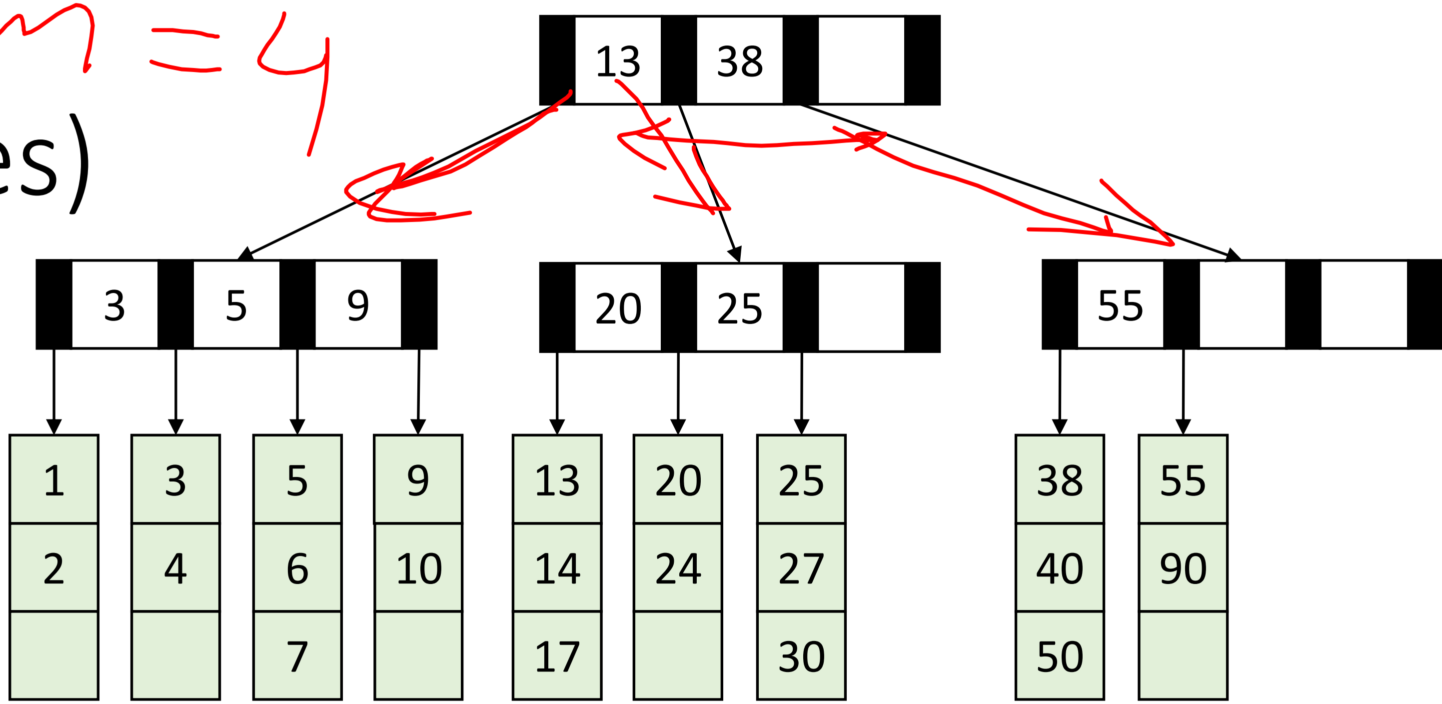
CSE 332 Autumn 2023
Lecture 11: B Trees and Hashing

Nathan Brunelle

http://www.cs.uw.edu/332

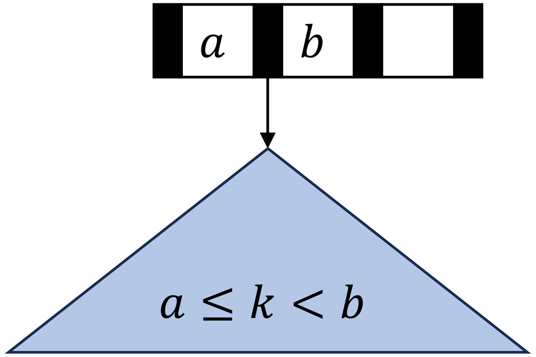
B Trees (aka B+ Trees)

$M = 4$



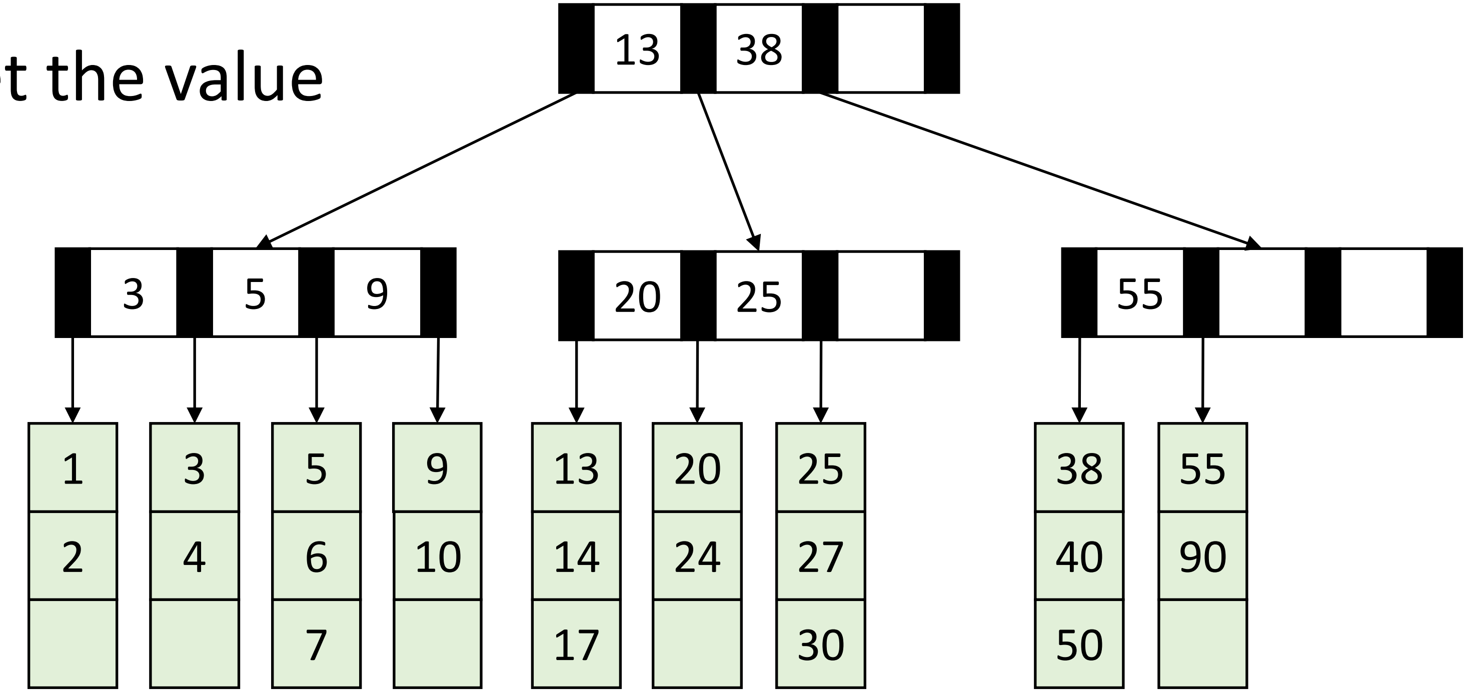
$L = 3$

- Two types of nodes:
  - Internal Nodes
    - Sorted array of $M - 1$ keys
    - Has M children
    - No other data!
  - Leaf Nodes
    - Sorted array of L key-value pairs
- Subtree between values a and b must contain only keys that are $\geq a$ and $< b$
  - If a is missing use $-\infty$
  - If b is missing use ∞



Find

- Start at the root node
- Binary search to identify correct subtree
- Repeat until you reach a leaf node
- Binary search the leaf to get the value



B Tree Structure Requirements

- Root:
  - If the tree has $\leq L$ items then root is a leaf node
  - Otherwise it is an internal node
- Internal Nodes:
  - Must have at least $\left\lceil \frac{M}{2} \right\rceil$ children (at least half full)
- Leaf Nodes:
  - Must have at least Must have at least $\left\lceil \frac{L}{2} \right\rceil$ items (at least half full)
  - All leaves are at the same depth

