# CSE332: Data Abstractions

# Lecture 23: Minimum Spanning Trees

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Winter 2013

# *Announcements*

- **Homework 7** – due NOW at the BEGINNING of lecture!
- **Homework 8** – coming soon!

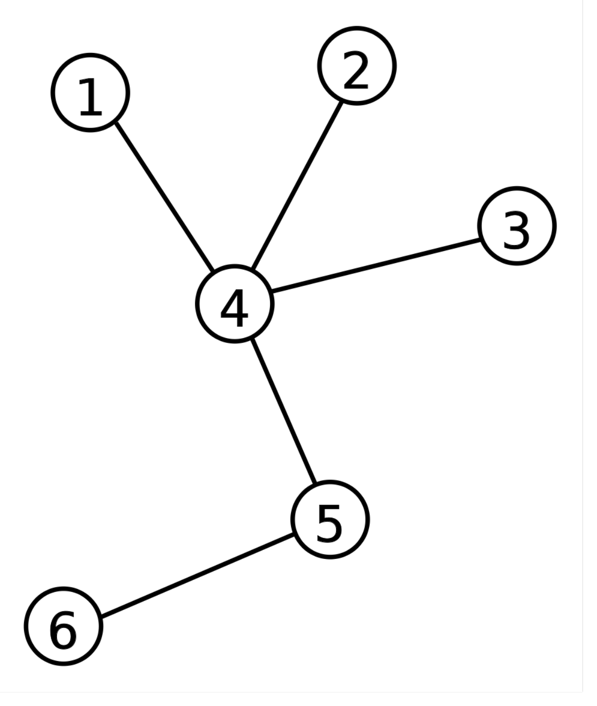
- **Project 3** – the last programming project!
  - **ALL Code - Tues March 12, 2013 11PM** - (65% of overall grade):
  - Writeup - Thursday March 14, 2013, 11PM - (25% of overall grade)

# *“Scheduling note”*

- “We now return to our interrupted program” on graphs
  - Last “graph lecture” was lecture 16
    - Shortest-path problem
    - Dijkstra’s algorithm for graphs with non-negative weights
- Why this strange schedule?
  - Needed to do parallelism and concurrency in time for project 3 and homeworks 6 and 7
  - But cannot delay all of graphs because of the CSE312 co-requisite
- So: not the most logical order, but hopefully not a big deal

# *Trees*

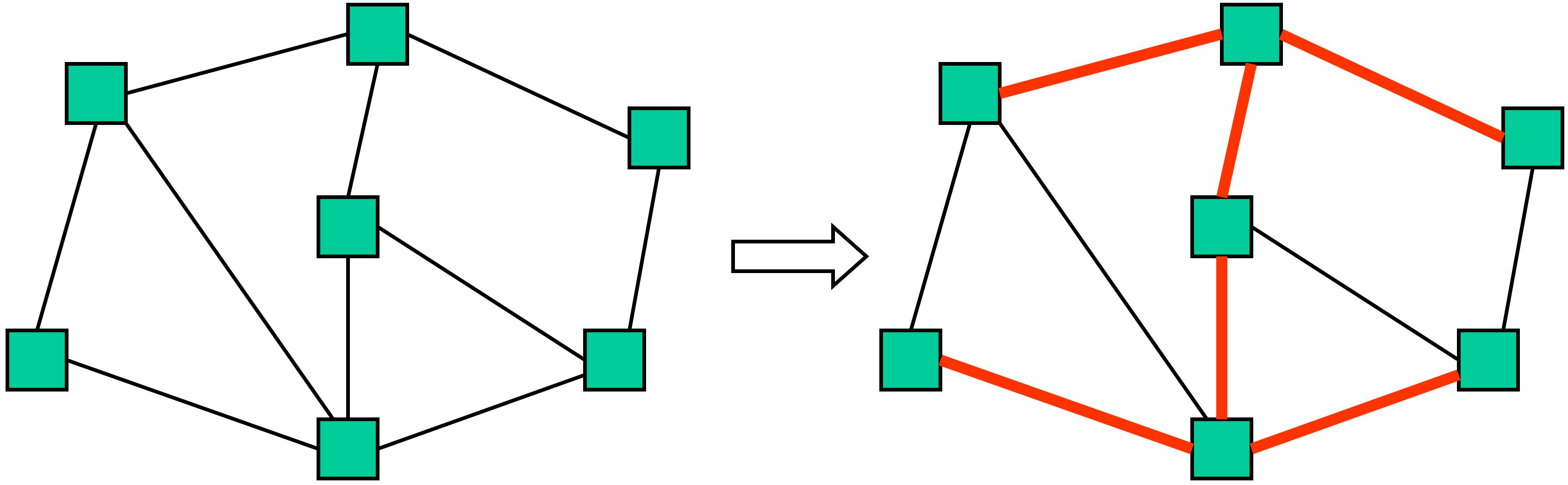
A tree is a graph with exactly one path between any two nodes.




No need for a root, hierarchy, or ordered children.

# *Spanning Trees*

- A simple problem: Given a *connected* graph **G**=(**V**,**E**), find a minimal subset of the edges such that the graph is still connected
  - A graph **G2**=(**V**,**E2**) such that **G2** is connected and removing any edge from **E2** makes **G2** disconnected



# *Observations*

1. Any solution to this problem is a tree
   - Recall a tree does not need a root; just means acyclic
   - For any cycle, could remove an edge and still be connected

2. Solution not unique unless original graph was already a tree

3. Problem ill-defined if original graph not connected

4. A tree with **|V|** nodes has **|V|-1** edges
   - So every solution to the spanning tree problem has **|V|-1** edges

# *Minimum Spanning Trees*

Given an undirected graph **G**=(**V**,**E**), find a graph **G'=(V, E')** such that:

- G**'** is a spanning tree.
- Sum of edge weights in G' is minimal

**G' is a minimum spanning tree.**

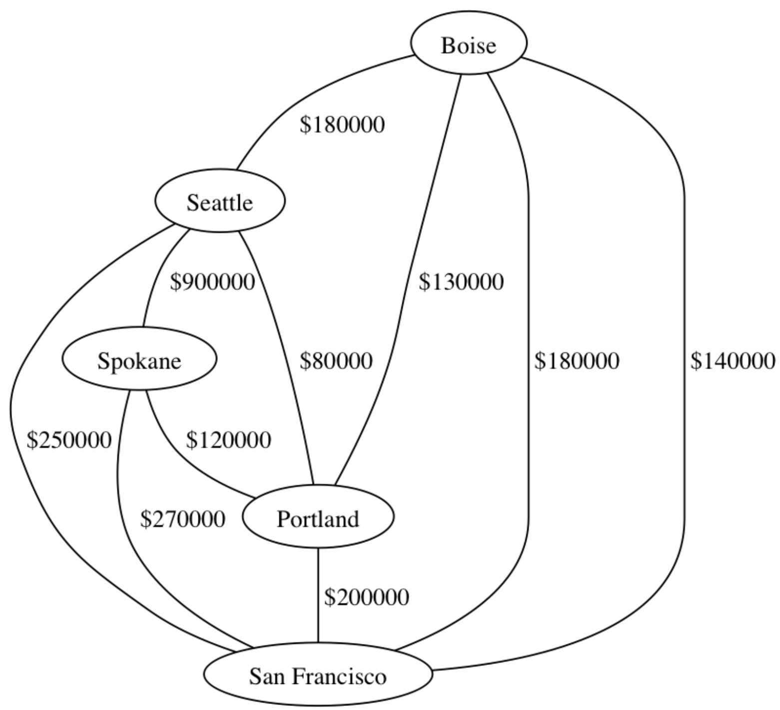
**Applications**:

- Example: Electrical wiring for a house or clock wires on a chip
- Example: A road network if you cared about asphalt cost rather than travel time

# *An application*

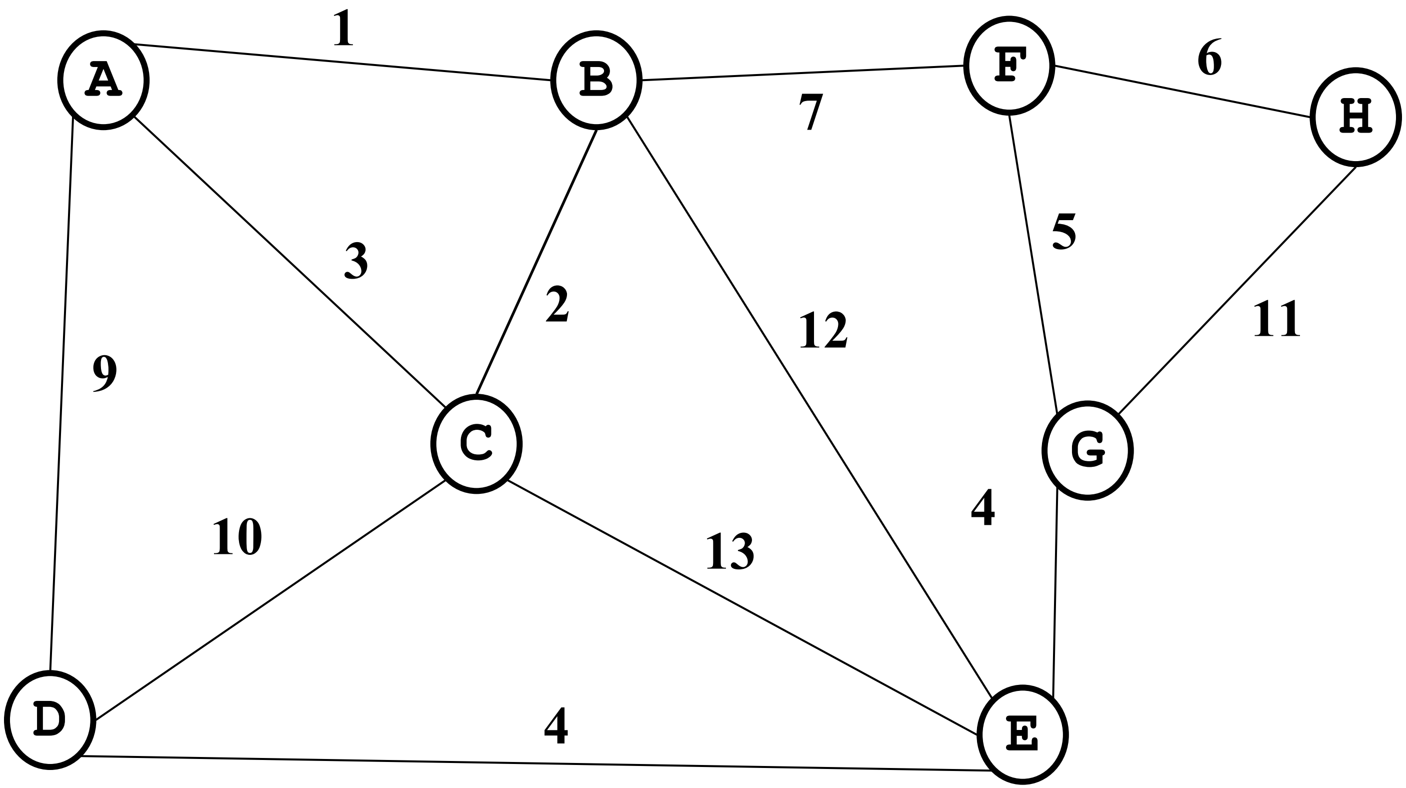
Bell systems was **the** telephone company for 100 years.

They want to connect everyone in the US to their telephone network as cheaply as possible.

