8
leftSum:

range: [5,6)
sum: 2
leftSum:

range: [6,7)
sum: 14
leftSum:

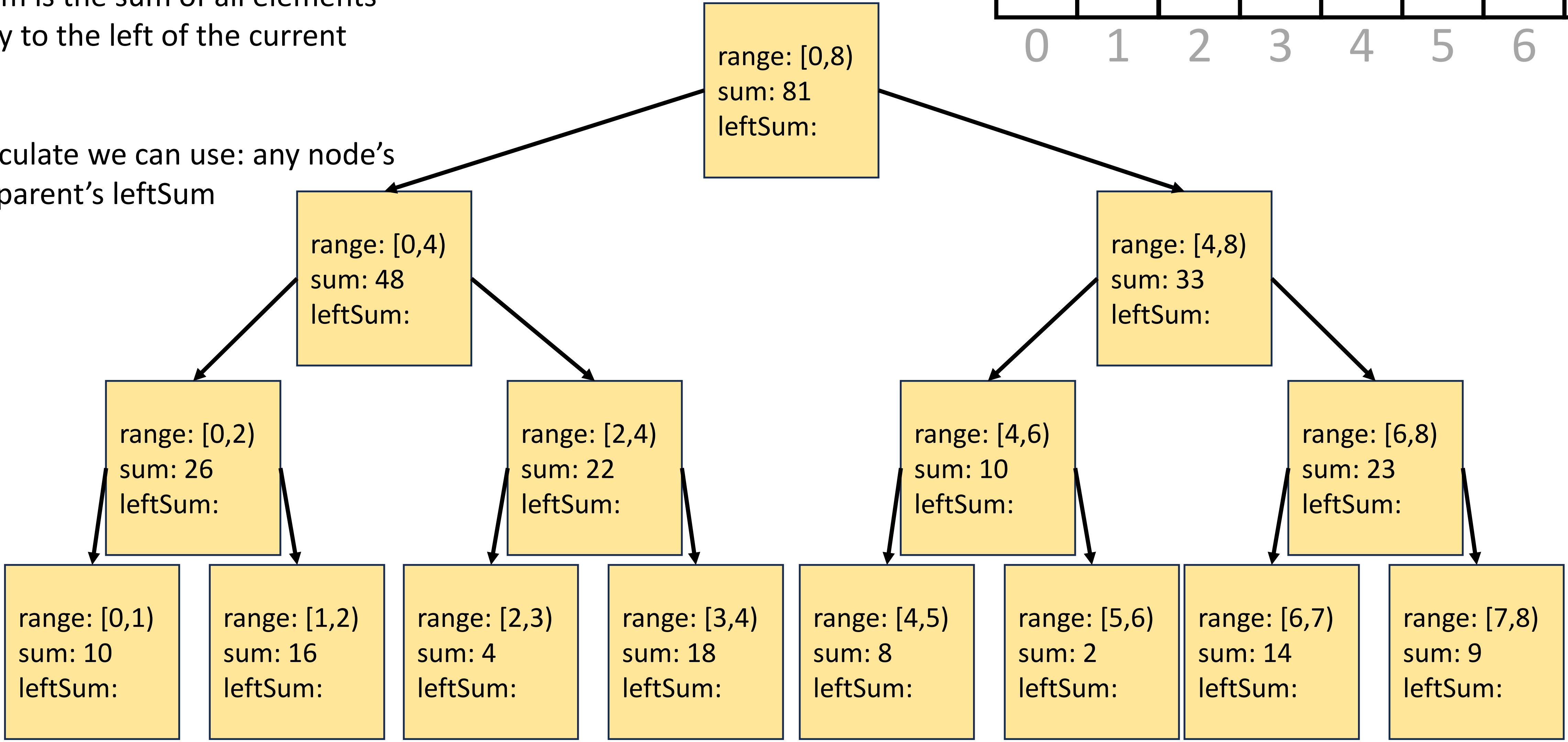
range: [7,8)
sum: 9
leftSum:

Step 2: fill in leftSum and Output

leftSum is the sum of all elements strictly to the left of the current range

To calculate we can use: any node's sum, parent's leftSum

Input:	10	16	4	18	8	2	14	9
Output:								
	0	1	2	3	4	5	6	7



range: [0,8)
sum: 81
leftSum:

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leftSum:

