# CSE 332: Data Abstractions

# Lecture 12: Comparison Sorting

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Winter 2013

# *Announcements*

- **Project 2** – Phase A due Wed Feb 6th at 11pm
  - Clarifications posted, check Msg board, email cse332-staff
  - Office Hours today, Tues, Wed
- (No homework due Friday)
- **Midterm – Monday Feb 11th during lecture,** info about midterm posted soon
- **Homework 4** – due Friday Feb 15th at the BEGINNING of lecture

# *Today*

- Dictionaries
  - Hashing
- Sorting
  - Comparison sorting

# *Introduction to sorting*

- Stacks, queues, priority queues, and dictionaries all focused on providing one element at a time
- But often we know we want “all the data items” in some order
  – Anyone can sort, but a computer can sort faster
  – Very common to need data sorted somehow
    - Alphabetical list of people
    - Population list of countries
    - Search engine results by relevance
    - …
- Different algorithms have different asymptotic and constant-factor trade-offs
  – No single ‘best’ sort for all scenarios
  – Knowing one way to sort just isn’t enough

# *More reasons to sort*

General technique in computing:

*Preprocess* (e.g. sort) data to make subsequent operations faster

Example: Sort the data so that you can

- Find the **`k`**$^{th}$ largest in constant time for any **`k`**
- Perform binary search to find an element in logarithmic time

Whether the benefit of the preprocessing depends on

- How often the data will change
- How much data there is

# *The main problem, stated carefully*

For now we will assume we have *n* comparable elements in an array and we want to rearrange them to be in increasing order

Input:

- An array **`A`** of data records
- A key value in each data record
- A comparison function (consistent and total)
  - Given keys a & b, what is their relative ordering? <, =, >?
  - Ex: keys that implement Comparable or have a Comparator that can handle them

Effect:

- Reorganize the elements of **`A`** such that for any **`i`** and **`j`**,

  if **`i < j`** then **`A[i]`** $\leq$ **`A[j]`**
- Usually unspoken assumption: **`A`** must have all the same data it started with
- Could also sort in reverse order, of course

An algorithm doing this is a comparison sort

# *Variations on the basic problem*

1. Maybe elements are in a linked list (could convert to array and back in linear time, but some algorithms needn't do so)
2. Maybe in the case of ties we should preserve the original ordering
   - Sorts that do this naturally are called stable sorts
   - One way to sort twice, Ex: Sort movies by year, then for ties, alphabetically
3. Maybe we must not use more than *O*(1) "auxiliary space"
   - Sorts meeting this requirement are called 'in-place' sorts
   - Not allowed to allocate extra array (at least not with size O(n)), but can allocate O(1) # of variables
   - All work done by swapping around in the array
4. Maybe we can do more with elements than just compare
   - Comparison sorts assume we work using a binary 'compare' operator
   - In special cases we can sometimes get faster algorithms
5. Maybe we have too much data to fit in memory
   - Use an "external sorting" algorithm

# *Sorting: The Big Picture*

| Simple algorithms: $O(n^2)$ | Fancier algorithms: $O(n \log n)$ | Comparison lower bound: $\Omega(n \log n)$ | Specialized algorithms: $O(n)$ | Handling huge data sets |
|---|---|---|---|---|
| Insertion sort | Heap sort | | Bucket sort | External sorting |
| Selection sort | Merge sort | | Radix sort | |
| Shell sort | Quick sort (avg) | | | |
| … | … | | | |

# *Insertion Sort*

- Idea: At step `k`, put the `k`th element in the correct position among the first `k` elements

- Alternate way of saying this:
  - Sort first two elements
  - Now insert 3rd element in order
  - Now insert 4th element in order
  - …

- "Loop invariant": when loop index is `i`, first `i` elements are sorted

- Time?

  Best-case _____ Worst-case _____ "Average" case ____

# *Insertion Sort*

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- Time?

  Best-case O(n) Worst-case $O(n^2)$ "Average" case $O(n^2)$

  start sorted start reverse sorted (see text)

# *Selection sort*

- Idea: At step **`k`**, find the smallest element among the not-yet-sorted elements and put it at position k

- Alternate way of saying this:
  - Find smallest element, put it 1st
  - Find next smallest element, put it 2nd
  - Find next smallest element, put it 3rd
  - …

- "Loop invariant": when loop index is **`i`**, first **`i`** elements are the **`i`** smallest elements in sorted order

- Time?

  Best-case _____ Worst-case _____ "Average" case ____

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- "Loop invariant": when loop index is **`i`**, first **`i`** elements are the **`i`** smallest elements in sorted order

- Time?

  Best-case $O(n^2)$ Worst-case $O(n^2)$ "Average" case $O(n^2)$

  *Always* $T(1) = 1$ and $T(n) = n + T(n-1)$

# *Insertion Sort vs. Selection Sort*

- Different algorithms

- Solve the same problem

- Have the same worst-case and average-case asymptotic complexity
  - Insertion-sort has better best-case complexity; preferable when input is “mostly sorted”

- Other algorithms are more efficient *for non-small arrays that are not already almost sorted*
  - Insertion sort may do well on small arrays

# *Aside: We won't cover Bubble Sort*

- It doesn't have good asymptotic complexity: $O(n^2)$
- It's not particularly efficient with respect to common factors
- Basically, almost everything it is good at, some other algorithm is at least as good at
- Some people seem to teach it just because someone taught it to them

- For fun see: "Bubble Sort: An Archaeological Algorithmic Analysis", Owen Astrachan, SIGCSE 2003 http://www.cs.duke.edu/~ola/bubble/bubble.pdf