Insertion Summary

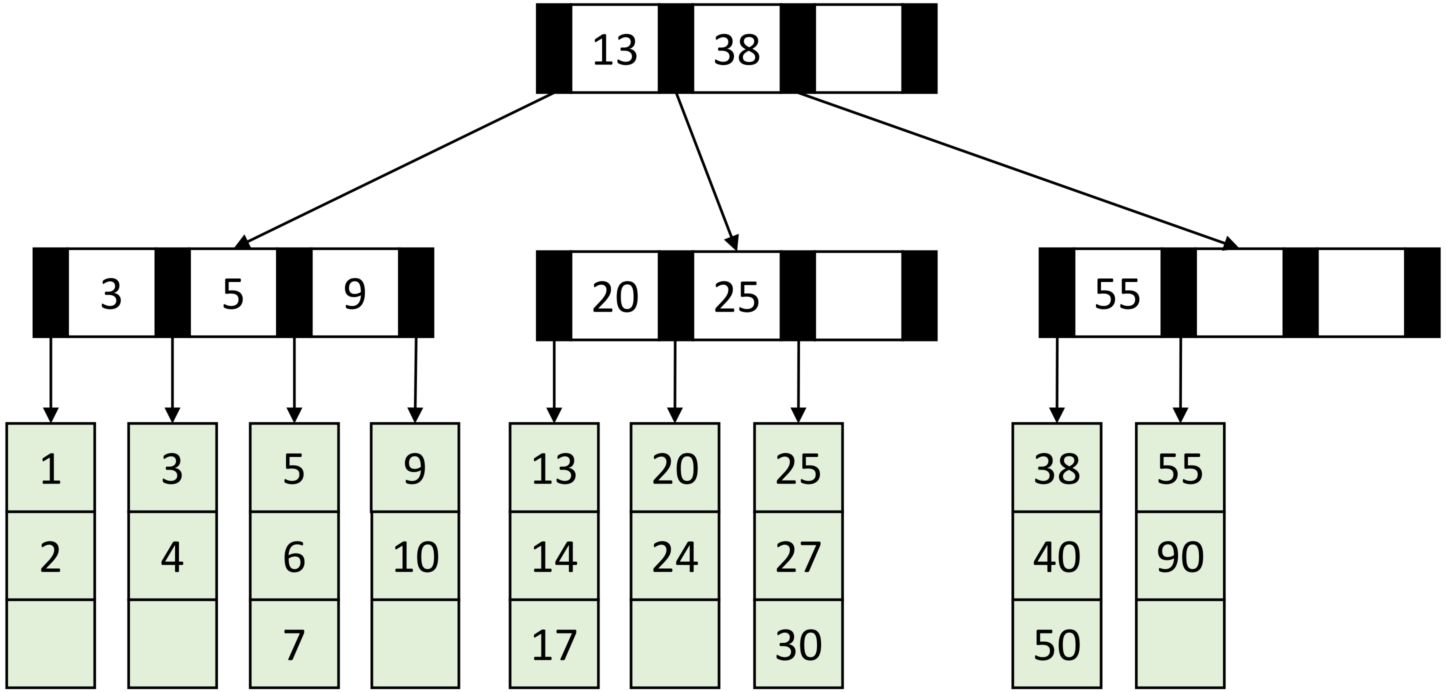
- Binary search to find which leaf should contain the new item
- If there's room, add it to the leaf array (maintaining sorted order)
- If there's not room, **split**
  - Make a new leaf node, move the larger $\left\lfloor \frac{L+1}{2} \right\rfloor$ items to it
  - If there's room in the parent internal node, add new leaf to it (with new key bound value)
  - If there's not room in the parent internal node, **split** that!
    - Make a new internal node and have it point to the larger $\left\lfloor \frac{M+1}{2} \right\rfloor$
    - If there's room in the parent internal node, add this internal node to it
    - If there's not room, repeat this process until there is!

Insertion TLDR

- Find where the item goes by repeated binary search
- If there’s room, just add it
- If there’s not room, split things until there is