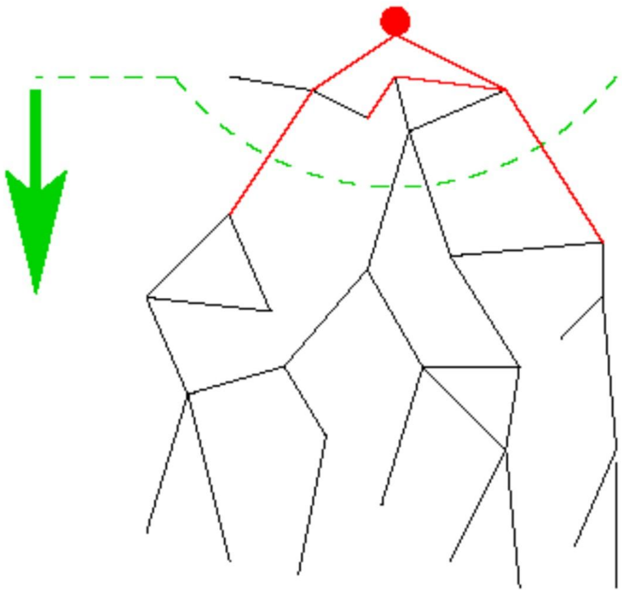


**Student Activity**

# *Find the MST*



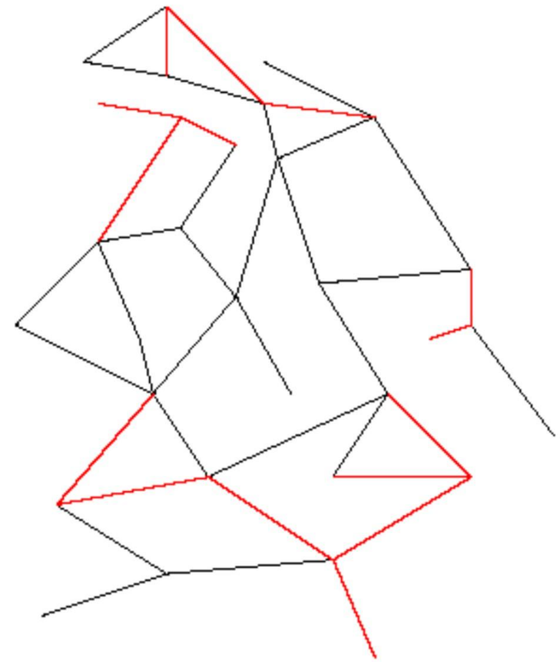
# *Two Different Approaches*



**Prim's Algorithm**

**Almost identical to Dijkstra's**

**One node, grow greedily**



**Kruskals's Algorithm**

**Completely different!**

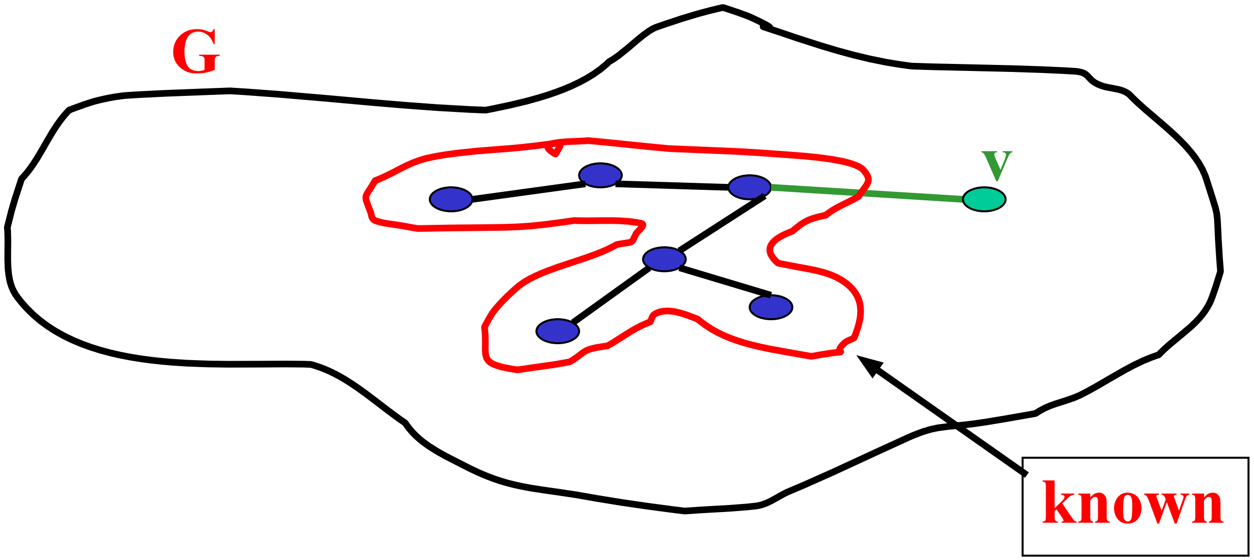
**Forest of MSTs,**
***Union* them together.**
**I wonder how to union…**

# *Prim's algorithm*

**Idea**: Grow a tree by picking a vertex from the unknown set that has the smallest cost. Here cost = cost of the edge that connects that vertex to the known set. *Pick the vertex with the smallest cost that connects "known" to "unknown."*

**A *node-based* greedy algorithm**

**Builds MST by greedily adding nodes**



# *Prim's Algorithm vs. Dijkstra's*

Recall:

Dijkstra picked the unknown vertex with smallest cost where
cost = ***distance to the source***.

Prim's pick the unknown vertex with smallest cost where
cost = ***distance from this vertex to the known set*** (in other words, the cost of the smallest edge connecting this vertex to the known set)

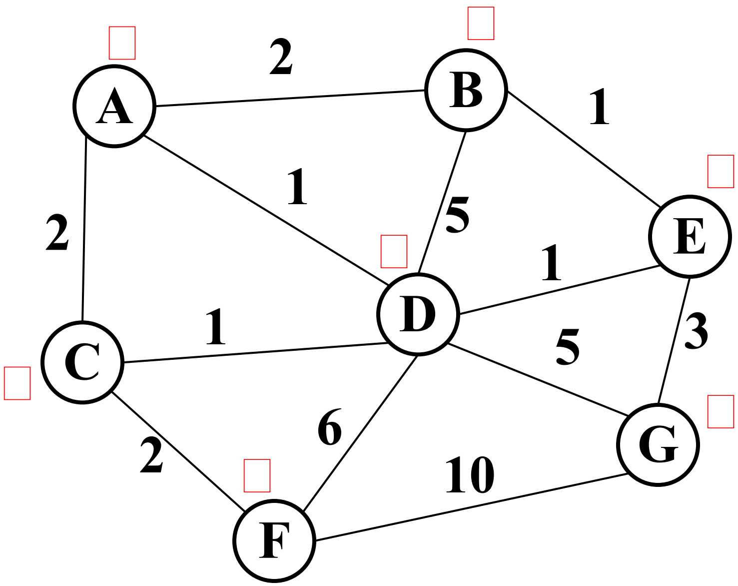
- Otherwise identical
- Compare to slides in lecture 16!

# *Prim's Algorithm for MST*

1. For each node **v**, set **v.cost = ∞** and **v.known = false**
2. Choose any node **v.** (this is like your "start" vertex in Dijkstra)
   a. Mark **v** as known
   b. For each edge **(v,u)** with weight **w**:
      set **u.cost=w** and **u.prev=v**
3. While there are unknown nodes in the graph
   c. Select the unknown node **v** with lowest cost
   d. Mark **v** as known and add **(v, v.prev)** to output (the MST)
   e. For each edge **(v,u)** with weight **w**,

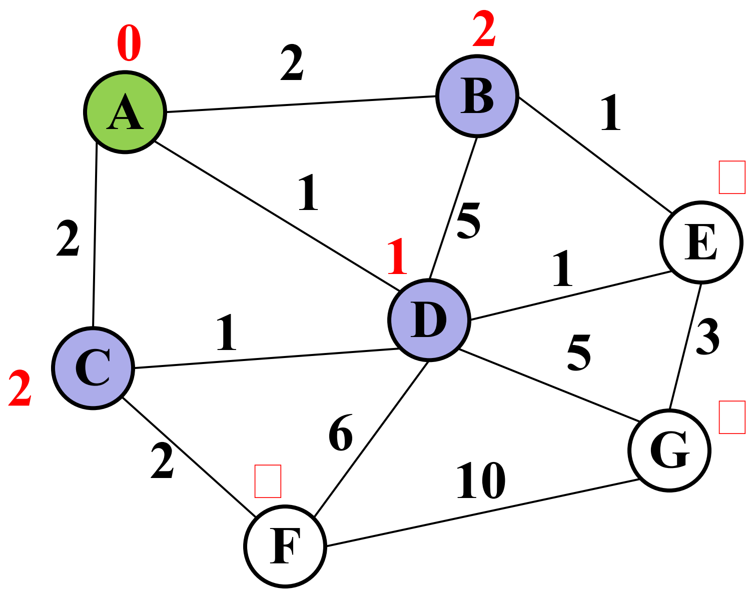
```
if(w < u.cost) {
  u.cost = w;
u.prev = v;
}
```