# *Sorting: The Big Picture*

**Simple algorithms: O($n^2$)**

- **Insertion sort**
- **Selection sort**
- **Shell sort**
- **…**

**Fancier algorithms: O($n \log n$)**

- **Heap sort**
- **Merge sort**
- **Quick sort (avg)**
- **…**

**Comparison lower bound: $\Omega(n \log n)$**

**Specialized algorithms: O($n$)**

- **Bucket sort**
- **Radix sort**

**Handling huge data sets**

- **External sorting**

# *Heap sort*

- As you saw on project 2, sorting with a heap is easy:
  - **`insert`** each **`arr[i]`**, better yet use **`buildHeap`**
  - ```
    for(i=0; i < arr.length; i++)
        arr[i] = deleteMin();
    ```
- Worst-case running time:
- We have the array-to-sort and the heap
  - So this is not an in-place sort
  - There’s a trick to make it in-place…
    ```

# *Heap sort*

- As you saw on project 2, sorting with a heap is easy:
  - **`insert`** each **`arr[i]`**, better yet use **`buildHeap`**
  - **`for(i=0; i < arr.length; i++)`**

    **`arr[i] = deleteMin();`**

- Worst-case running time: $O(n \mathbf{\ log\ } n)$  why?

- We have the array-to-sort and the heap
  - So this is not an in-place sort
  - There's a trick to make it in-place…

# *In-place heap sort*

But this reverse sorts – how would you fix that?

- Treat the initial array as a heap (via **`buildHeap`**)
- When you delete the **`i`**th element, put it at **`arr[n-i]`**
  - It's not part of the heap anymore!

| 4 | 7 | 5 | 9 | **8** | **6** | **10** | **3** | **2** | **1** |
|---|---|---|---|---|---|---|---|---|---|

heap part (first 7 cells) | sorted part (last 3 cells)

**`arr[n-i]=`**
**`deleteMin()`**

| 5 | 7 | 6 | 9 | **8** | **10** | **4** | **3** | **2** | **1** |
|---|---|---|---|---|---|---|---|---|---|

heap part (first 6 cells) | sorted part (last 4 cells)

# *“AVL sort”*

- How?

# *"AVL sort"*

- We can also use a balanced tree to:
  - `insert` each element: total time $O(n \log n)$
  - Repeatedly `deleteMin`: total time $O(n \log n)$

- But this cannot be made in-place and has worse constant factors than heap sort
  - both are $O(n \log n)$ in worst, best, and average case
  - neither parallelizes well