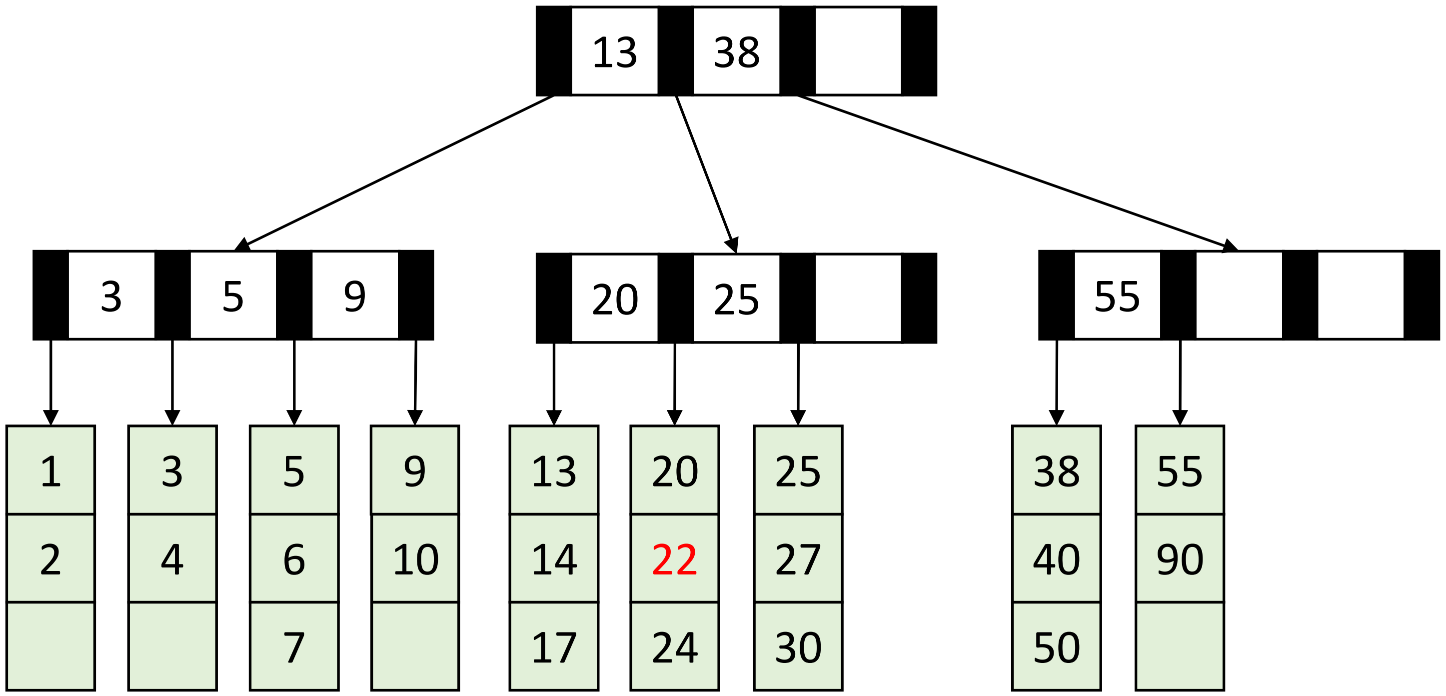
Insert Example

Insert 22



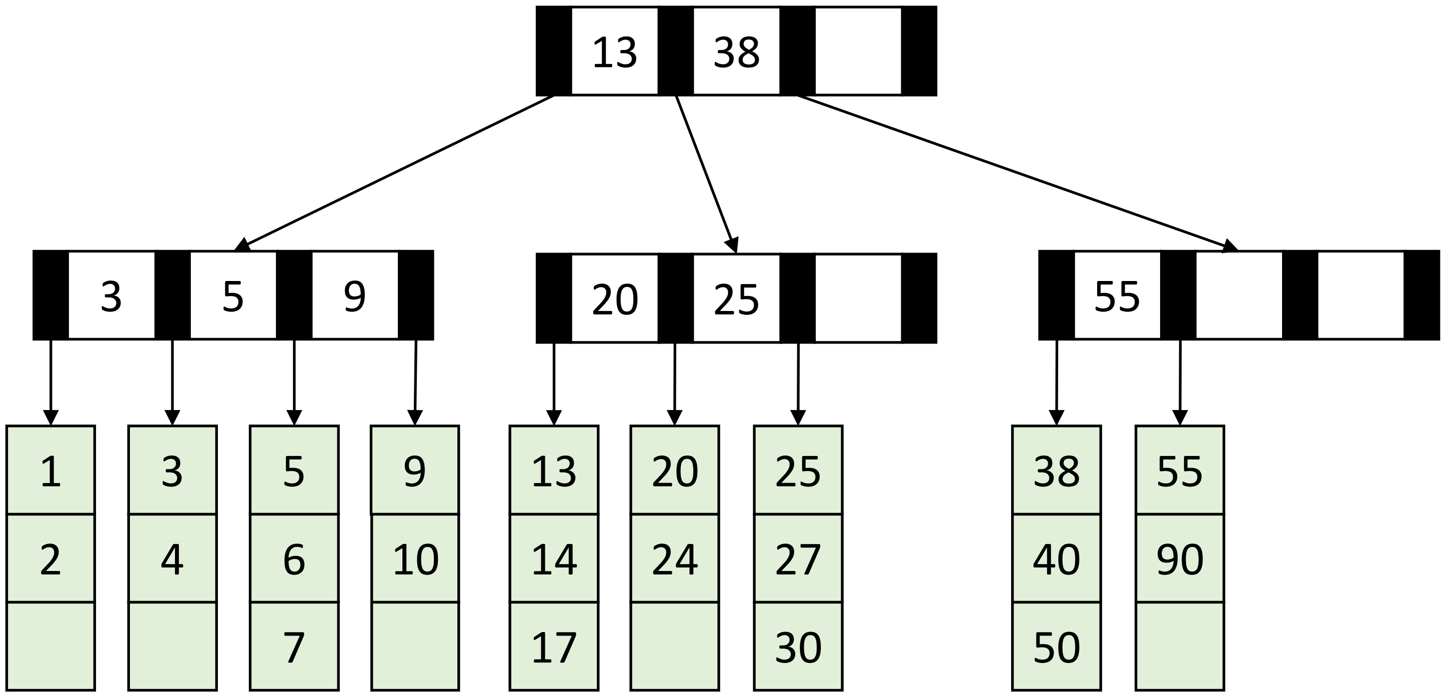
Insert Example

Insert 22



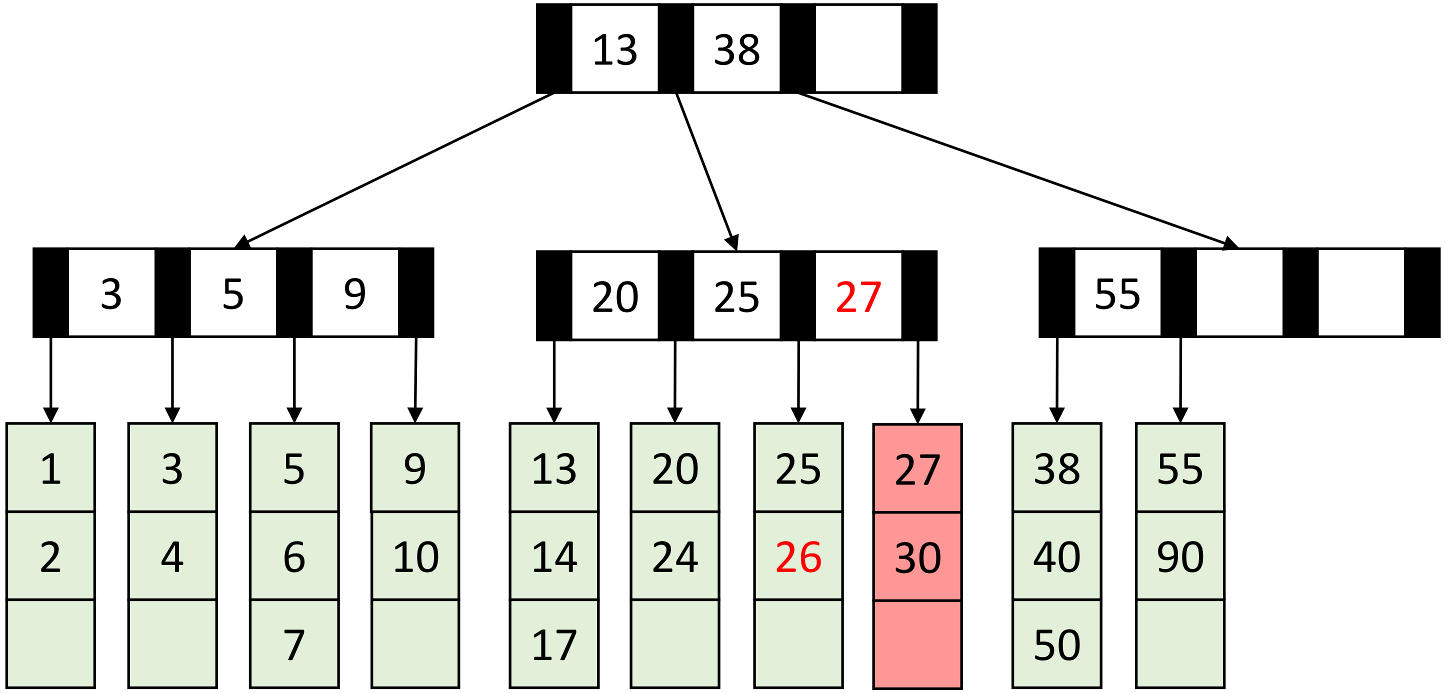
Insert Example

Insert 26



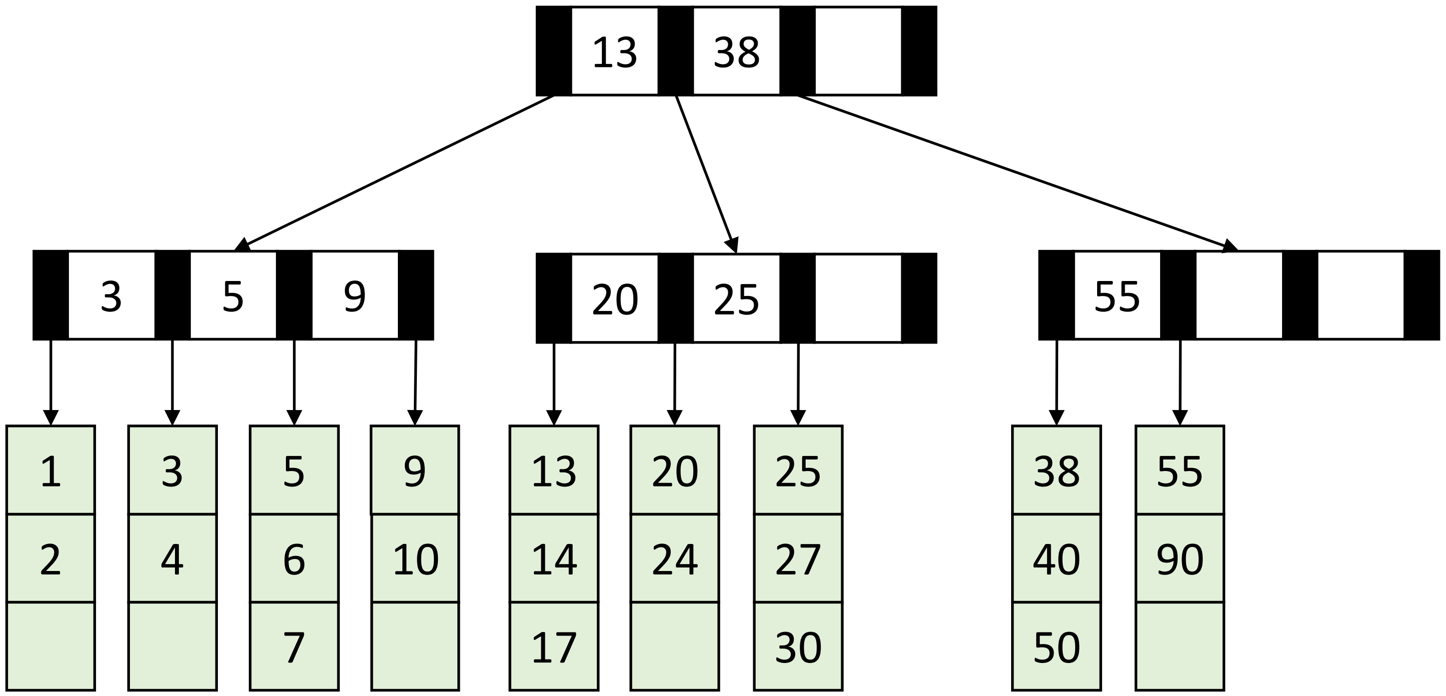
Insert Example

Insert 26



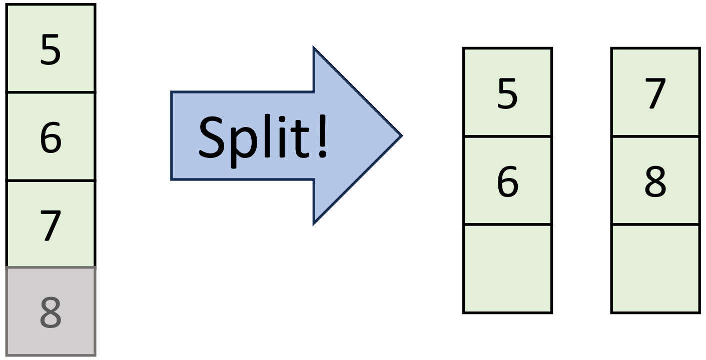
Insert Example

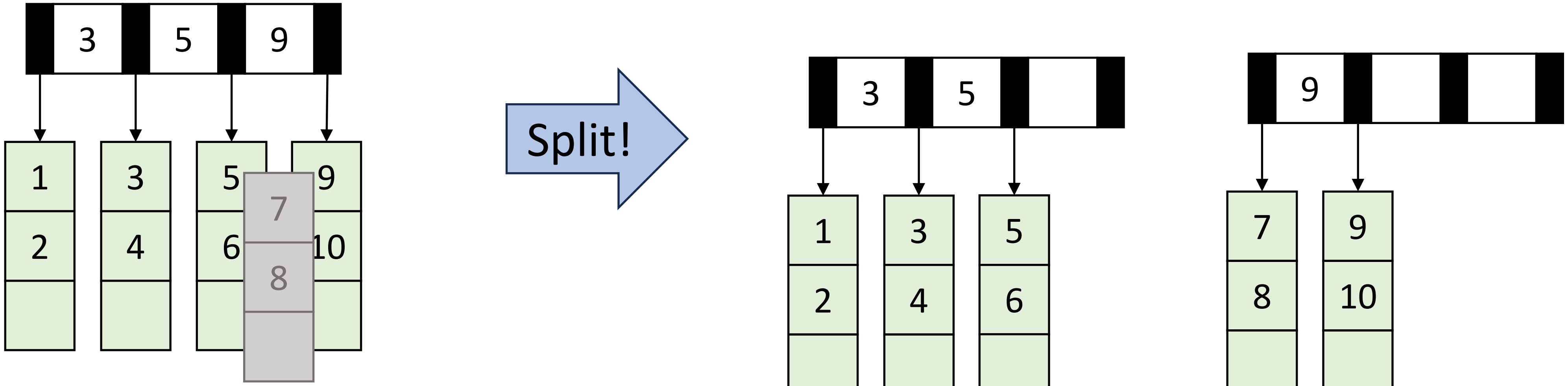
Insert 8



Insert Example

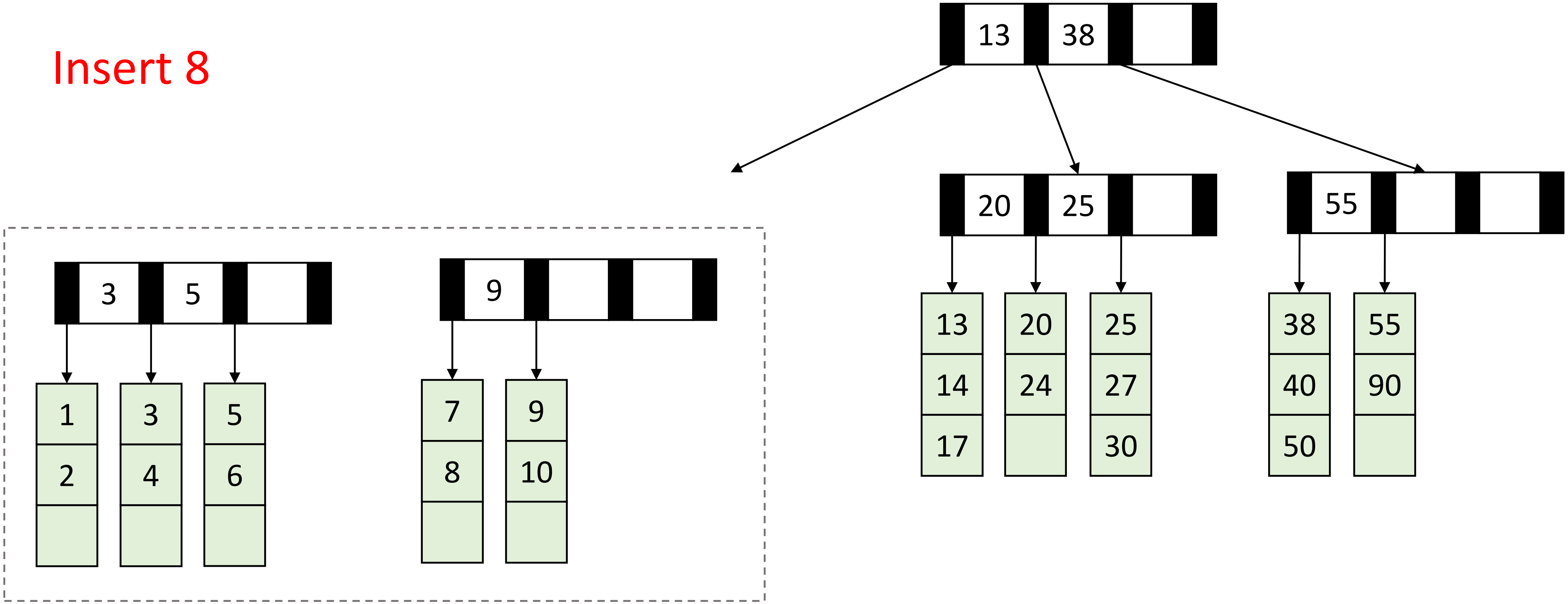
Insert 8





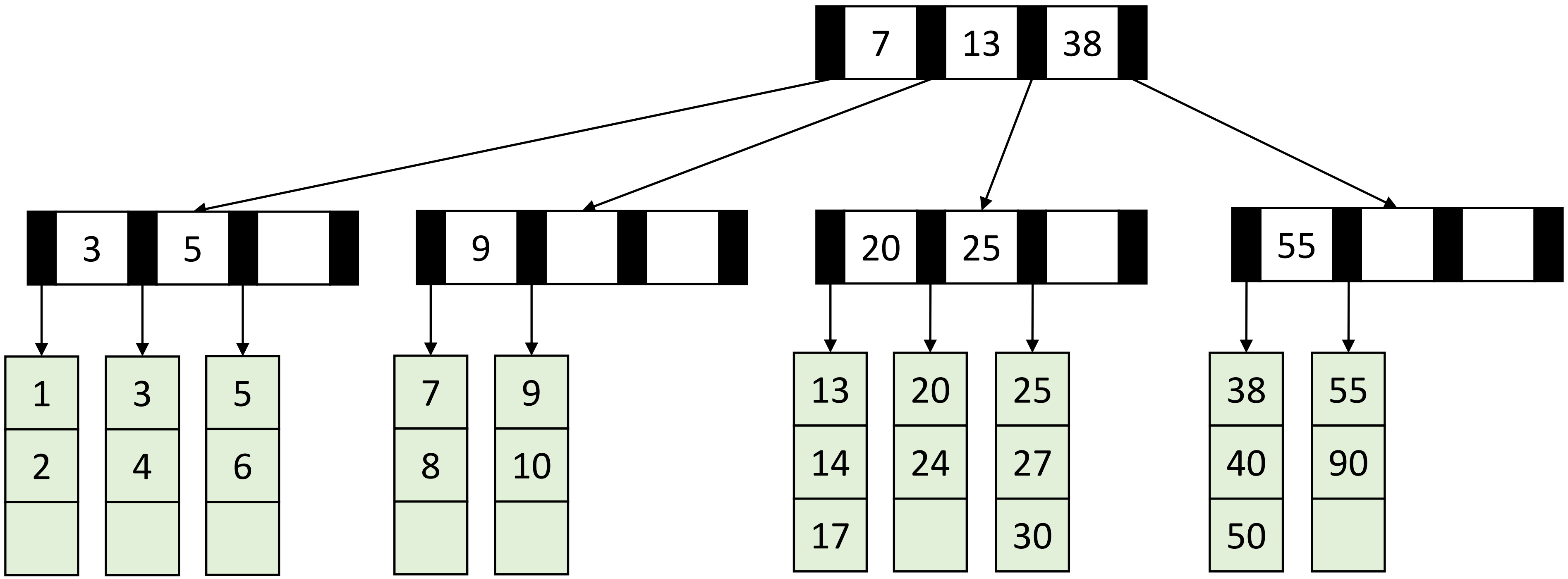
Insert Example

Insert 8



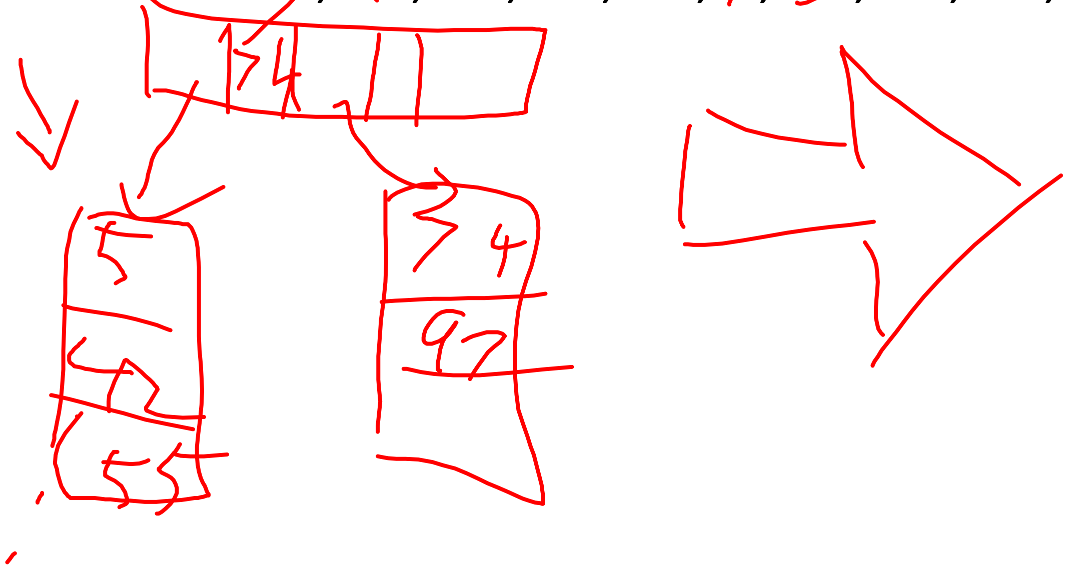
Insert Example