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sum: 22
leftSum:

range: [4,6)
sum: 10
leftSum:

range: [6,8)
sum: 23
leftSum:

range: [0,1)
sum: 10
leftSum:

range: [1,2)
sum: 16
leftSum:

range: [2,3)
sum: 4
leftSum:

range: [3,4)
sum: 18
leftSum:

range: [4,5)
sum: 8
leftSum:

range: [5,6)
sum: 2
leftSum:

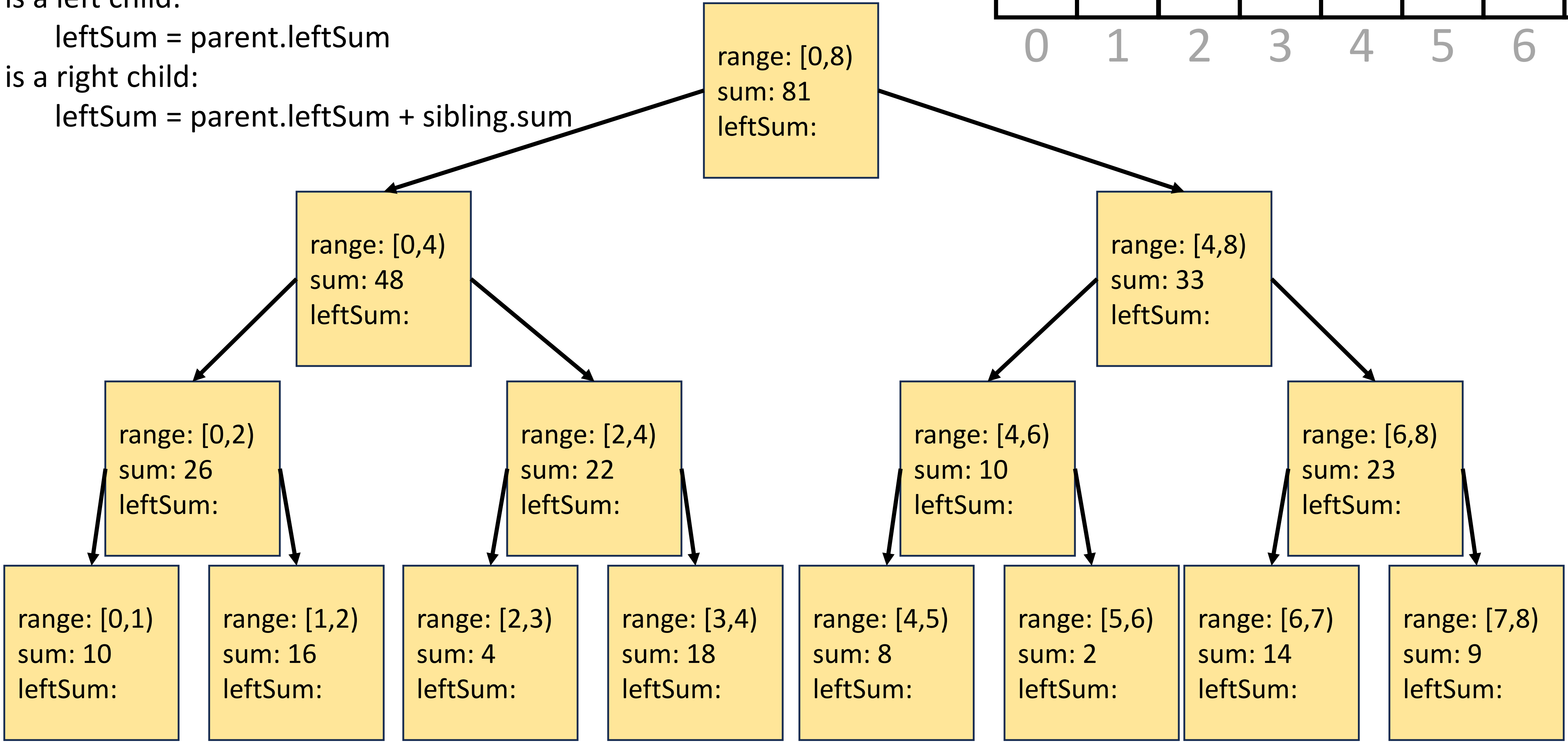
range: [6,7)
sum: 14
leftSum:

range: [7,8)
sum: 9
leftSum:

Step 2: fill in leftSum and Output

Input:	10	16	4	18	8	2	14	9
Output:								
	0	1	2	3	4	5	6	7

If this is a left child:
leftSum = parent.leftSum
If this is a right child:
leftSum = parent.leftSum + sibling.sum



range: [0,8)
sum: 81
leftSum:

range: [0,4)
sum: 48
leftSum:

range: [4,8)
sum: 33
leftSum:

range: [0,2)
sum: 26
leftSum:

range: [2,4)
sum: 22
leftSum:

range: [4,6)
sum: 10
leftSum:

range: [6,8)
sum: 23
leftSum:

range: [0,1)
sum: 10
leftSum:

range: [1,2)
sum: 16
leftSum:

range: [2,3)
sum: 4
leftSum:

range: [3,4)
sum: 18
leftSum:

range: [4,5)
sum: 8
leftSum:

range: [5,6)
sum: 2
leftSum:

range: [6,7)
sum: 14
leftSum:

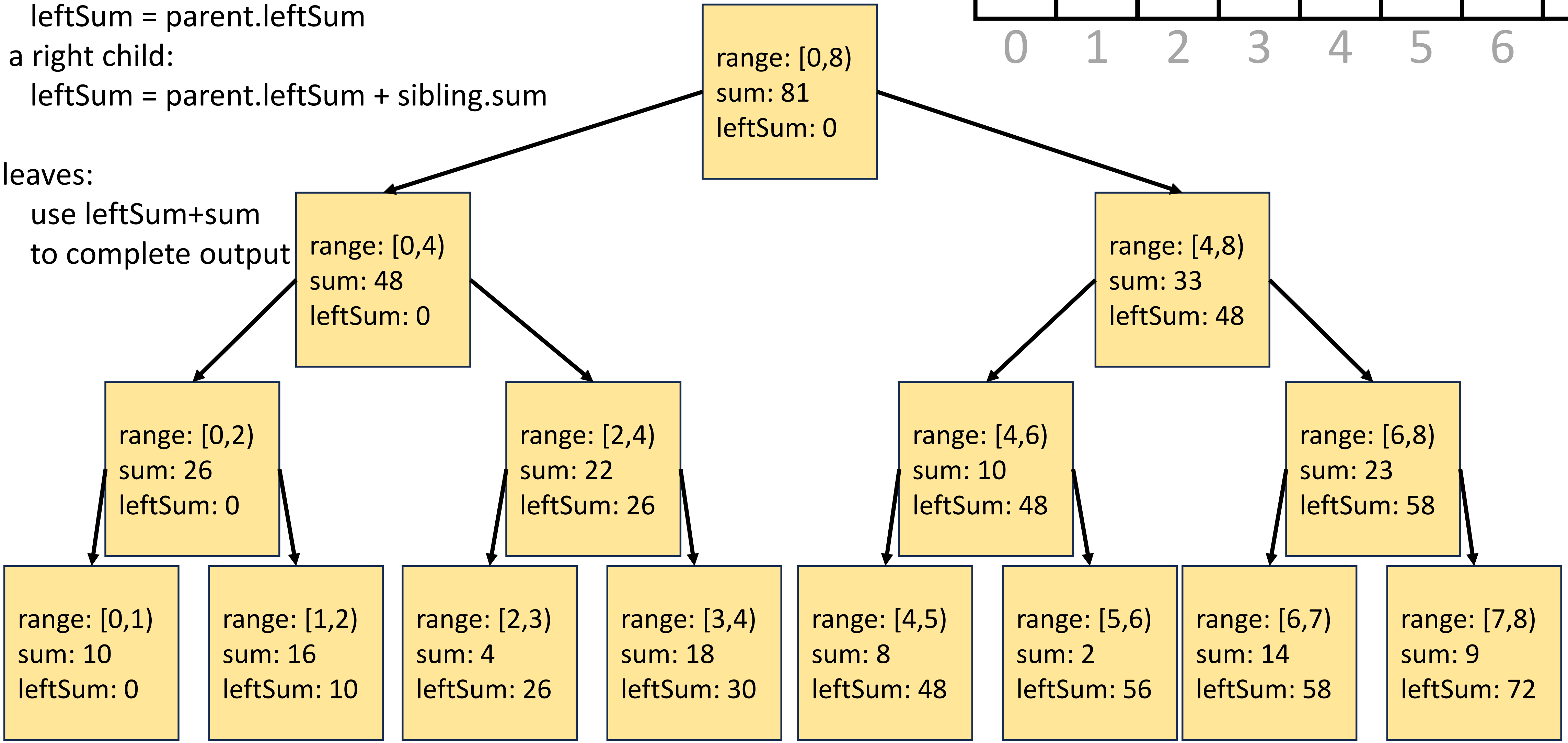
range: [7,8)
sum: 9
leftSum:

Step 2: fill in leftSum and Output

Input:	10	16	4	18	8	2	14	9
Output:								
	0	1	2	3	4	5	6	7

If this is a left child:
leftSum = parent.leftSum
If this is a right child:
leftSum = parent.leftSum + sibling.sum

For the leaves:
use leftSum+sum
to complete output



range: [0,8)
sum: 81
leftSum: 0

range: [0,4)
sum: 48
leftSum: 0

range: [4,8)
sum: 33
leftSum: 48

range: [0,2)
sum: 26
leftSum: 0

range: [2,4)
sum: 22
leftSum: 26

range: [4,6)
sum: 10
leftSum: 48

range: [6,8)
sum: 23
leftSum: 58

range: [0,1)
sum: 10
leftSum: 0

range: [1,2)
sum: 16
leftSum: 10

range: [2,3)
sum: 4
leftSum: 26

range: [3,4)
sum: 18
leftSum: 30

range: [4,5)
sum: 8
leftSum: 48

range: [5,6)
sum: 2
leftSum: 56

range: [6,7)
sum: 14
leftSum: 58

range: [7,8)
sum: 9
leftSum: 72

Step 2: fill in leftSum and Output

If this is a left child:

leftSum = parent.leftSum

If this is a right child:

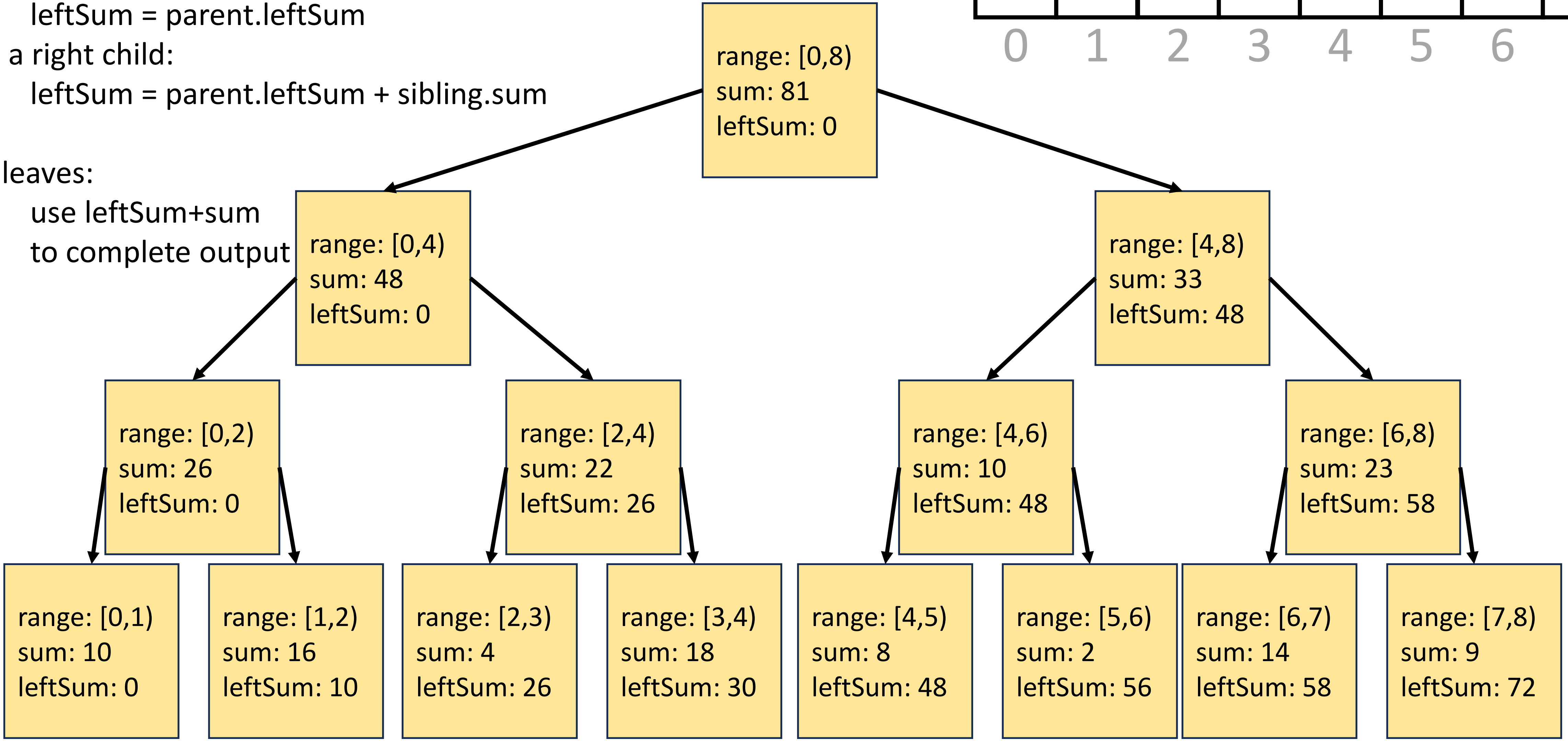
leftSum = parent.leftSum + sibling.sum

For the leaves:

use leftSum+sum

to complete output

	0	1	2	3	4	5	6	7
Input:	10	16	4	18	8	2	14	9
Output:	10	26	30	48	56	58	72	81



range: [0,8)
sum: 81
leftSum: 0

range: [0,4)
sum: 48
leftSum: 0

range: [4,8)
sum: 33
leftSum: 48

range: [0,2)
sum: 26
leftSum: 0

range: [2,4)
sum: 22
leftSum: 26

range: [4,6)
sum: 10
leftSum: 48

range: [6,8)
sum: 23
leftSum: 58

range: [0,1)
sum: 10
leftSum: 0

range: [1,2)
sum: 16
leftSum: 10

range: [2,3)
sum: 4
leftSum: 26

range: [3,4)
sum: 18
leftSum: 30

range: [4,5)
sum: 8
leftSum: 48

range: [5,6)
sum: 2
leftSum: 56

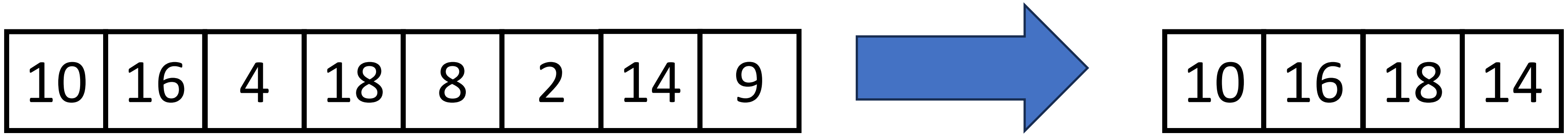
range: [6,7)
sum: 14
leftSum: 58

range: [7,8)
sum: 9
leftSum: 72

```
class CompleteTree extends RecursiveAction {
    public CompleteTree(Node curr, Node parent, Node sibling, boolean isLeftChild, int[] output, int[] input){…}
    protected Void compute(){
        if(isLeftChild)
            curr.sumLeft = parent.sumLeft;
        else
             curr.sumLeft = parent.sumLeft + sibling.sum;
        if (curr.leftChild != Null && curr.rightChild != Null){  // if this isn’t a leaf
            CompleteTree left = new CompleteTree(curr.leftChild, curr, curr.rightChild, true, output, input);
            left.fork();
            CompleteTree right = new CompleteTree(curr.rightChild, curr, curr.leftChild, false, output, input);
            right.compute();
            left.join();
        }
        else{
            output[curr.lo] = curr.sumLeft + input[curr.lo];
            for(int i = curr.lo; i < curr.hi; i++){
                output[i] = output[i-1] + input[i]
            }
        }
    }
}
```

Whew! Back to Pack/Filter

- Given an array of values and a Boolean function, return a new array which contains only elements that were “true



10	16	4	18	8	2	14	9

10	16	18	14

$f(x) = x > 9$

Parallel Pack

Input:	10	16	4	18	8	2	14	9
Output:	10	16	18	14				
	0	1	2	3	4	5	6	7

, $f(x) = x > 9$

1. Do a map to identify the true elements

1	1	0	1	0	0	1	0

2. Do prefix sum on the result of the map to identify the count of true elements seen to the left of each position

1	2	2	3	3	3	4	4

3. Do a map using the previous results fill in the output

10	16	18	14

3. Do a map using the result of the prefix sum to fill in the output

Input:	10	16	4	18	8	2	14	9
Map Result:	1	1	0	1	0	0	1	0
Prefix Result:	1	2	2	3	3	3	4	4

- Because the last value in the prefix result is 4, the length of the output is 4
- Each time there is a 1 in the map result, we want to include that element in the output
- If element i should be included, its position matches prefixResult[i]-1

```
Int[] output = new int[prefixResult[input.length-1]];
FORALL(int i = 0; i < input.length; i++){
        if (mapResult[i] == 1)
                output[prefixResult[i]-1] = input[i];
}
```