  - heap sort is better

- Don't even think about trying to sort with a hash table…

# *Divide and conquer*

Very important technique in algorithm design

1. Divide problem into smaller parts

2. Solve the parts independently
   - Think recursion
   - Or potential parallelism

3. Combine solution of parts to produce overall solution

Ex: Sort each half of the array, combine together; to sort each half, split into halves…

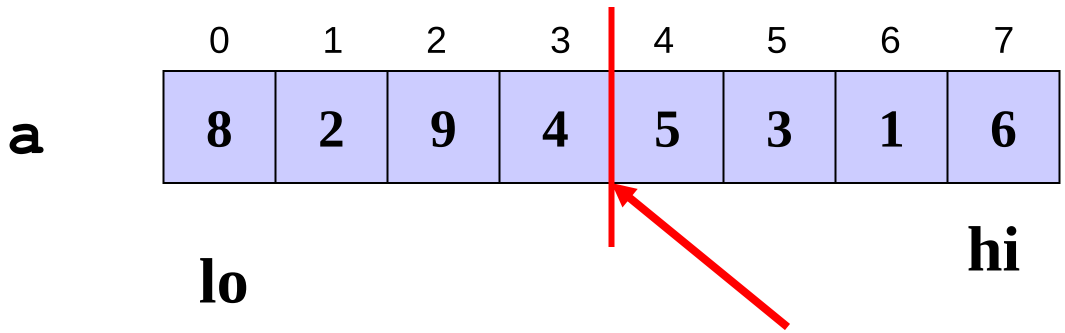
# *Divide-and-conquer sorting*

Two great sorting methods are fundamentally divide-and-conquer

1. Mergesort: Sort the left half of the elements (recursively)
   Sort the right half of the elements (recursively)
   Merge the two sorted halves into a sorted whole

2. Quicksort: Pick a "pivot" element
   Divide elements into those less-than pivot
   and those greater-than pivot
   Sort the two divisions (recursively on each)
   Answer is [*sorted-less-than* then *pivot* then
   *sorted-greater-than*]

# *Mergesort*




- To sort array from position **`lo`** to position **`hi`**:
  - If range is 1 element long, it's sorted! (Base case)
  - Else, split into two halves:
    - Sort from **`lo`** to **`(hi+lo)/2`**
    - Sort from **`(hi+lo)/2`** to **`hi`**
    - Merge the two halves together

- Merging takes two sorted parts and sorts everything
  - *O*(*n*) but requires auxiliary space…

# *Example, focus on merging*

Start with:

**a**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **8** | **2** | **9** | **4** | **5** | **3** | **1** | **6** |

After we return from left and right recursive calls (pretend it works for now)

**a**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **2** | **4** | **8** | **9** | **1** | **3** | **5** | **6** |

Merge:

Use 3 “fingers”

and 1 more array

**aux**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | | | | | |

(After merge, copy back to original array)

# *Example, focus on merging*

Start with:

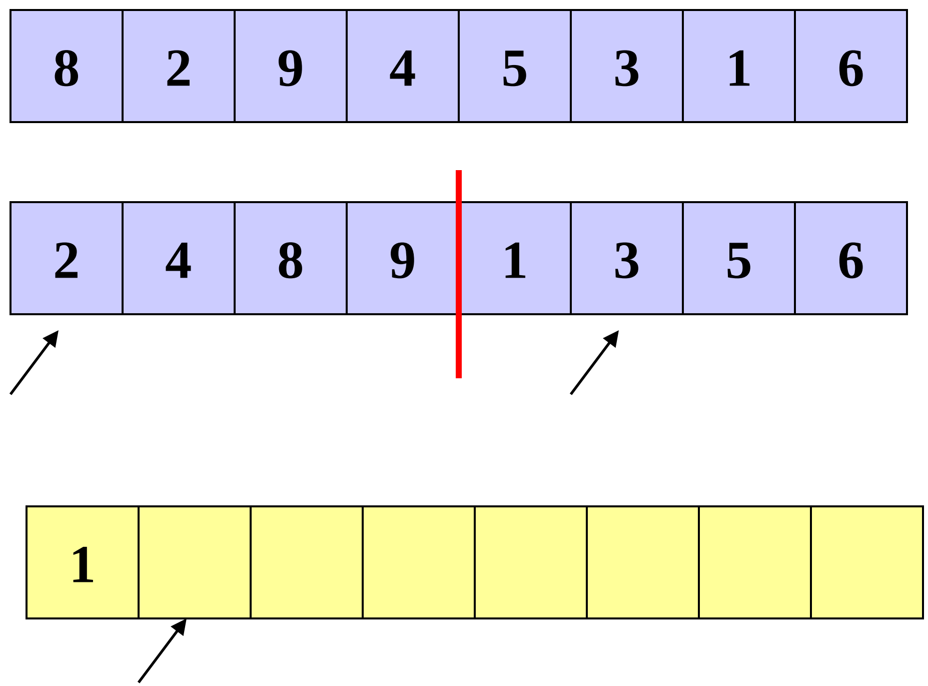
After recursion:

(not magic ☺)

Merge:

Use 3 “fingers”

and 1 more array

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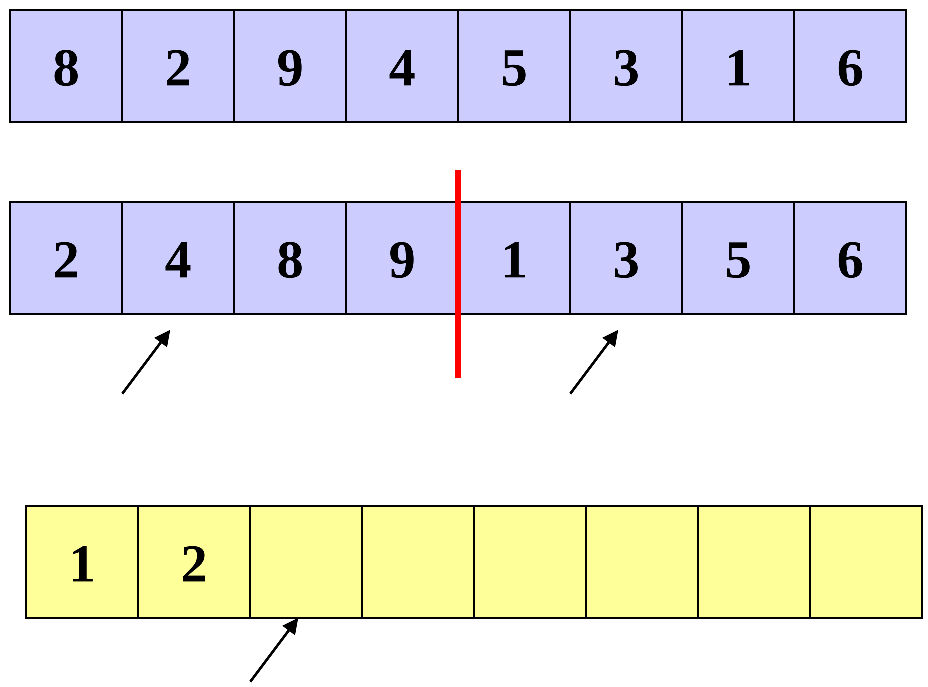