# *Example: Find MST using Prim's*



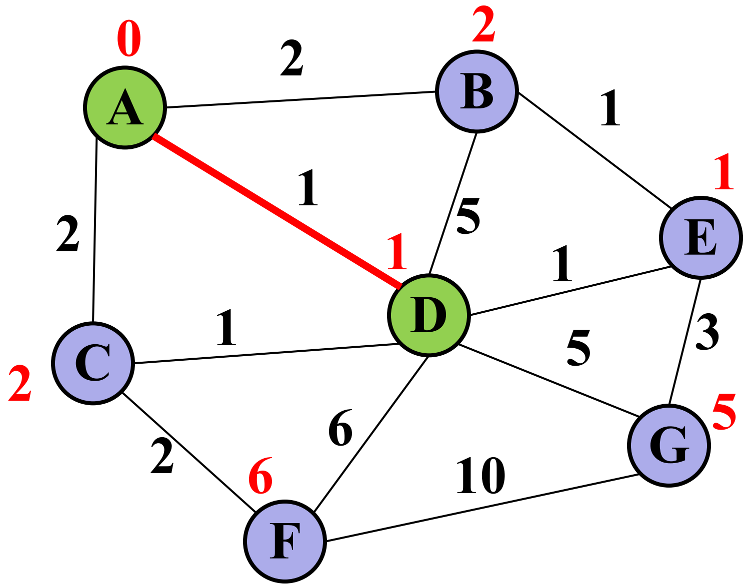

| vertex | known? | cost | prev |
|---|---|---|---|
| A | | ?? | |
| B | | ?? | |
| C | | ?? | |
| D | | ?? | |
| E | | ?? | |
| F | | ?? | |
| G | | ?? | |

# *Example: Find MST using Prim's*



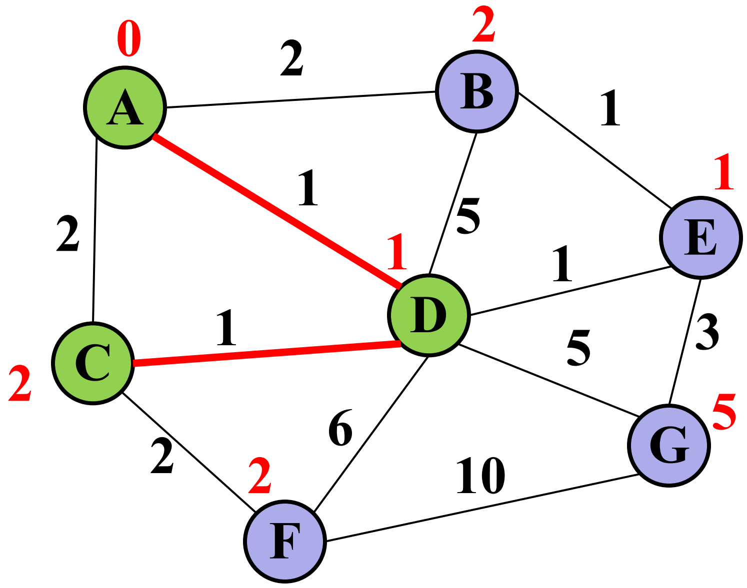

| vertex | known? | cost | prev |
|---|---|---|---|
| A | Y | 0 | |
| B | | 2 | A |
| C | | 2 | A |
| D | | 1 | A |
| E | | ?? | |
| F | | ?? | |
| G | | ?? | |

# *Example: Find MST using Prim's*



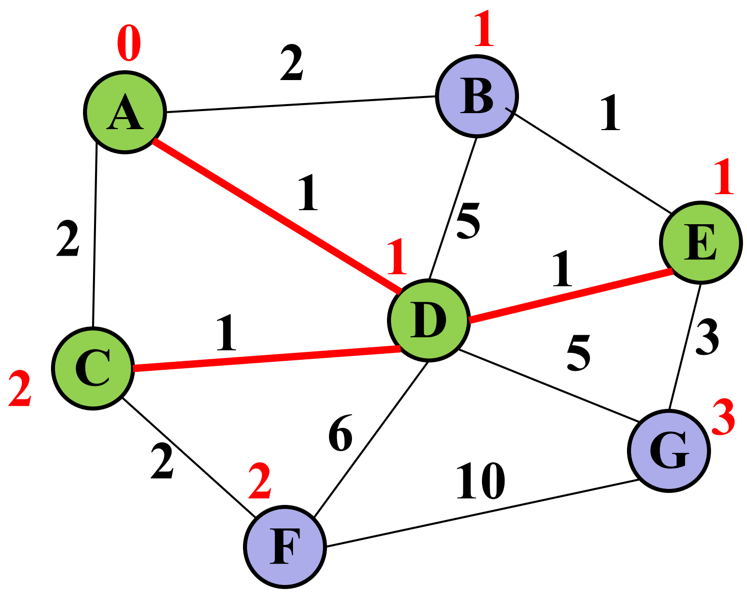

| vertex | known? | cost | prev |
|---|---|---|---|
| A | Y | 0 | |
| B | | 2 | A |
| C | | 1 | D |
| D | Y | 1 | A |
| E | | 1 | D |
| F | | 6 | D |
| G | | 5 | D |