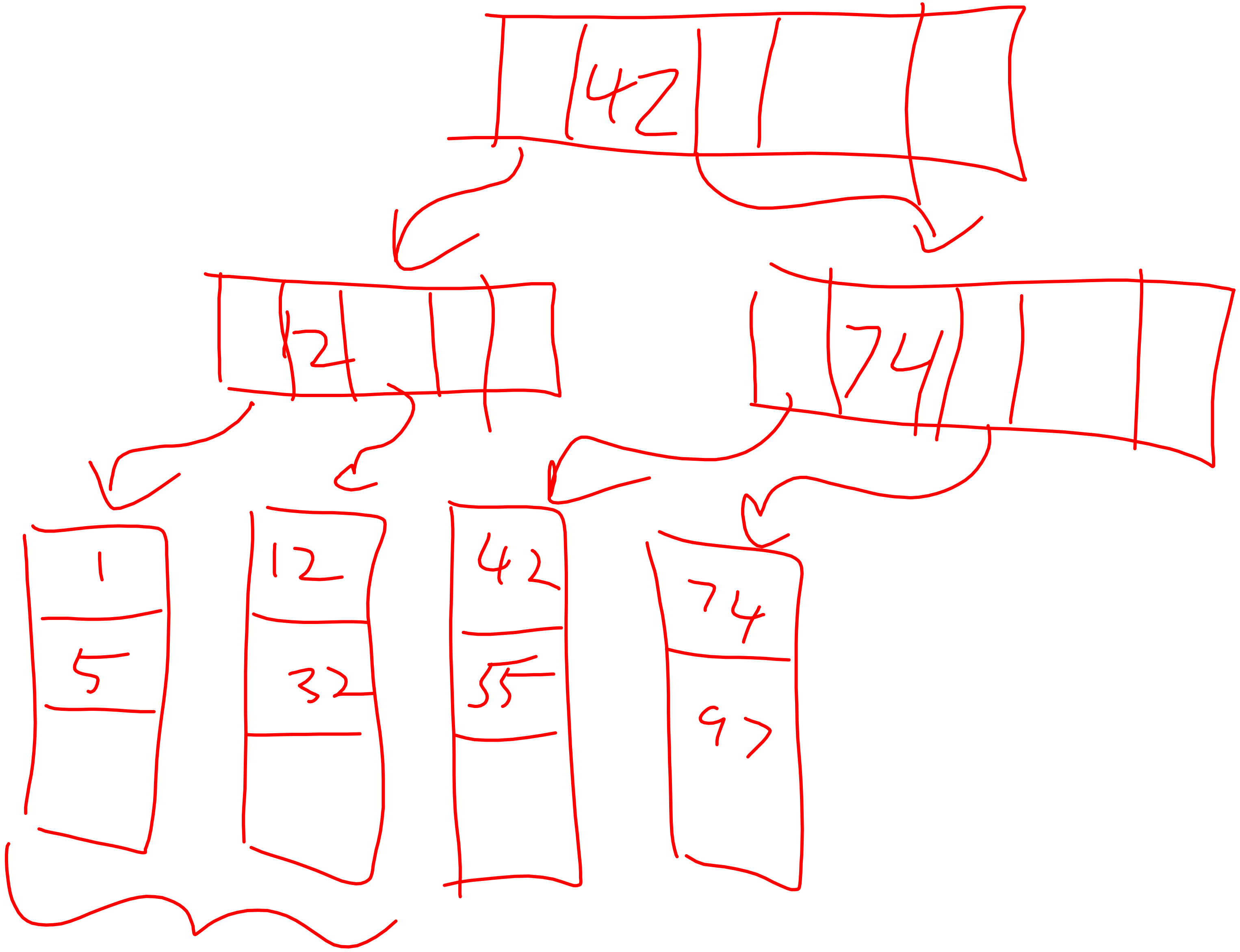
Insert 8



Let’s do it together!

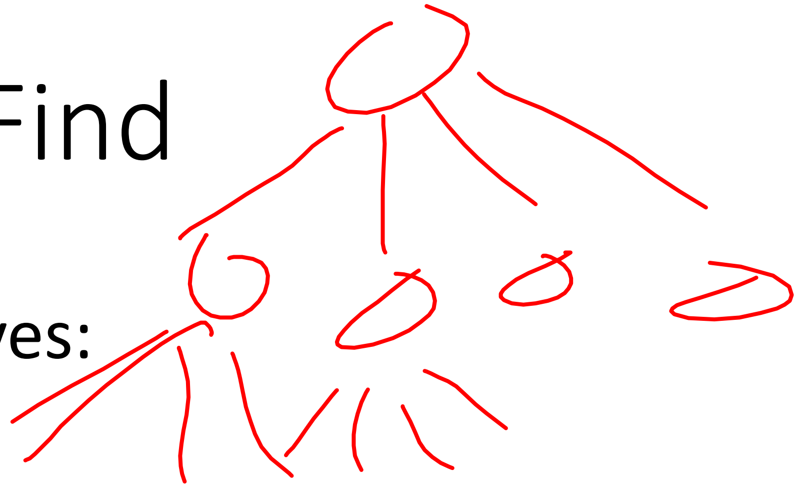
- $M = 3, L = 3$
- Inserts all of these:
  - ~~5~~, ~~42~~, ~~74~~, ~~97~~, ~~55~~, ~~1~~, ~~12~~, 32, 34, 18





42
12
74
1
5
12
32
42
55
74
97

Running Time of Find



- Maximum number of leaves:
  - $\frac{2n}{L}$
  - $\Theta\left(\frac{n}{L}\right)$
- Maximum height of the tree:
  - $2\log_M \frac{2n}{L}$
  - $\Theta\left(\log_M \frac{n}{L}\right)$
- Find:
  - One binary search per level of the tree
    - $\Theta(\log_2 M)$ per search
  - One binary search in the leaf