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After recursion:
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| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **1** | **2** | **3** | | | | | |

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(not magic ☺)

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Merge:
Use 3 “fingers”
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| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **1** | **2** | **3** | **4** | **5** | **6** | **8** | |

(After merge,
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# *Example, focus on merging*

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(After merge,
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Merge:
Use 3 “fingers”
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|---|---|---|---|---|---|---|---|
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(After merge,
copy back to
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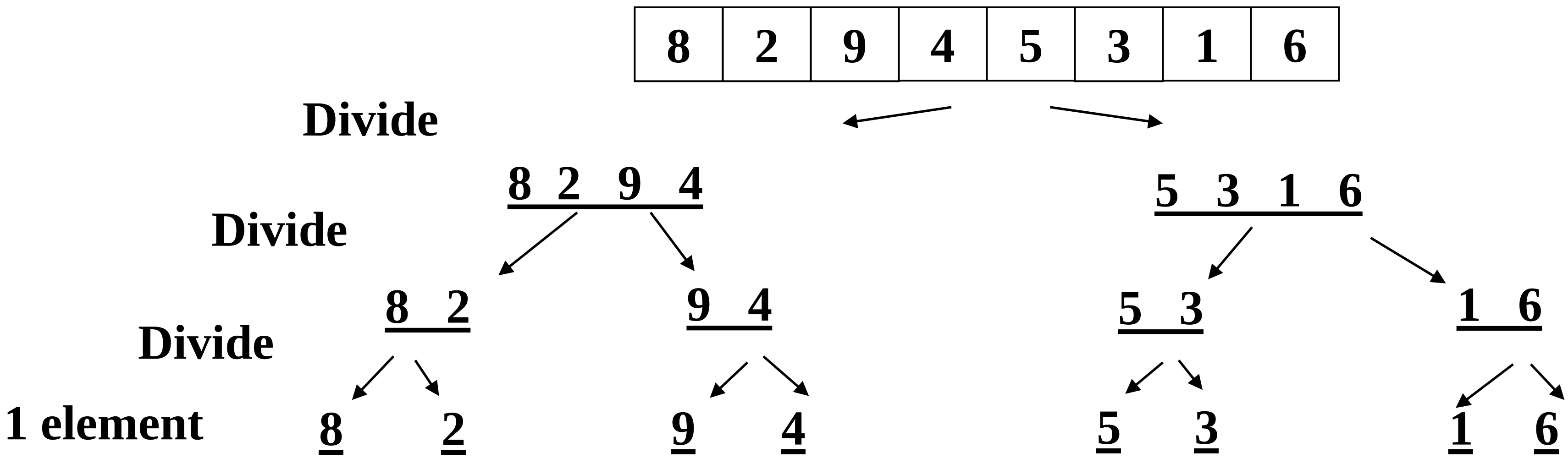
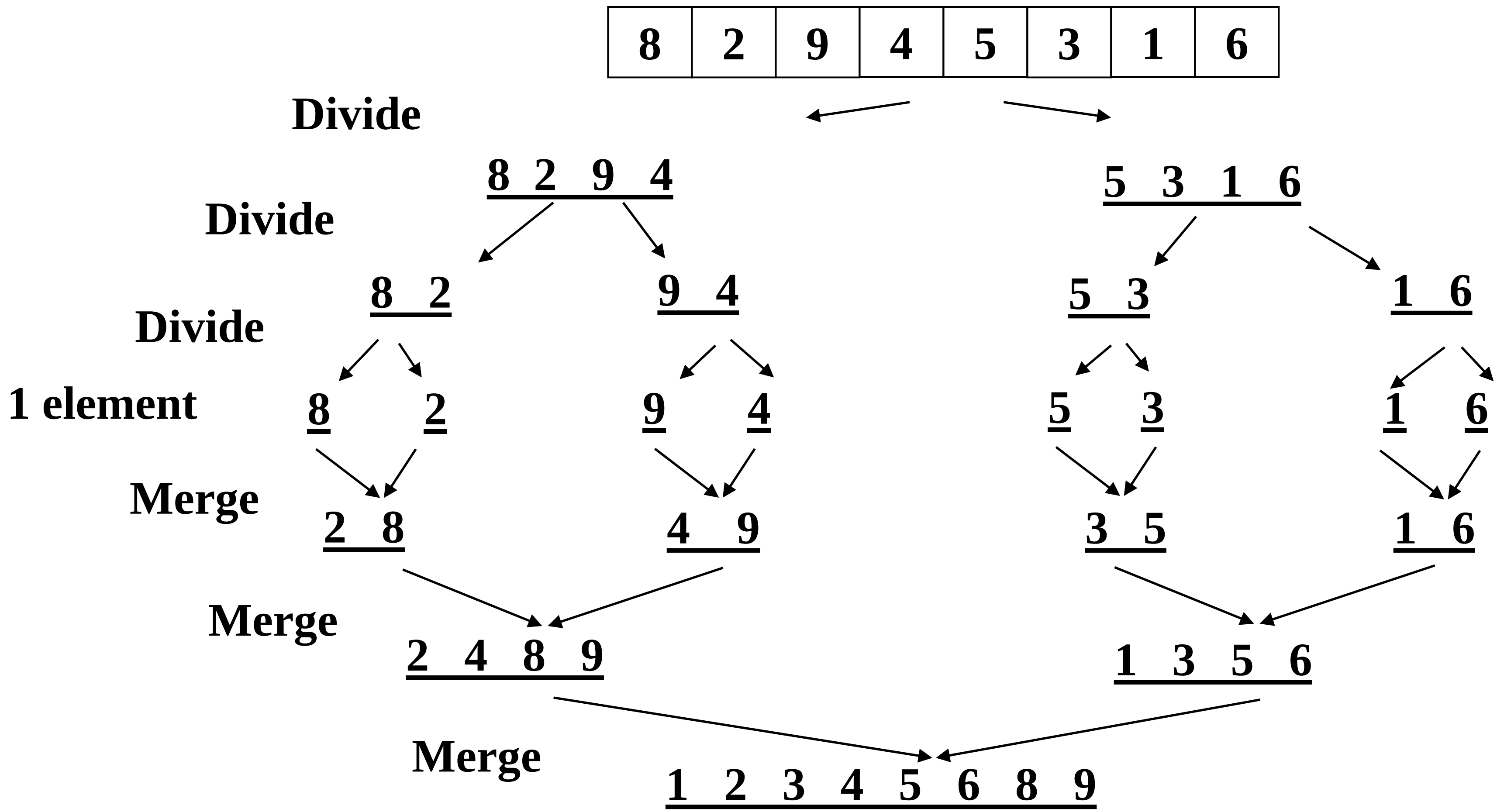
| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **1** | **2** | **3** | **4** | **5** | **6** | **8** | **9** |

# *Mergesort example: Recursively splitting list in half*

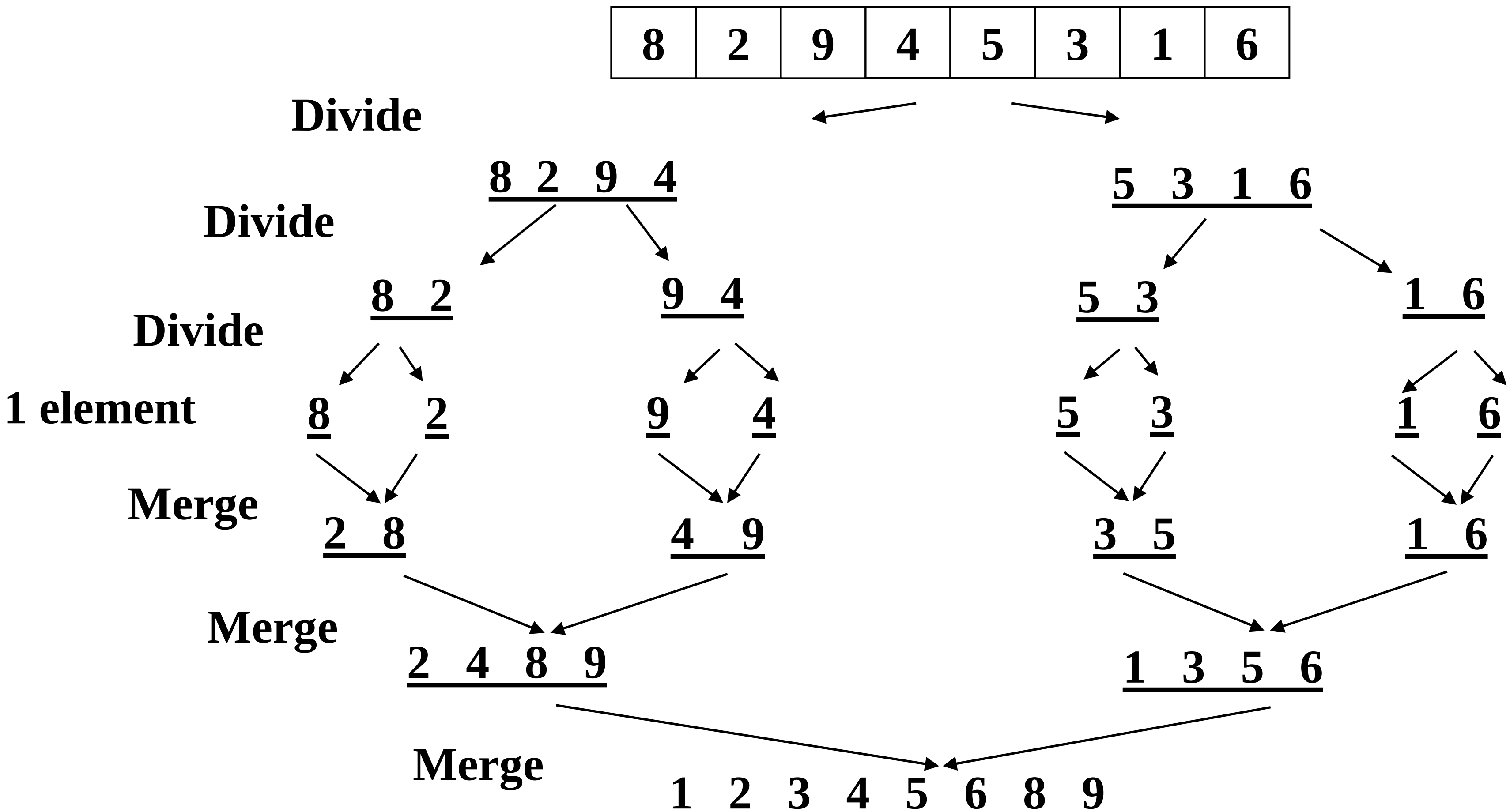


# *Mergesort example: Merge as we return from recursive calls*




**When a recursive call ends, it's sub-arrays are each in order; just need to merge them in order together**

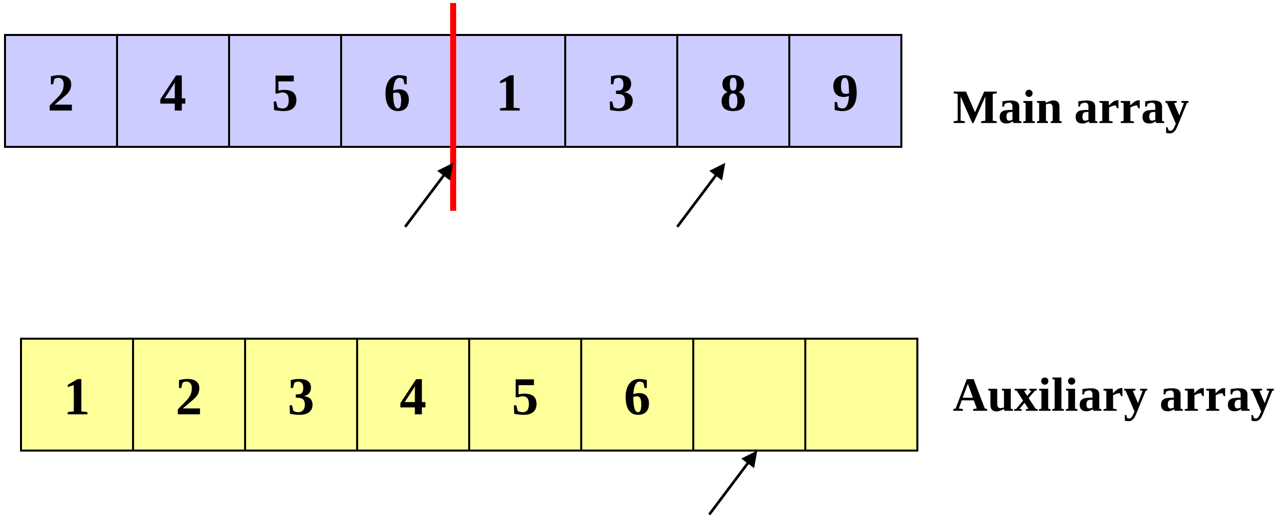
# *Mergesort example: Merge as we return from recursive calls*




**We need another array in which to do each merging step; merge results into there, then copy back to original array**

# *Mergesort, some details: saving a little time*

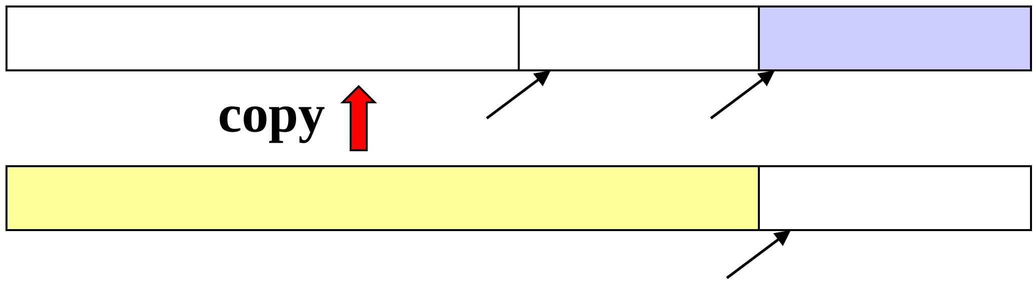
- What if the final steps of our merging looked like the following:



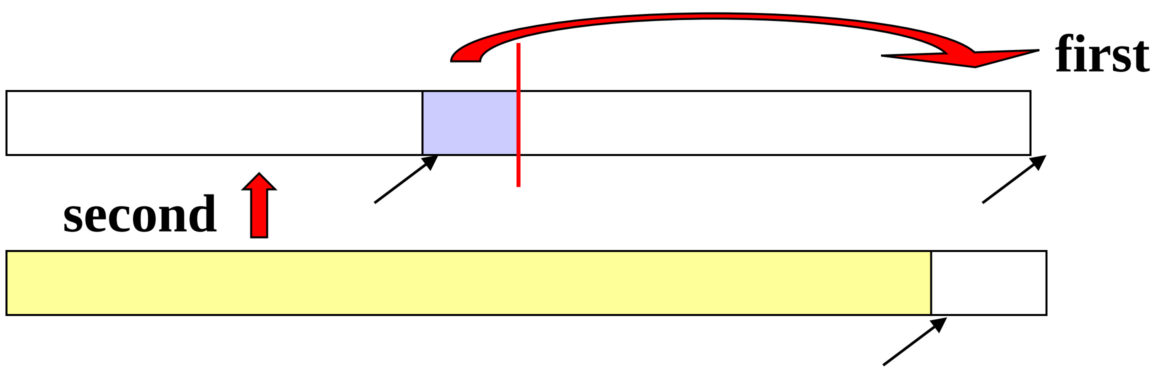

- Seems kind of wasteful to copy 8 & 9 to the auxiliary array just to copy them immediately back…

# *Mergesort, some details: saving a little time*

- Unnecessary to copy ‘dregs’ over to auxiliary array