# *Example: Find MST using Prim's*



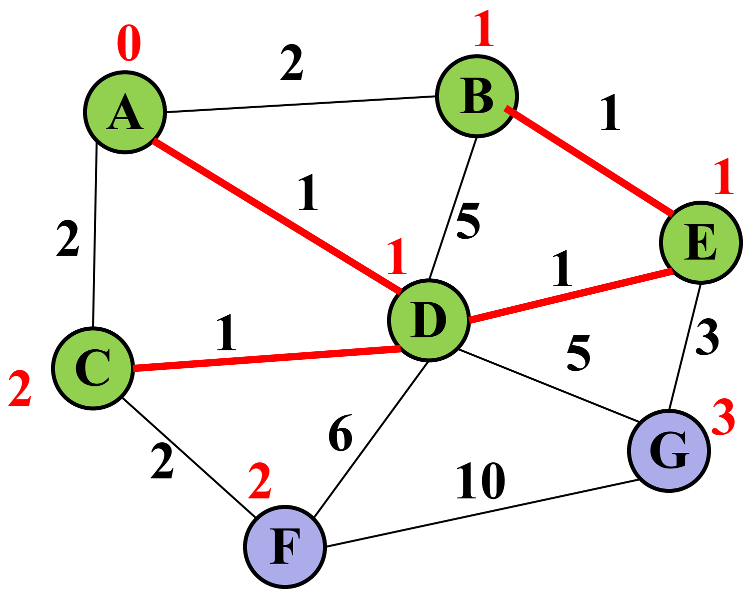

| vertex | known? | cost | prev |
|---|---|---|---|
| A | Y | 0 | |
| B | | 2 | A |
| C | Y | 1 | D |
| D | Y | 1 | A |
| E | | 1 | D |
| F | | 2 | C |
| G | | 5 | D |

# *Example: Find MST using Prim's*



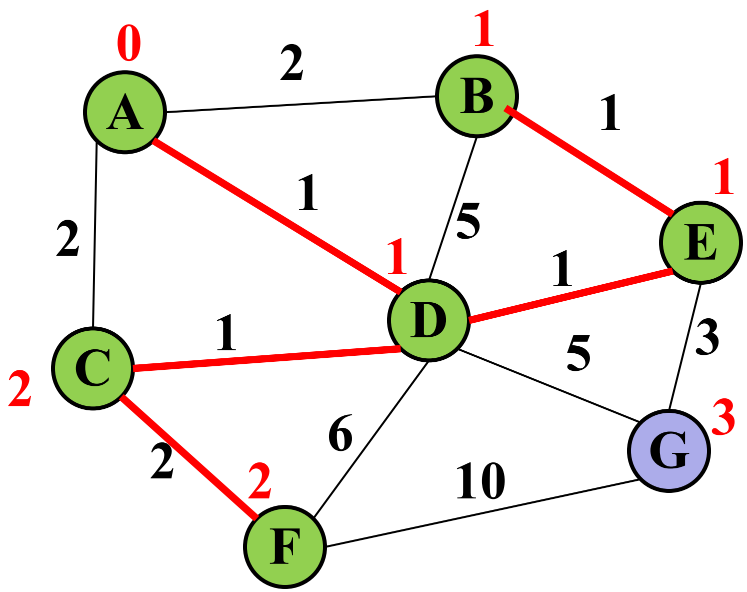

| vertex | known? | cost | prev |
|---|---|---|---|
| A | Y | 0 | |
| B | | 1 | E |
| C | Y | 1 | D |
| D | Y | 1 | A |
| E | Y | 1 | D |
| F | | 2 | C |
| G | | 3 | E |

# *Example: Find MST using Prim's*



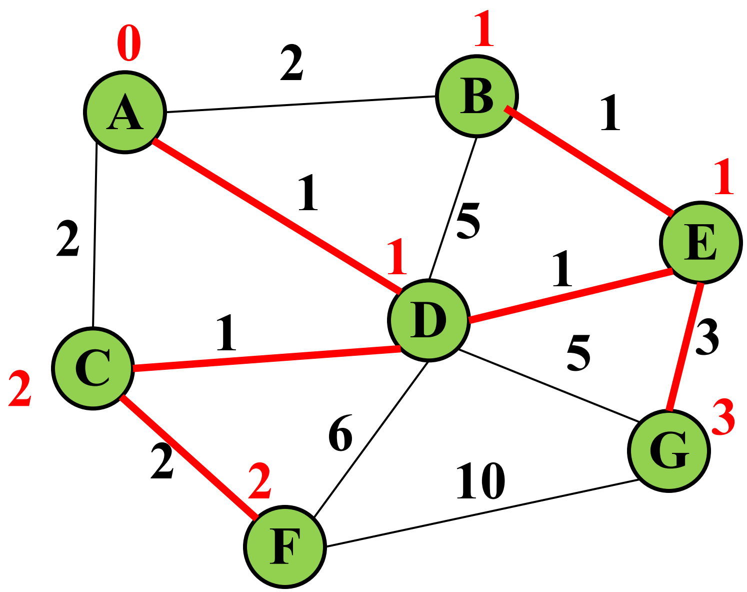

| vertex | known? | cost | prev |
|---|---|---|---|
| A | Y | 0 | |
| B | Y | 1 | E |
| C | Y | 1 | D |
| D | Y | 1 | A |
| E | Y | 1 | D |
| F | | 2 | C |
| G | | 3 | E |

# *Example: Find MST using Prim's*




| vertex | known? | cost | prev |
|---|---|---|---|
| A | Y | 0 | |
| B | Y | 1 | E |
| C | Y | 1 | D |
| D | Y | 1 | A |
| E | Y | 1 | D |
| F | Y | 2 | C |
| G | | 3 | E |