    - $\Theta(\log_2 L)$

Overall: $\Theta\left(\log_2 M \cdot \log_M \frac{n}{L} + \log_2 L\right)$

Usually simplified to:

$$\Theta(\log_2 M \cdot \log_M n)$$

Running Time of Insert

- Find: 1
  - $\Theta(\log_2 M \cdot \log_M n)$
- Add item to leaf: 2
  - $\Theta(L)$
- Split a leaf 1
  - $\Theta(L)$
- Split one internal node:
  - $\Theta(M)$

$(\log_M n) \cdot M$

Overall: $\Theta(L + M \cdot \log_M n)$

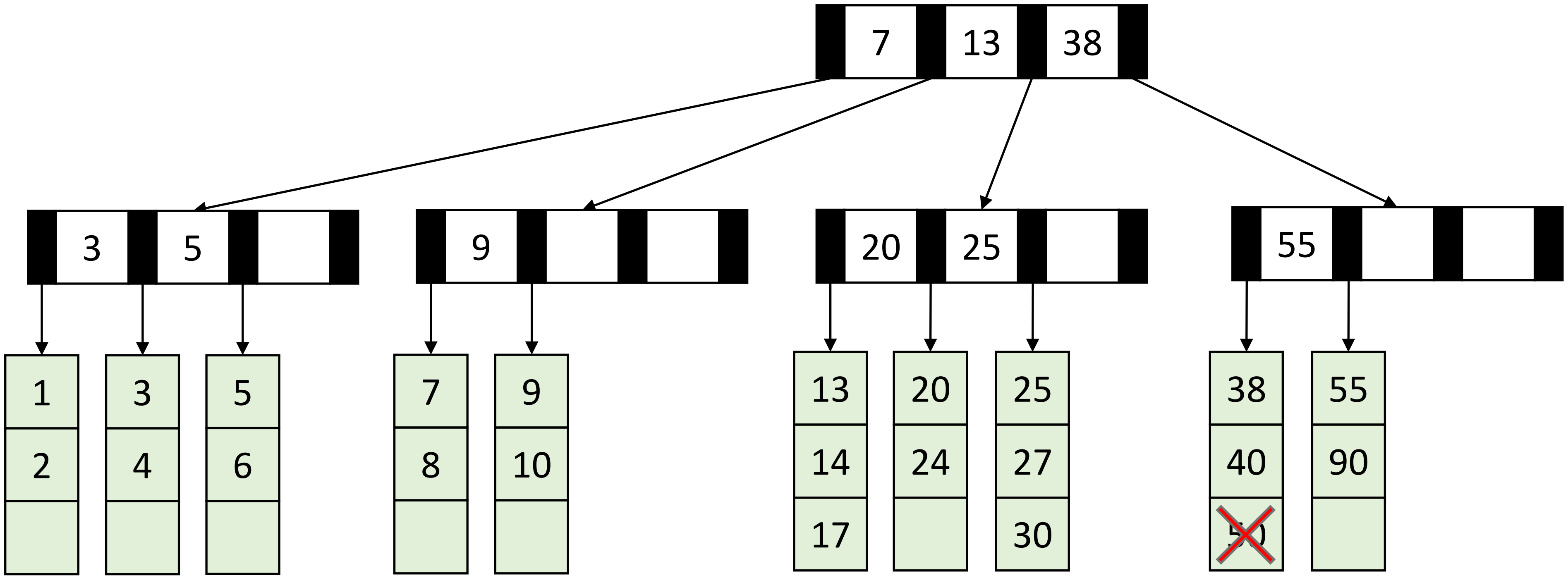
Usually simplified to:

$\Theta(\cancel{\log_2} M \cdot \log_M n)$

Delete

- Recall: all nodes must be at least half full (except root at startup)