  - If left-side finishes first, just stop the merge & copy the auxiliary array:




  - If right-side finishes first, copy dregs directly into right side, then copy auxiliary array



# *Some details: saving space / copying*

Simplest / worst approach:

Use a new auxiliary array of size **`(hi-lo)`** for every merge

Returning from a recursive call? Allocate a new array!

Better:

Reuse same auxiliary array of size **`n`** for every merging stage

Allocate auxiliary array at beginning, use throughout

Best (but a little tricky):

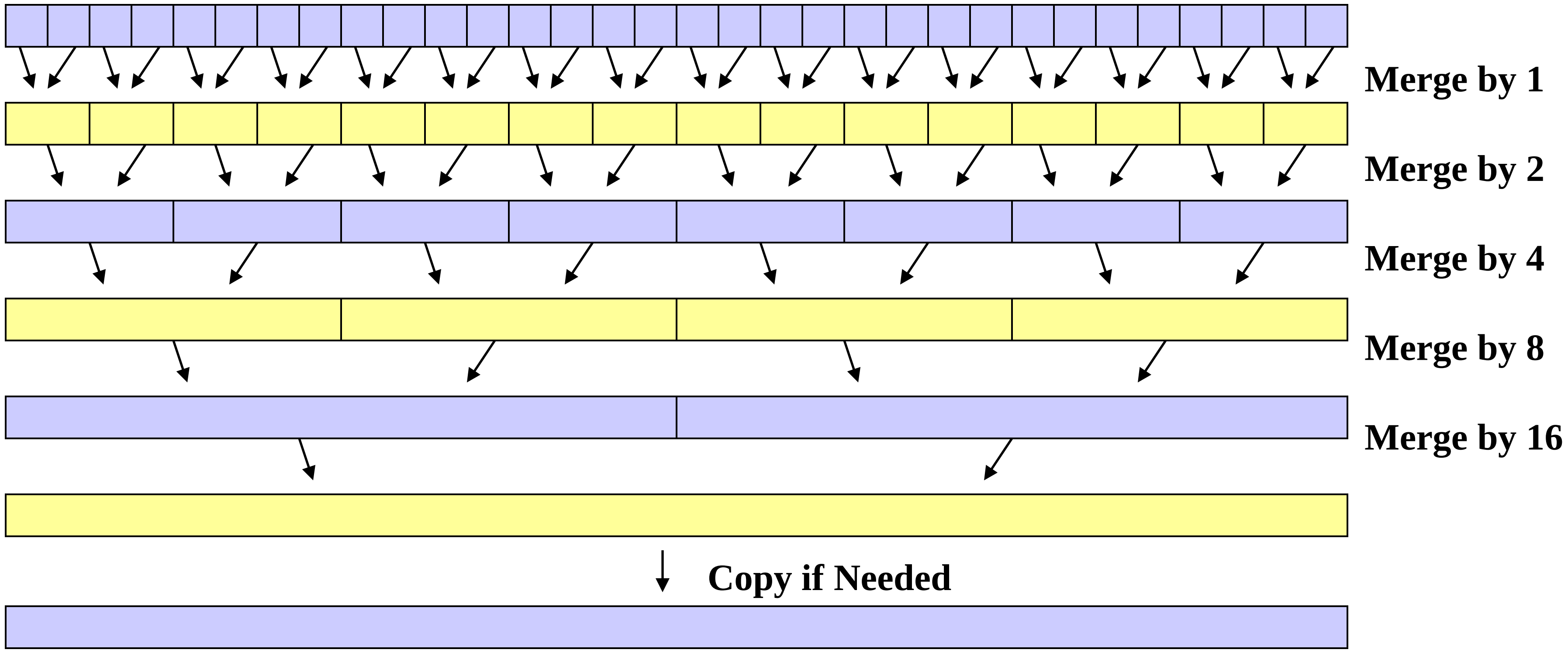
Don’t copy back – at 2nd, 4th, 6th, … merging stages, use the original array as the auxiliary array and vice-versa

- Need one copy at end if number of stages is odd

# *Picture of the "best" from previous slide: Allocate one auxiliary array, switch each step*

First recurse down to lists of size 1

As we return from the recursion, switch off arrays




Arguably easier to code up without recursion at all

# *Linked lists and big data*

We defined the sorting problem as over an array, but sometimes you want to sort linked lists

One approach:

- Convert to array: $O(n)$
- Sort: $O(n \log n)$
- Convert back to list: $O(n)$

Or: mergesort works very nicely on linked lists directly

- heapsort and quicksort do not
- insertion sort and selection sort do but they're slower

Mergesort is also the sort of choice for external sorting

- Linear merges minimize disk accesses

# *Mergesort Analysis*

Having defined an algorithm and argued it is correct, we should analyze its running time (and space):

To sort $n$ elements, we:

- Return immediately if $n$=1
- Else do 2 subproblems of size $n$/2 and then an $O(n)$ merge

Recurrence relation?

# *Mergesort Analysis*

Having defined an algorithm and argued it is correct, we should analyze its running time (and space):

To sort $n$ elements, we:

- Return immediately if $n$=1
- Else do 2 subproblems of size $n$/2 and then an $O(n)$ merge

Recurrence relation:

$$T(1) = c_1$$
$$T(n) = 2T(n/2) + c_2 n$$

# *MergeSort Recurrence*

(For simplicity let constants be 1 – no effect on asymptotic answer)