# *Example: Find MST using Prim's*

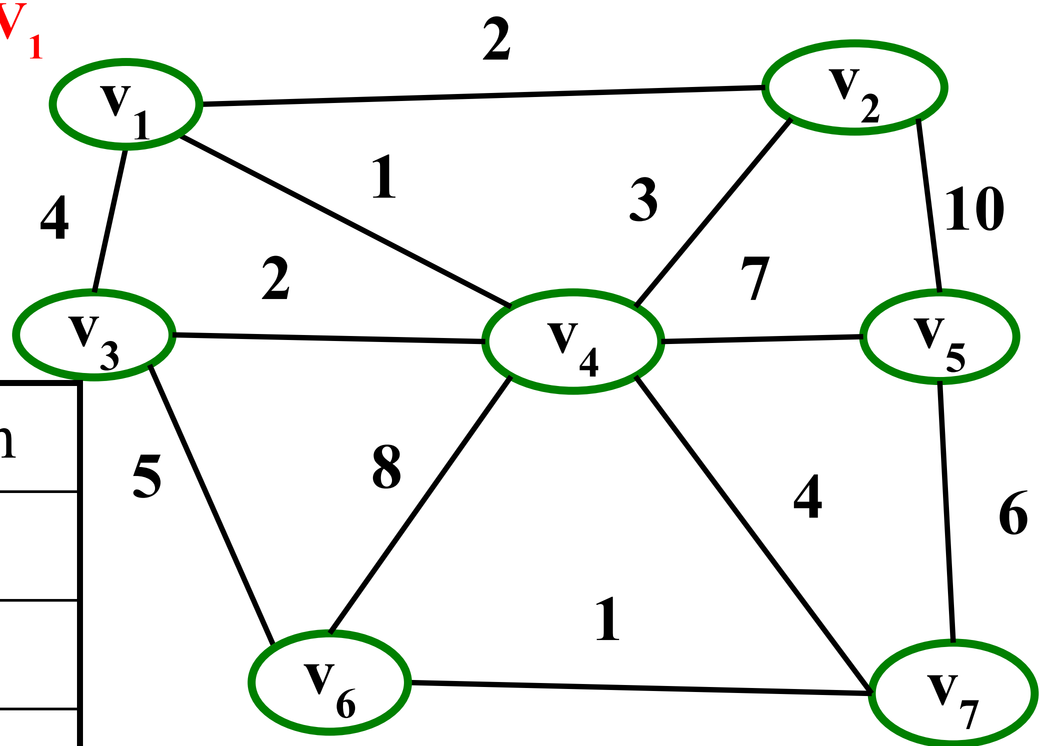



| vertex | known? | cost | prev |
|---|---|---|---|
| A | Y | 0 | |
| B | Y | 1 | E |
| C | Y | 1 | D |
| D | Y | 1 | A |
| E | Y | 1 | D |
| F | Y | 2 | C |
| G | Y | 3 | E |

Student Activity

Start with $V_1$

# Find MST using Prim's




| V | Kwn | Distance | path |
|---|---|---|---|
| v1 | | | |
| v2 | | | |
| v3 | | | |
| v4 | | | |
| v5 | | | |
| v6 | | | |
| v7 | | | |

**Order Declared Known:**

$V_1$

**Total Cost:**

# *Prim's Analysis*

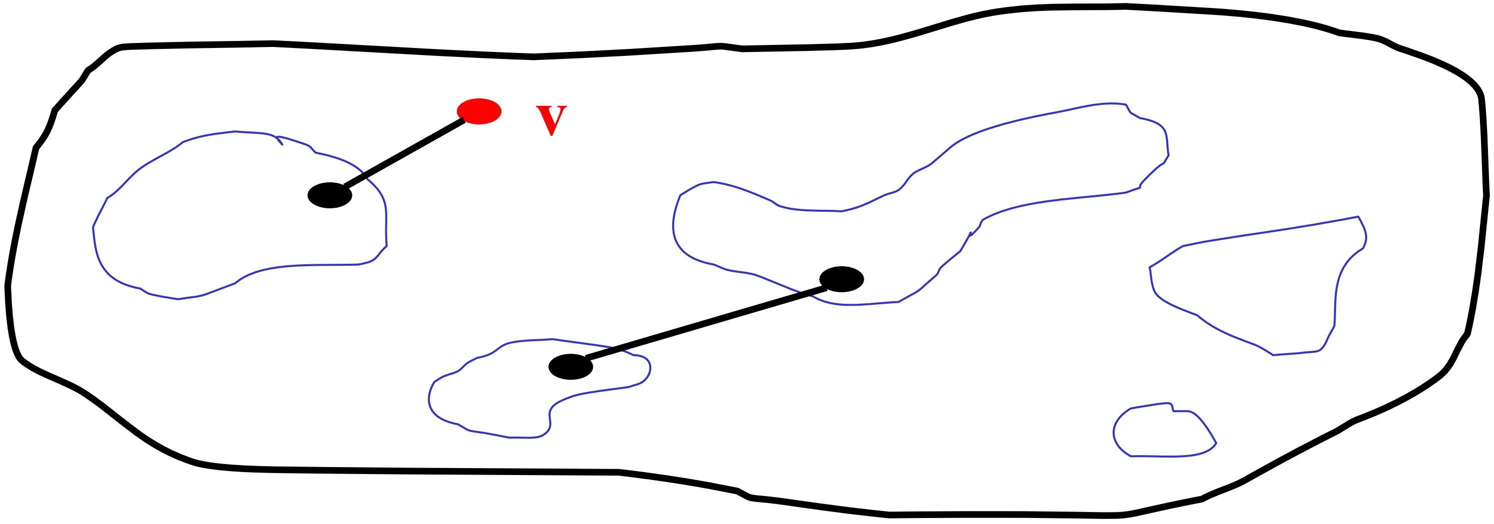
- Correctness ??
  - A bit tricky
  - Intuitively similar to Dijkstra
  - Might return to this time permitting (unlikely)

- Run-time
  - Same as Dijkstra
  - $O(|E| \log |V|)$ using a heap

# *Kruskal's MST Algorithm*

Idea: Grow a forest out of edges that do not create a cycle. Pick an edge with the smallest weight.

G=(V,E)



# *Kruskal's Algorithm for MST*

## An *edge-based* greedy algorithm

**Builds MST by greedily adding edges**

1. Initialize with
   - empty MST
   - all vertices marked unconnected
   - all edges unmarked
2. While there are still unmarked edges
   a. Pick the <u>lowest cost edge</u> **`(u,v)`** and mark it
   b. If **`u`** and **`v`** are not already connected, add **`(u,v)`** to the MST and mark **`u`** and **`v`** as connected to each other

**Maze construction used random edge order.**
**Otherwise the same!**

# *Aside: Union-Find aka Disjoint Set ADT*

- **Union(x,y)** – take the union of two sets named x and y
  - Given sets: {3,5,7} , {4,2,8}, {9}, {1,6}
  - **Union(5,1)**

    Result: {3,5,7,1,6}, {4,2,8}, {9},

  To perform the union operation, we replace sets x and y by (x ∪ y)

- **Find(x)** – return the name of the set containing x.
  - Given sets: {3,5,7,1,6}, {4,2,8}, {9},
  - **Find(1)** returns 5
  - **Find(4)** returns 8

- We can do Union in constant time.
- We can get Find to be ***amortized*** constant time

  (worst case O(log n) for an individual Find operation).

# *Kruskal's pseudo code*

On heap of edges
Deletemin = log |E|

```
void Graph::kruskal(){
  int edgesAccepted = 0;
  DisjSet s(NUM_VERTICES);
```