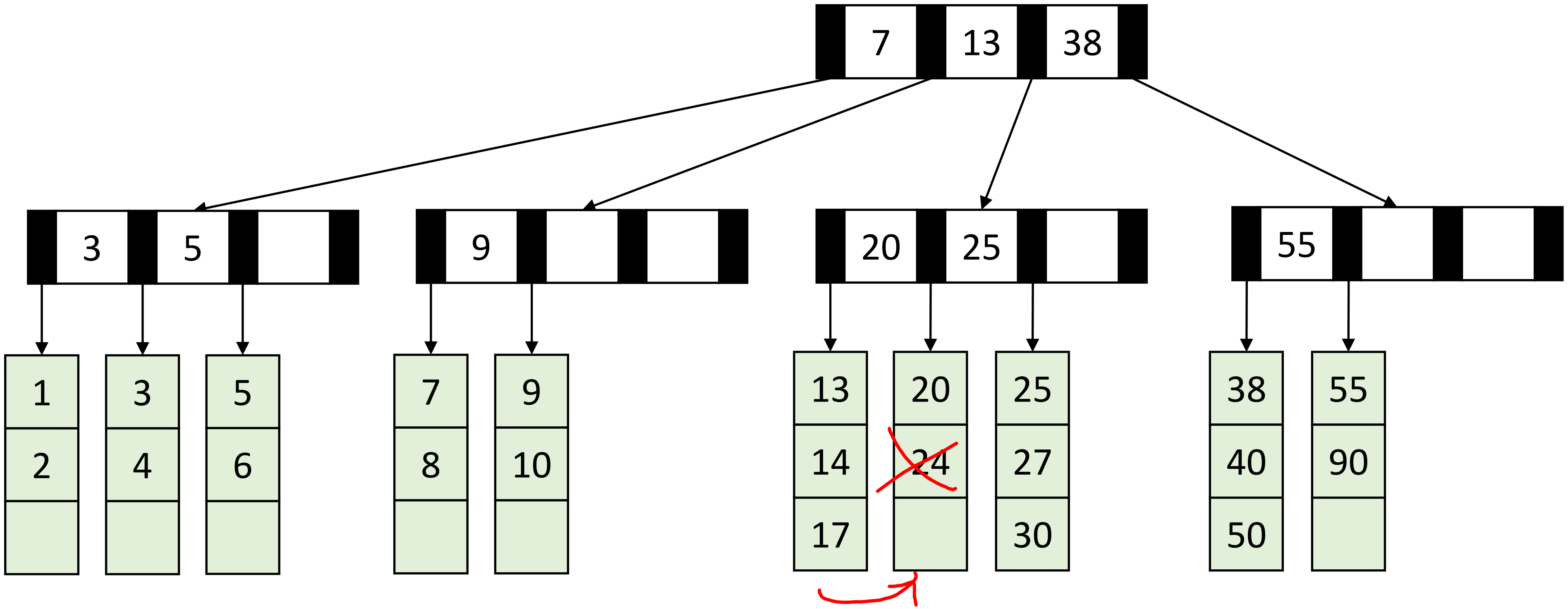
delete 50



Delete

- Recall: all nodes must be at least half full (except root at startup)

delete 24



Delete

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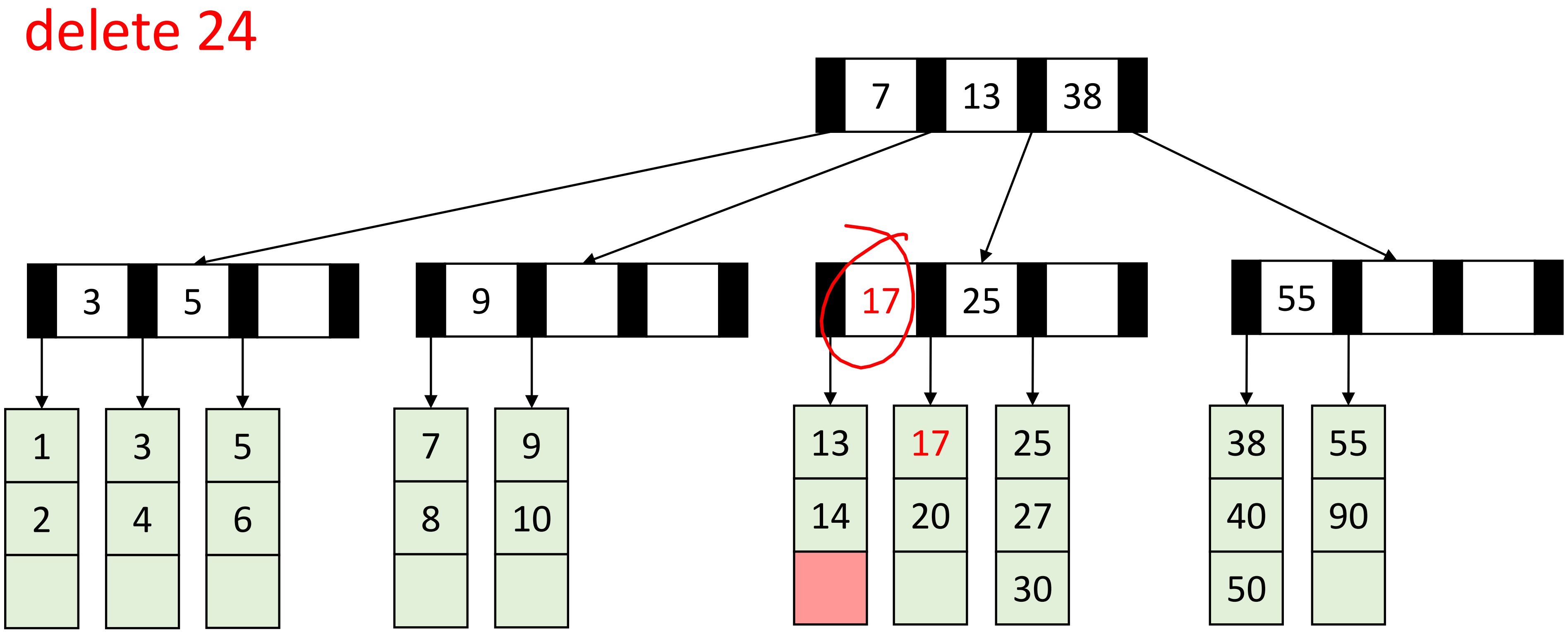
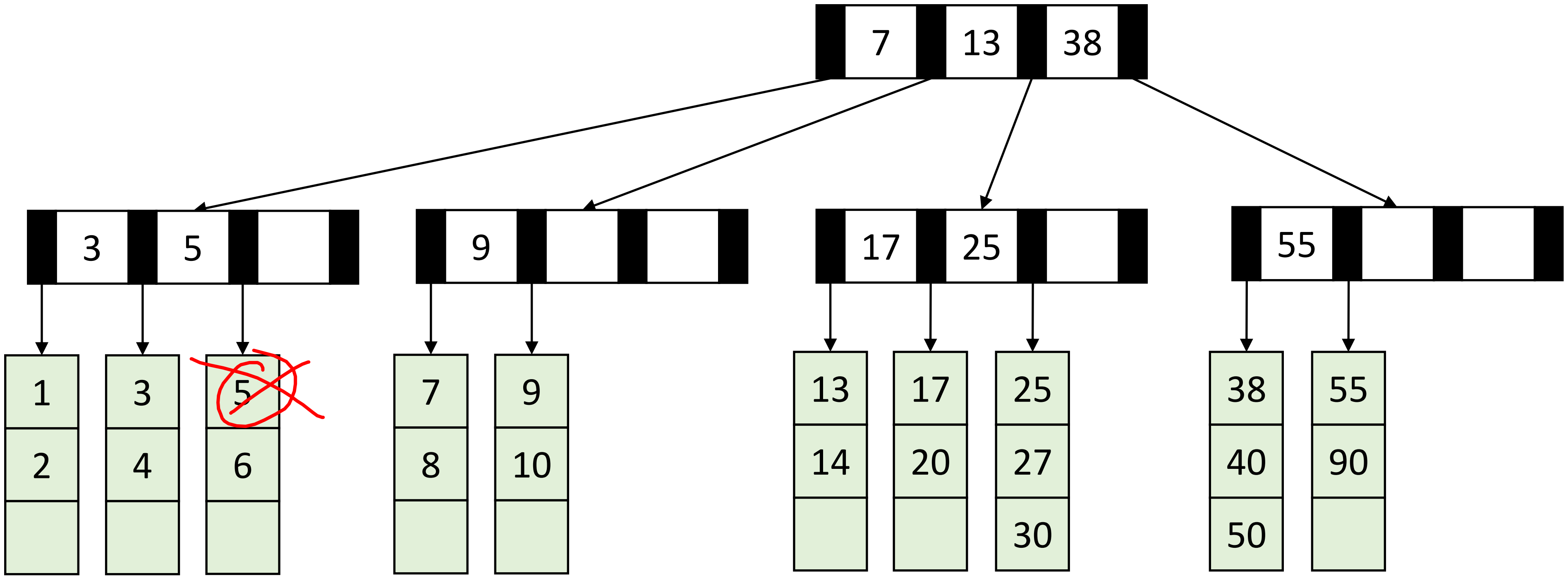