$T(1) = 1$

$T(n) = 2T(n/2) + n$

$= 2(2T(n/4) + n/2) + n$

$= 4T(n/4) + 2n$

$= 4(2T(n/8) + n/4) + 2n$

$= 8T(n/8) + 3n$

…. (after k expansions)

$= 2^{\mathbf{k}}T(n/2^{\mathbf{k}}) + kn$

So total is $2^{\mathbf{k}}T(n/2^{\mathbf{k}}) + kn$ where

$n/2^{\mathbf{k}} = 1$, i.e., $\log n = k$

That is, $2^{\mathbf{log\ n}}\, T(1) + n \log n$

$= n + n \log n$

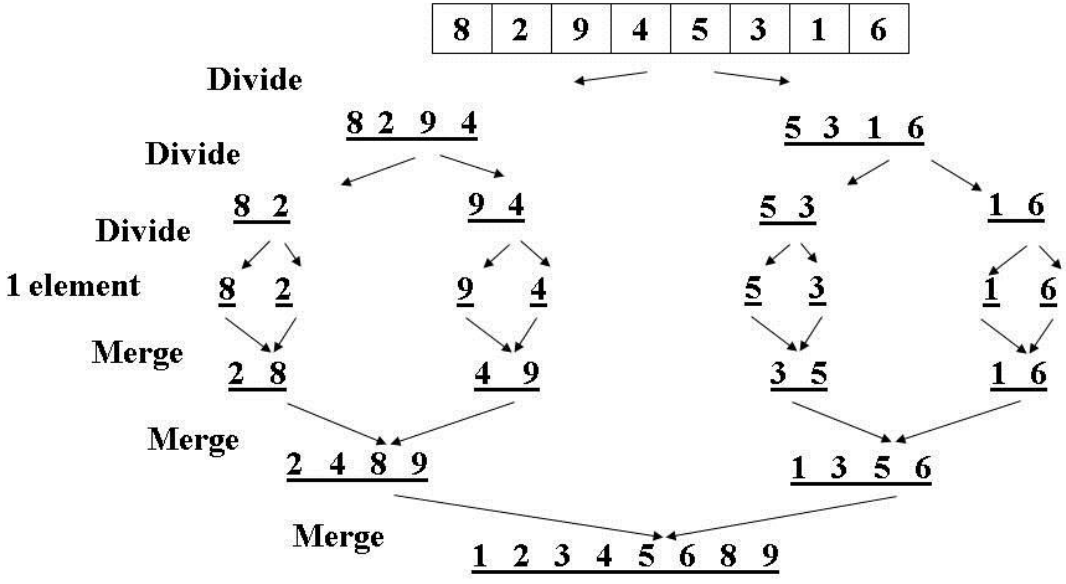
$= O(n \log n)$

# *Or more intuitively…*

This recurrence comes up often enough you should just “know” it’s *O*(*n* **`log`** *n*)

Merge sort is relatively easy to intuit (best, worst, and average):

- The recursion “tree” will have **`log`** *n* height
- At each level we do a *total* amount of merging equal to *n*



# *Quicksort*

- Also uses divide-and-conquer
  - Recursively chop into halves
  - But, instead of doing all the work as we merge together, we'll do all the work as we recursively split into halves
  - Also unlike MergeSort, does not need auxiliary space

- $O(n \log n)$ on average ☺, but $O(n^2)$ worst-case ☹
  - MergeSort is always O(nlogn)
  - So why use QuickSort?

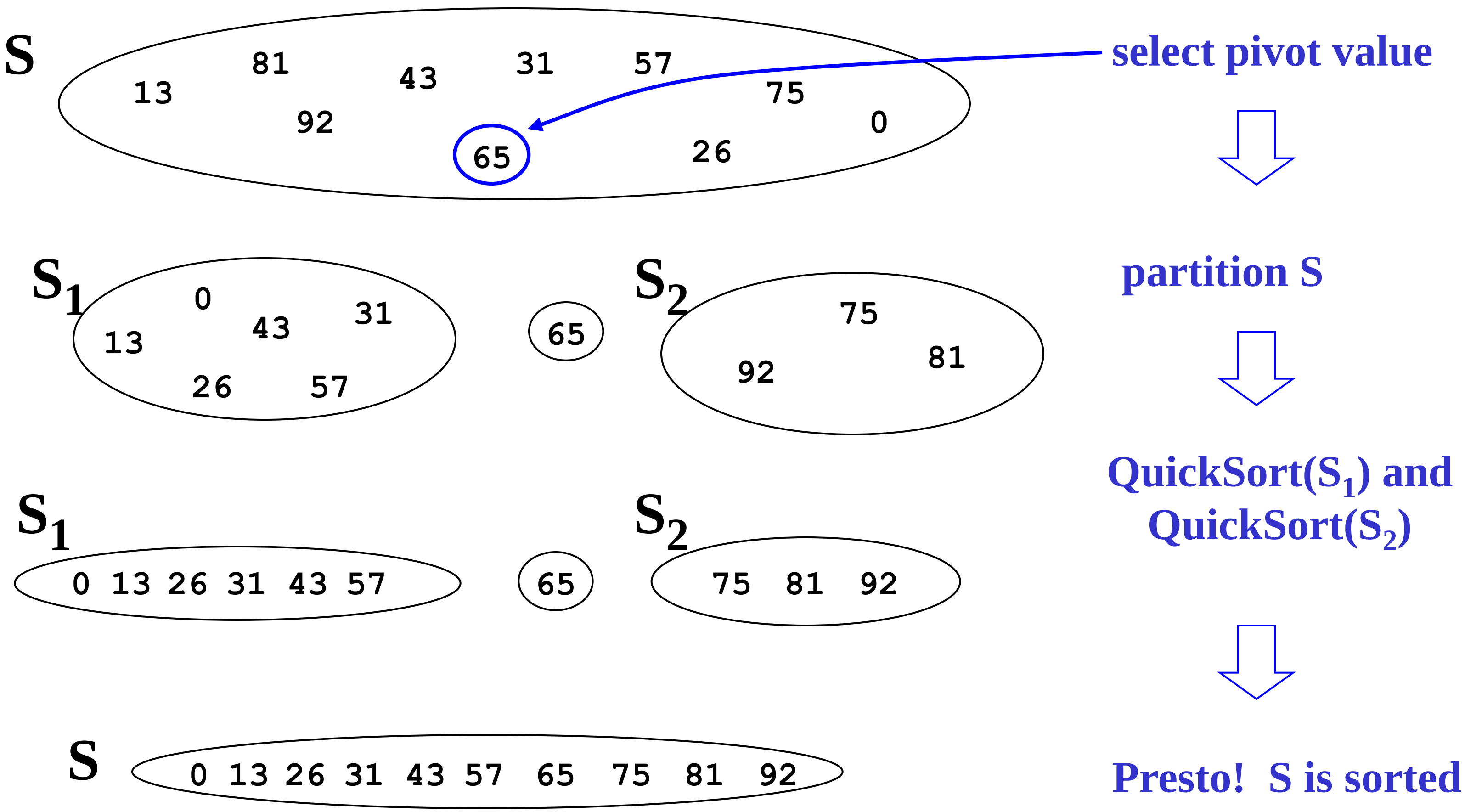
- Can be faster than mergesort
  - Often believed to be faster
  - Quicksort does fewer copies and more comparisons, so it depends on the relative cost of these two operations!

# *Quicksort overview*

1. Pick a pivot element
   - Hopefully an element ~median
   - Good QuickSort performance depends on good choice of pivot; we'll see why later, and talk about good pivot selection later
2. Partition all the data into:
   A. The elements less than the pivot
   B. The pivot
   C. The elements greater than the pivot
3. Recursively sort A and C
4. The answer is, "as simple as A, B, C"

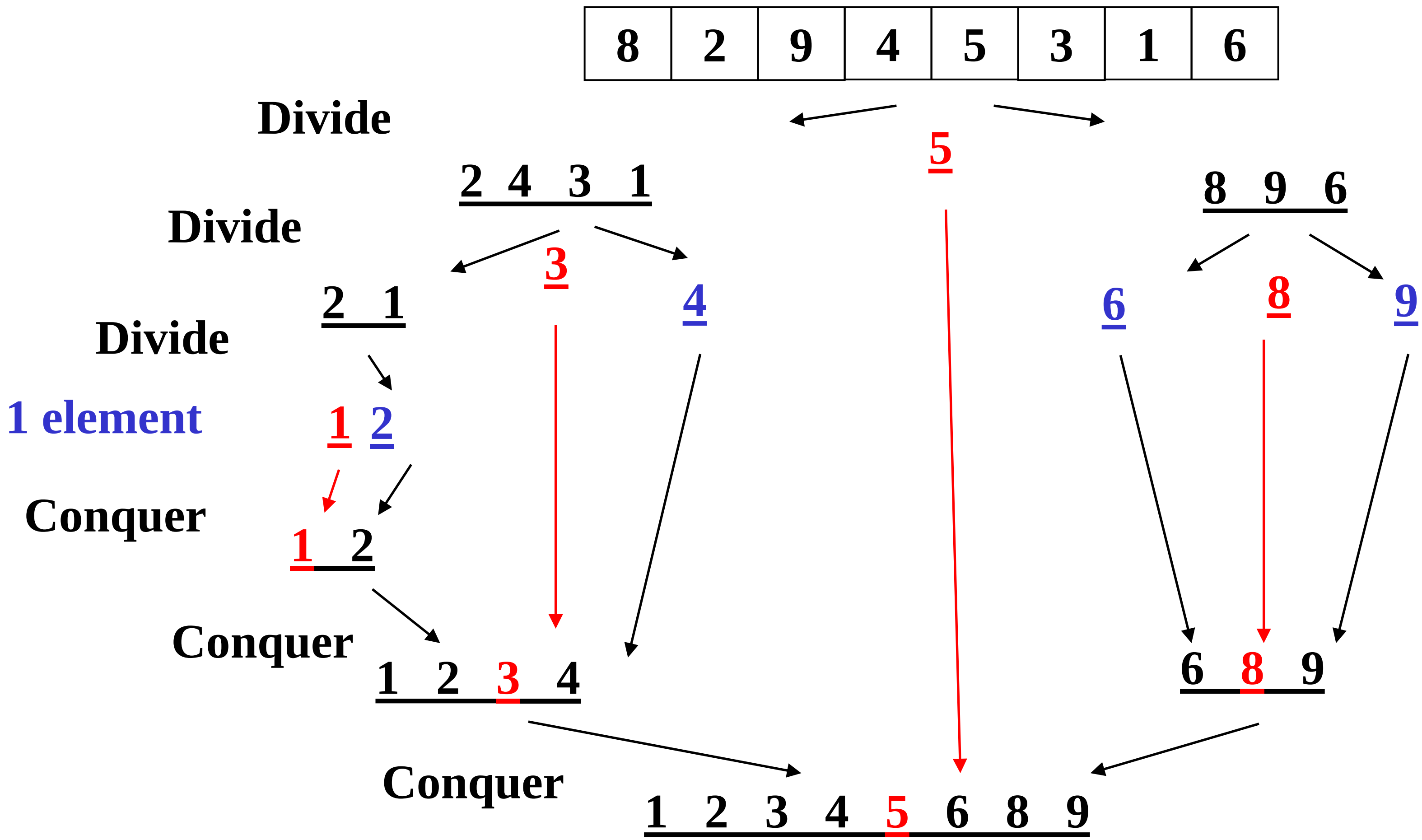
(Alas, there are some details lurking in this algorithm)

# *Quicksort: Think in terms of sets*

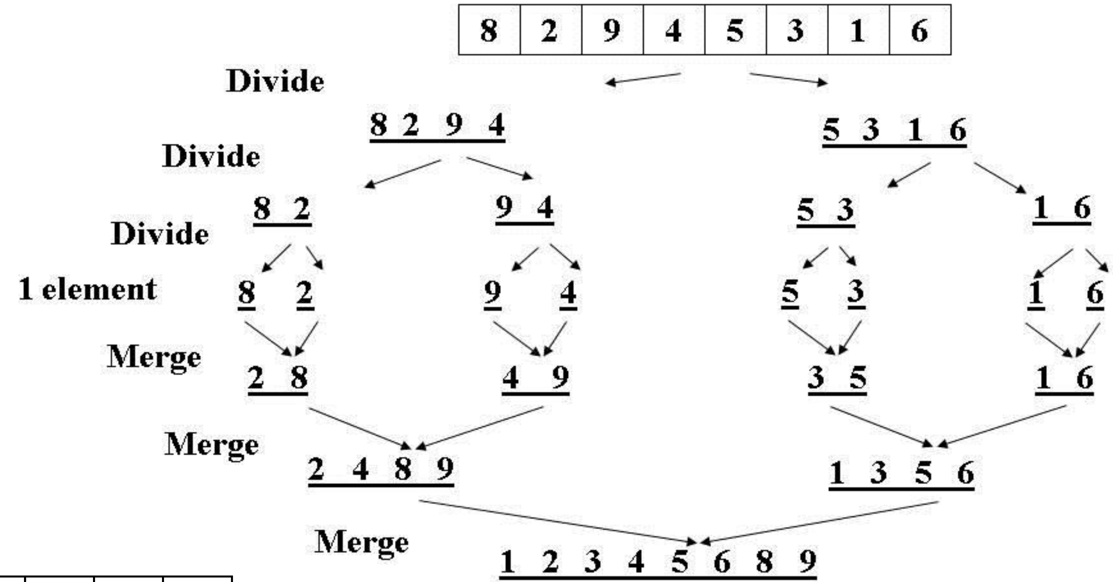



**[Weiss]**

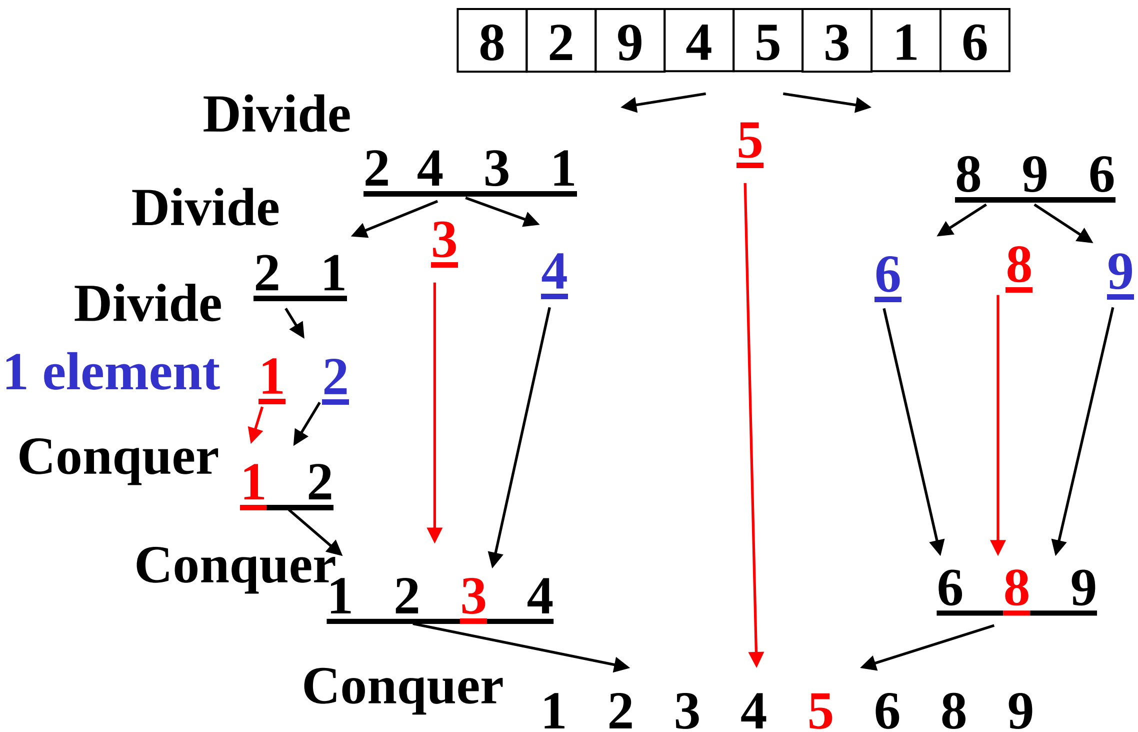
# *Quicksort Example, showing recursion*



# MergeSort Recursion Tree




# QuickSort Recursion Tree



# *Quicksort Details*

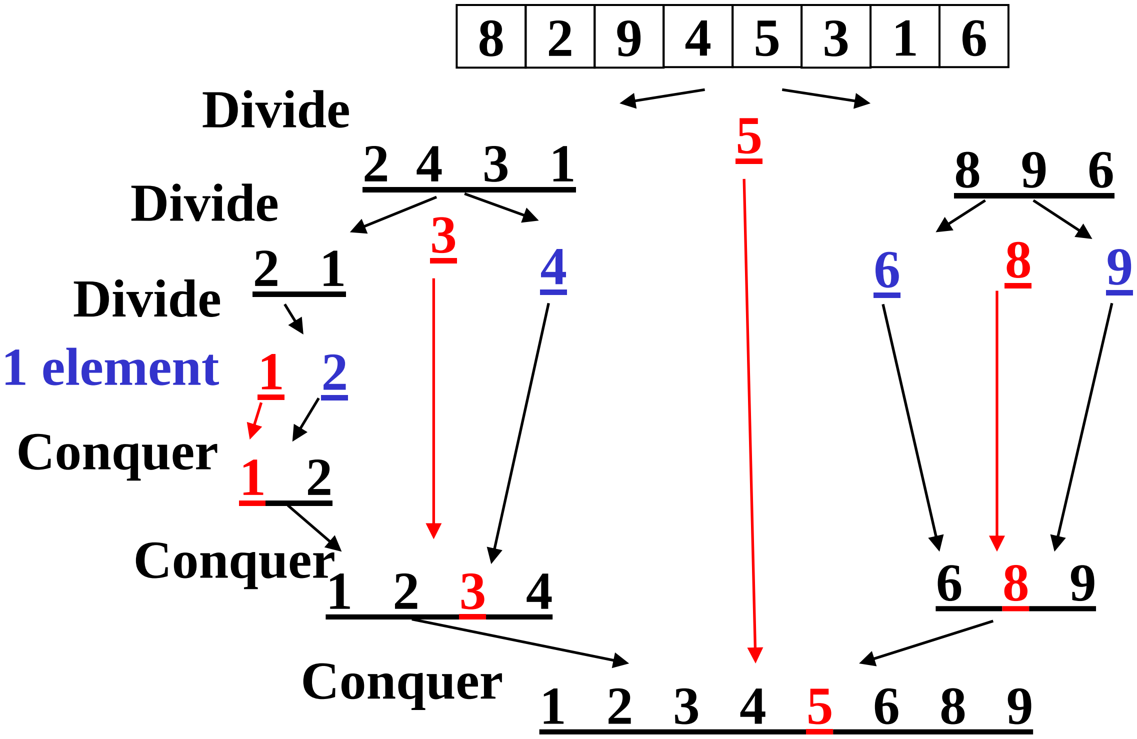