Sort of ignore this loop in calc run-time…

```
  while (edgesAccepted < NUM_VERTICES – 1){
    e = smallest weight edge not deleted yet;
    // edge e = (u, v)
    uset = s.find(u);
    vset = s.find(v);
    if (uset != vset){
      edgesAccepted++;
      s.unionSets(uset, vset);
    }
  }
}
```

|E| heap ops

2|E| finds

One for each vertex in the edge
Find = log |V|

|V| unions

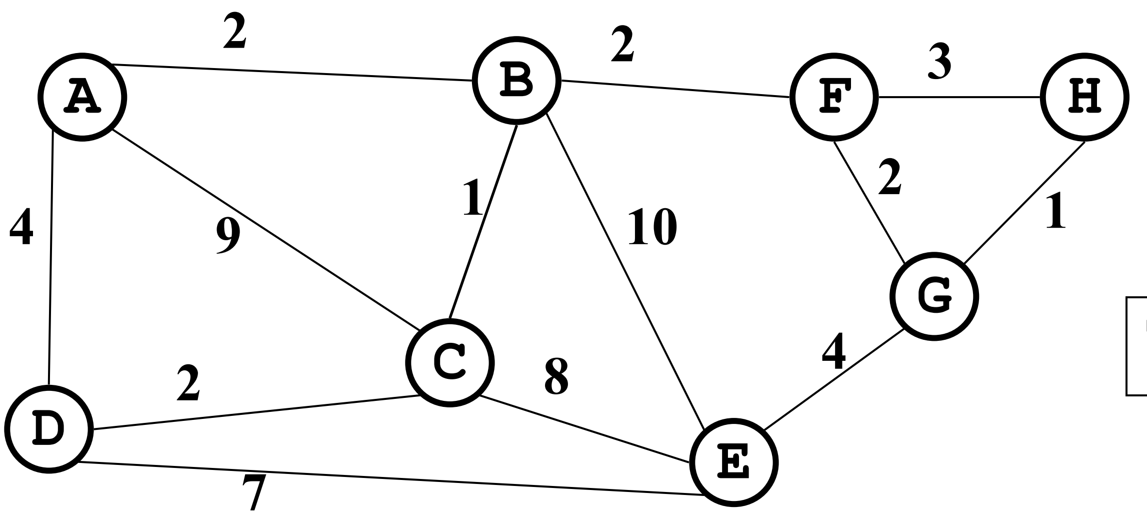
Union = O(1)

$|E| \log |E| + 2|E|\log|V|+|V|$

$O(|E|\log|E|) = O(|E|\log|V|)$
b/c $\log |E| < \log|V|^2 = 2\log|V|$

**Student Activity**

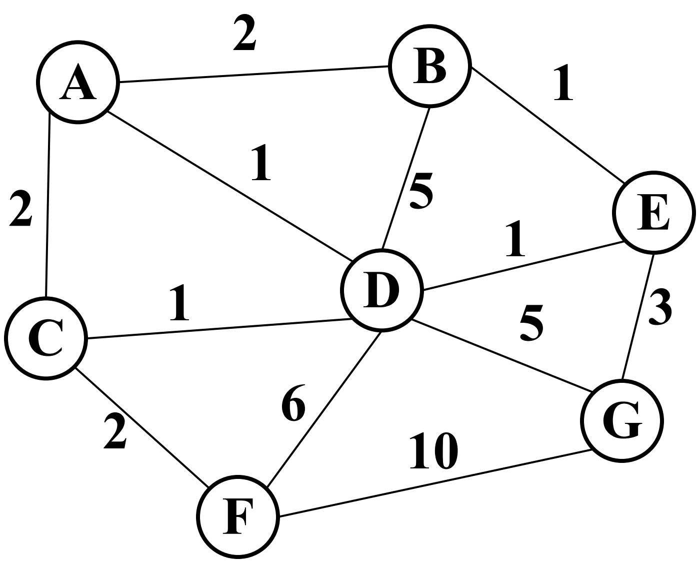
# *Find MST using Kruskal's*




**Total Cost:**

- **Now find the MST using Prim's method.**
- **Under what conditions will these methods give the same result?**

# *Example: Find MST using Kruskal's*




Edges in sorted order:

1: (A,D), (C,D), (B,E), (D,E)

2: (A,B), (C,F), (A,C)

3: (E,G)

5: (D,G), (B,D)

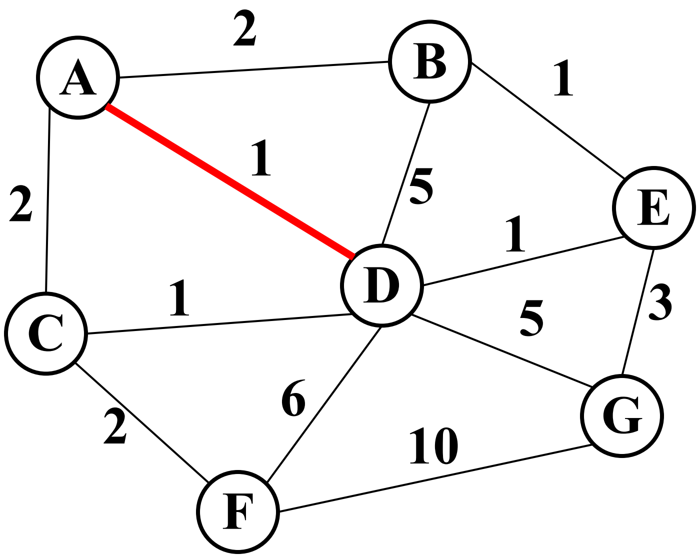
6: (D,F)

10: (F,G)

Output:

Note: At each step, the union/find sets are the trees in the forest

# *Example: Find MST using Kruskal's*

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3: (E,G)

5: (D,G), (B,D)

6: (D,F)

10: (F,G)

Output: (A,D)

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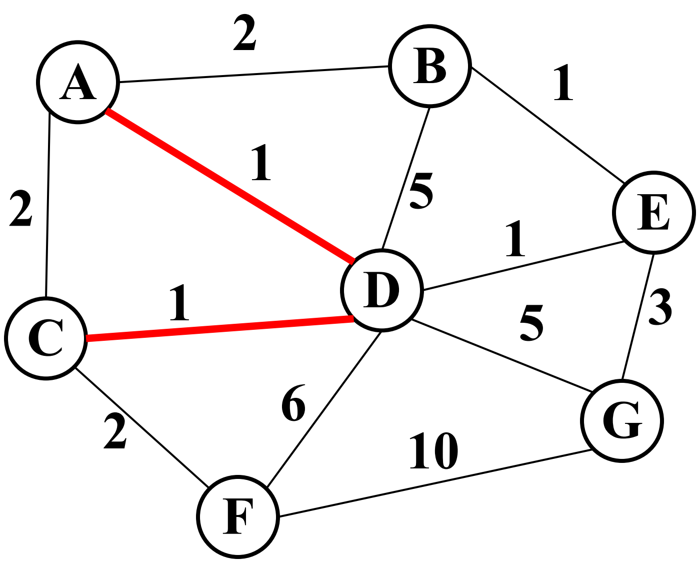
# *Example: Find MST using Kruskal's*