Delete

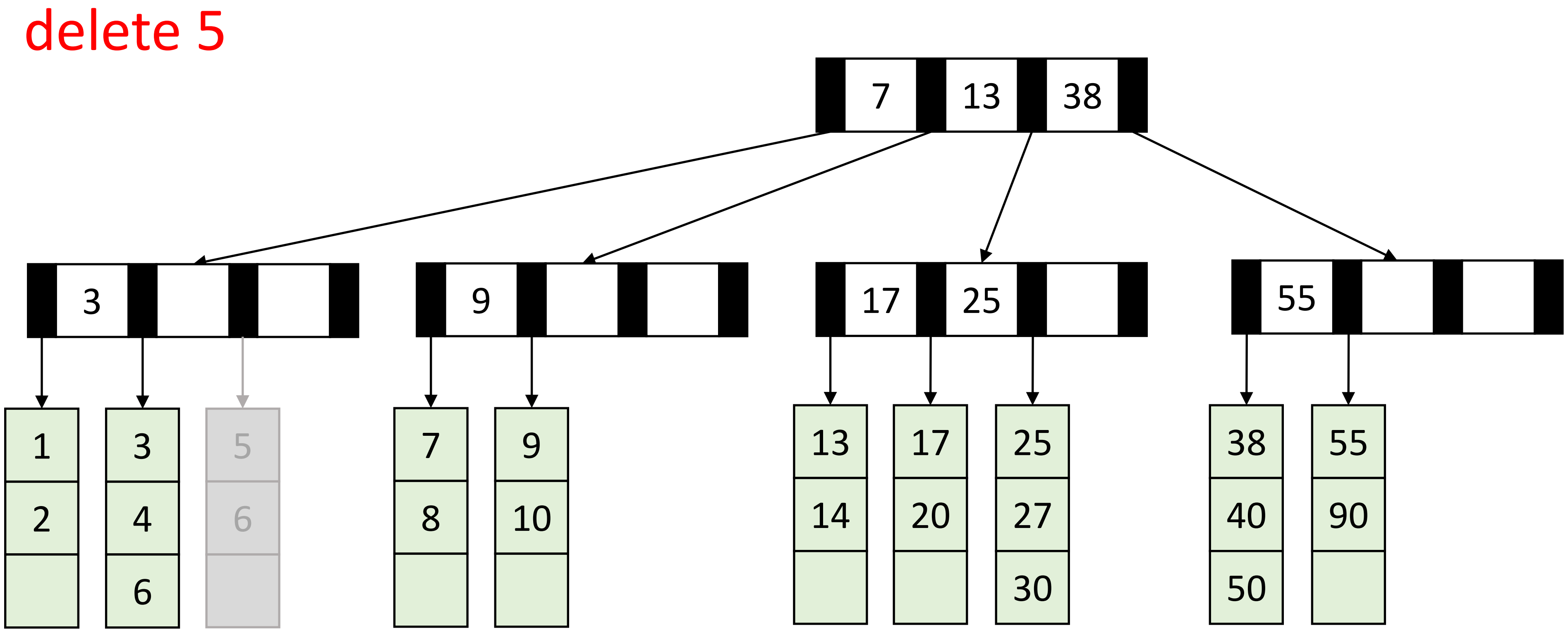
- Recall: all nodes must be at least half full (except root at startup)

delete 5



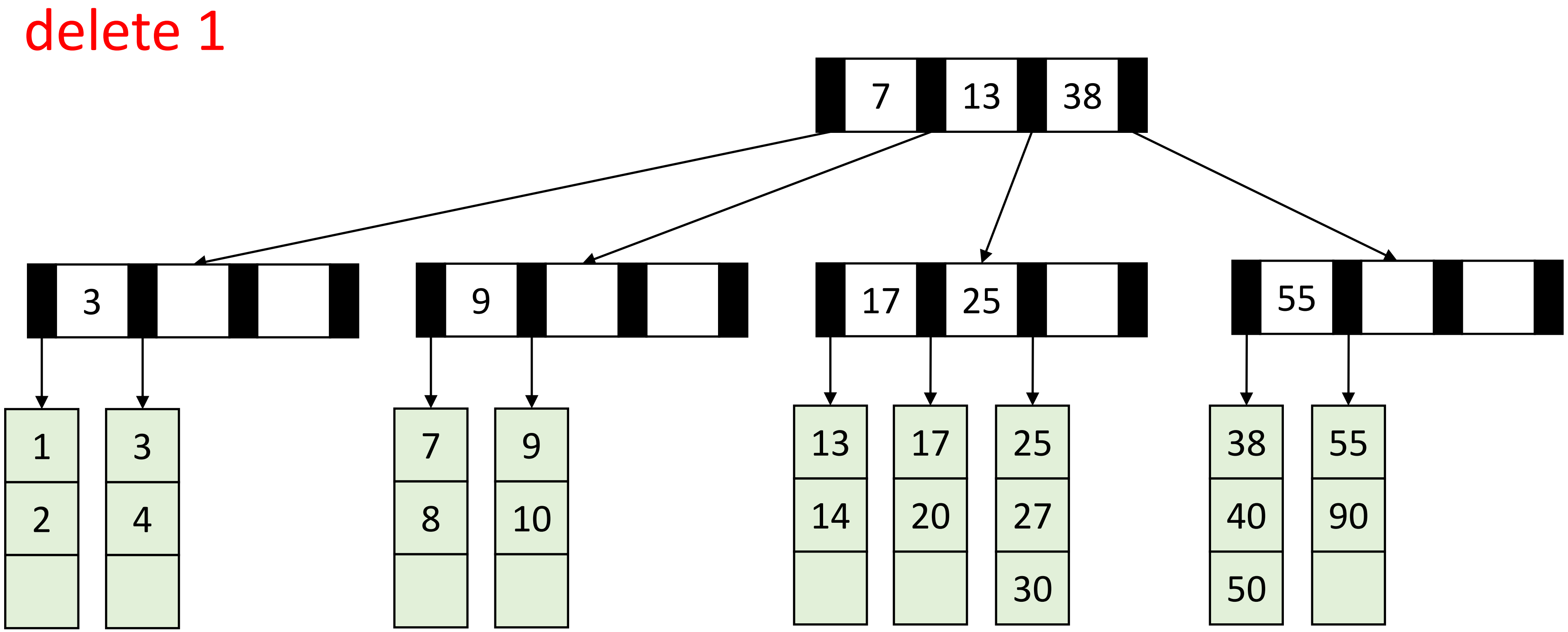
Delete

- Recall: all nodes must be at least half full (except root at startup)



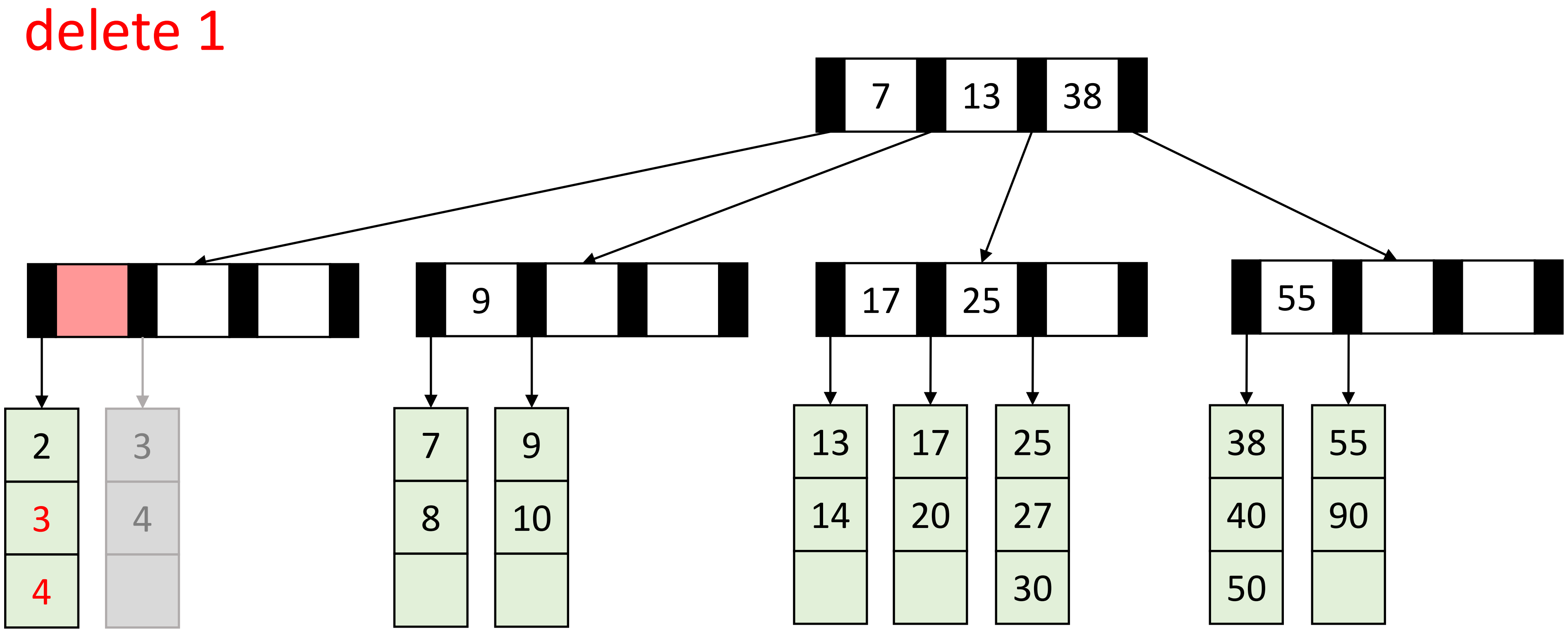
Delete

- Recall: all nodes must be at least half full (except root at startup)