We have not yet explained:

- How to pick the pivot element
  - Any choice is correct: data will end up sorted
  - But as analysis will show, want the two partitions to be about equal in size

- How to implement partitioning
  - In linear time
  - In place

# *Pivots*

- Best pivot?
  - Median
  - Halve each time




- Worst pivot?
  - Greatest/least element
  - Reduce to problem of size 1 smaller
  - $O(n^2)$

# *Quicksort: Potential pivot rules*

While sorting **`arr`** from **`lo`** (inclusive) to **`hi`** (exclusive)…

- Pick **`arr[lo]`** or **`arr[hi-1]`**
  - Fast, but worst-case is (mostly) sorted input

- Pick random element in the range
  - Does as well as any technique, but (pseudo)random number generation can be slow
  - (Still probably the most elegant approach)

- Median of 3, e.g., **`arr[lo], arr[hi-1], arr[(hi+lo)/2]`**
  - Common heuristic that tends to work well

# *Partitioning*

- That is, given 8, 4, 2, 9, 3, 5, 7 and pivot 5
  - Dividing into left half & right half (based on pivot)

- Conceptually simple, but hardest part to code up correctly
  - After picking pivot, need to partition
    - Ideally in linear time
    - Ideally in place

- Ideas?

# *Partitioning*

- One approach (there are slightly fancier ones):
  1. Swap pivot with **`arr[lo]`**; move it ‘out of the way’
  2. Use two fingers **`i`** and **`j`**, starting at **`lo+1`** and **`hi-1`** (start & end of range, apart from pivot)
  3. Move from right until we hit something less than the pivot; belongs on left side
     Move from left until we hit something greater than the pivot; belongs on right side
     Swap these two; keep moving inward
     ```
     while (i < j)
        if (arr[j] > pivot) j--
        else if (arr[i] < pivot) i++
        else swap arr[i] with arr[j]
     ```
  4. Put pivot back in middle (Swap with **`arr[i]`**)

# *Quicksort Example*

- Step one: pick pivot as median of 3
  - **`lo`** = 0, **`hi`** = 10

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 8 | 1 | 4 | 9 | 0 | 3 | 5 | 2 | 7 | 6 |

- Step two: move pivot to the **`lo`** position