Delete

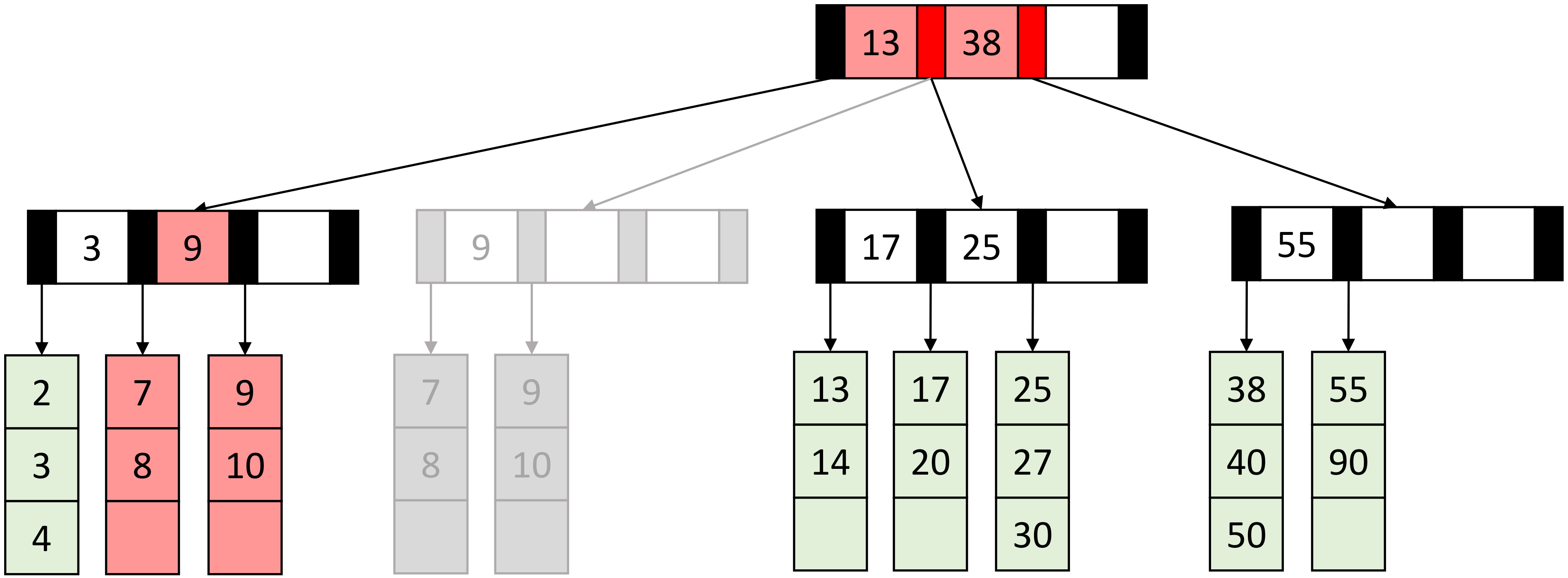
- Recall: all nodes must be at least half full (except root at startup)



Delete

- Recall: all nodes must be at least half full (except root at startup)

delete 1



Delete Summary

- Find the item
- Remove the item from the leaf
  - If that causes the leaf to be underfull, adopt from a neighbor
  - If that would cause the neighbor to be underfull, merge those two leaves
  - Update the parent
    - If that causes the parent to be underfull, adopt from a neighbor
    - If that causes the neighbor to be underfull, merge
    - Update the parent
      - …

Delete TLDR

- Find and remove from leaf
- Keep doing this until everything is “full enough”:
  - If the node is now too small, adopt from a neighbor
  - If the neighbor is too small then merge

Aside: Implementation

- What an internal node class might look like:
  - int M
  - int[] keys
  - Node[] children
  - int num_children
- What a leaf node class might look like:
  - int L
  - E[] data
  - int num_items

Next topic: Hash Tables

Data Structure	Time to insert	Time to find	Time to delete
Unsorted Array	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
Unsorted Linked List	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
Sorted Array	$\Theta(n)$	$\Theta(\log n)$	$\Theta(n)$
Sorted Linked List	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
Binary Search Tree	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
AVL Tree	$\Theta(\log n)$	$\Theta(\log n)$	$\Theta(\log n)$
Hash Table (Worst case)	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
Hash Table (Average)	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$

Two Different ideas of "Average"

- Expected Time
  - The expected number of operations a randomly-chosen input uses
  - Assumed randomness from somewhere
    - Most simply: from the input
    - Preferably: from the algorithm/data structure itself
  - $f(n) =$ sum of the running times for each input of size n divided by the number of inputs of size n
- Amortized Time
  - The long-term average per-execution cost (in the worst case)
  - Rather than look at the worst case of one execution, look at the total worst case of a sequential chain of many executions
    - Why? The worst case may be guaranteed to be rare
  - $f(n) =$ the sum of the running times from a sequence of n sequential calls to the function divided by n

Amortized Example

- ArrayList Insert:
  - Worst case: $\Theta(n)$

0	1	2	3	4	5	6	7	8

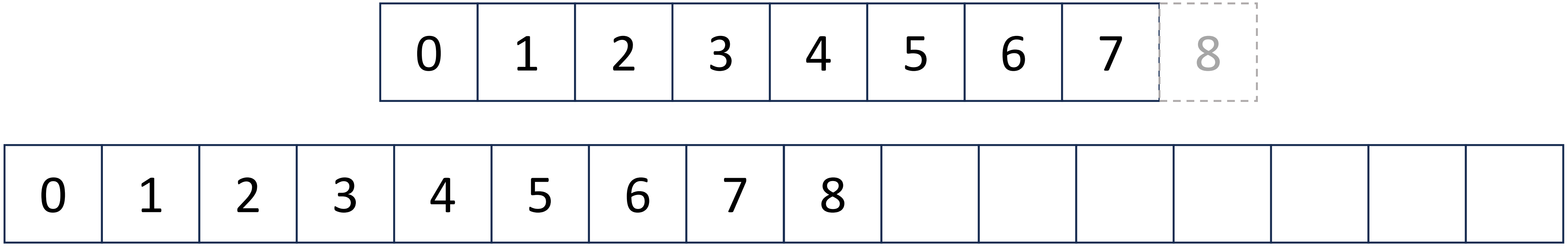
0	1	2	3	4	5	6	7	8							

Amortized Example

- ArrayList Insert:
  - First 8 inserts: 1 operation each
  - 9th insert: 9 operations
  - Next 7 inserts: 1 operation each
  - 17th insert: 17 operations
  - Next 15 inserts: 1 operation each
  - …

Do x operations with cost 1
Do 1 operation with cost x
Do x operations with cost 1
Do 1 operation with cost $2x$
Do $2x$ operations with cost 1
Do 1 operation with cost $4x$
Do $4x$ operations with cost 1
Do 1 operation with cost $8x$
…
Amortized: each operation cost 2 operations

$\Theta(1)$

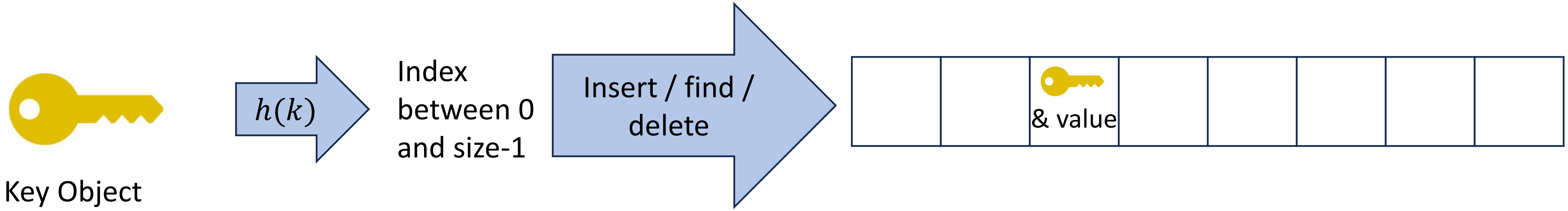


Hash Tables

- Motivation:
  - Why not just have a gigantic array?

Hash Tables

- Idea:
  - Have a small array to store information
  - Use a **hash function** to convert the key into an index
    - Hash function should “scatter” the keys, behave as if it randomly assigned keys to indices
  - Store key at the index given by the hash function
  - Do something if two keys map to the same place (should be very rare)
    - Collision resolution



Example

0	1	2	3	4	5	6	7	8	9

- Key: Phone Number
- Value: People
- Table size: 10
- $h(phone) =$ number as an integer % 10
- $h(8675309) = 9$

What Influences Running time?