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 6 | 1 | 4 | 9 | 0 | 3 | 5 | 2 | 7 | 8 |

# *Quicksort Example*

Often have more than one swap during partition – this is a short example

Now partition in place

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 6 | 1 | 4 | 9 | 0 | 3 | 5 | 2 | 7 | 8 |

Move fingers

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 6 | 1 | 4 | 9 | 0 | 3 | 5 | 2 | 7 | 8 |

Swap

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 6 | 1 | 4 | 2 | 0 | 3 | 5 | 9 | 7 | 8 |

Move fingers

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 6 | 1 | 4 | 2 | 0 | 3 | 5 | 9 | 7 | 8 |

Move pivot

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 5 | 1 | 4 | 2 | 0 | 3 | 6 | 9 | 7 | 8 |

# *Quicksort Analysis*

- Best-case?

- Worst-case?

- Average-case?

# *Quicksort Analysis*

- Best-case: Pivot is always the median

  T(0)=T(1)=1

  $T(n)=2T(n/2) + n$ -- linear-time partition

  Same recurrence as mergesort: $O(n$ **`log`** $n)$

- Worst-case: Pivot is always smallest or largest element

  T(0)=T(1)=1

  $T(n) = 1T(n\text{-}1) + n$

  Basically same recurrence as selection sort: $O(n^2)$

- Average-case (e.g., with random pivot)
  - O($n$ **`log`** $n$), not responsible for proof (in text)

# *Quicksort Cutoffs*

- For small *n*, all that recursion tends to cost more than doing a quadratic sort
  - Remember asymptotic complexity is for large *n*
  - Also, recursive calls add a lot of overhead for small n
- Common engineering technique: switch to a different algorithm for subproblems below a cutoff
  - Reasonable rule of thumb: use insertion sort for $n < 10$
- Notes:
  - Could also use a cutoff for merge sort
  - Cutoffs are also the norm with parallel algorithms
    - switch to sequential algorithm
  - None of this affects asymptotic complexity

# Quicksort Cutoff skeleton

```
void quicksort(int[] arr, int lo, int hi) {
  if(hi - lo < CUTOFF)
     insertionSort(arr,lo,hi);
  else
     …
}
```

Notice how this cuts out the vast majority of the recursive calls

- Think of the recursive calls to quicksort as a tree
- Trims out the bottom layers of the tree