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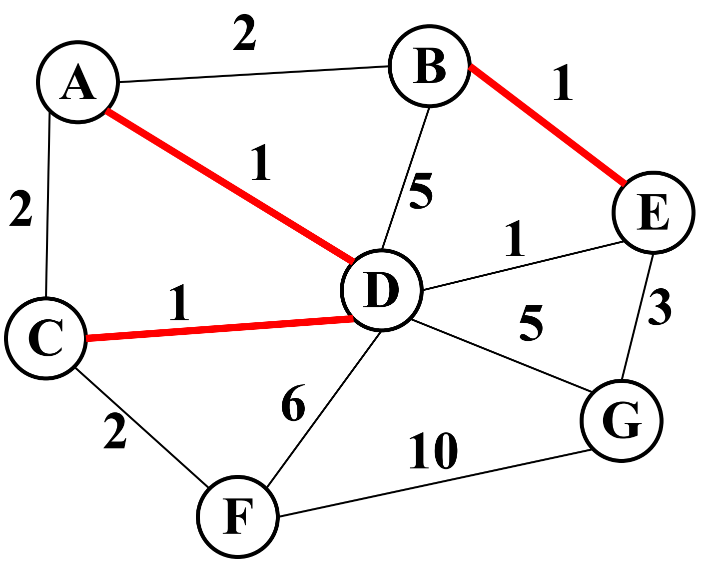
6: (D,F)

10: (F,G)

Output: (A,D), (C,D)

Note: At each step, the union/find sets are the trees in the forest

# *Example: Find MST using Kruskal's*

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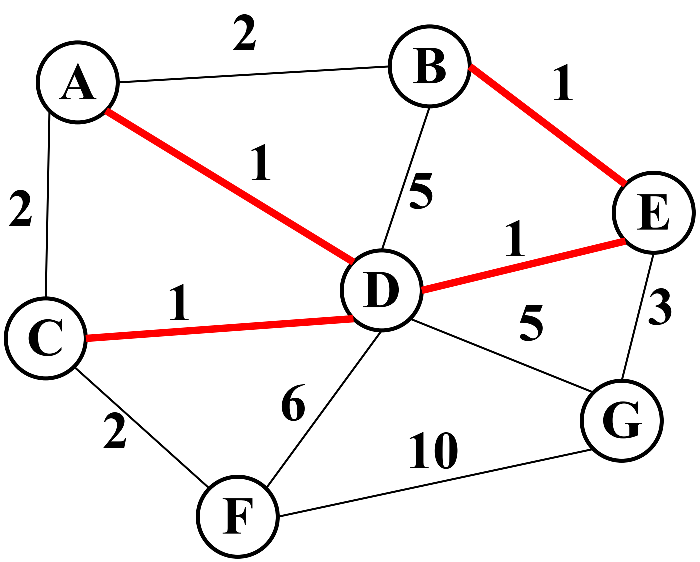
Output: (A,D), (C,D), (B,E)

Note: At each step, the union/find sets are the trees in the forest

# *Example: Find MST using Kruskal's*

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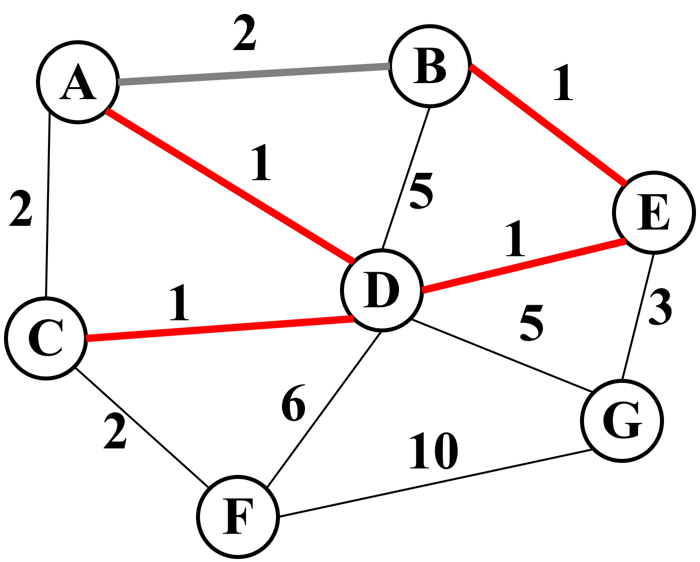


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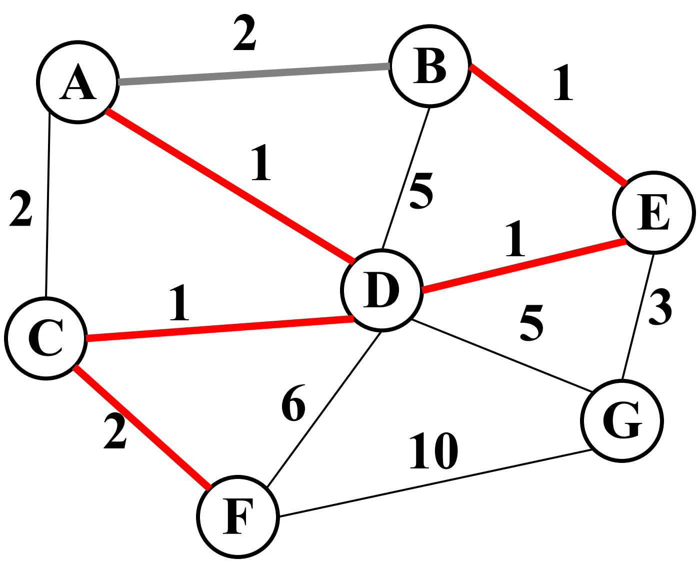
# *Example: Find MST using Kruskal's*

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6: (D,F)

10: (F,G)

Output: (A,D), (C,D), (B,E), (D,E), (C,F)

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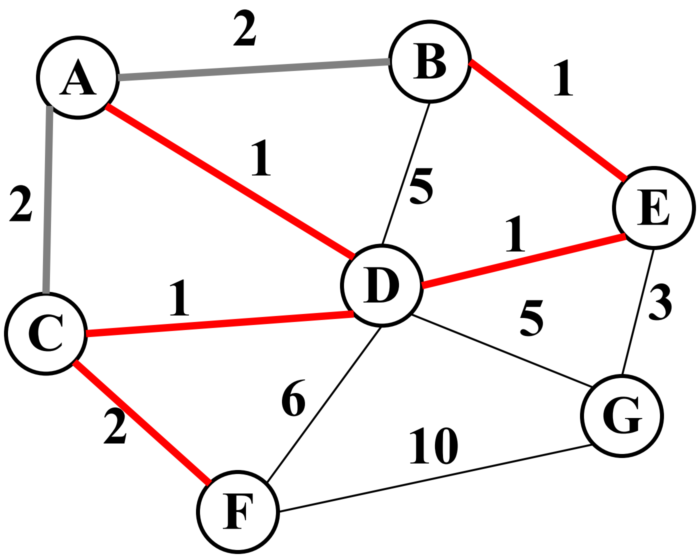
# *Example: Find MST using Kruskal's*

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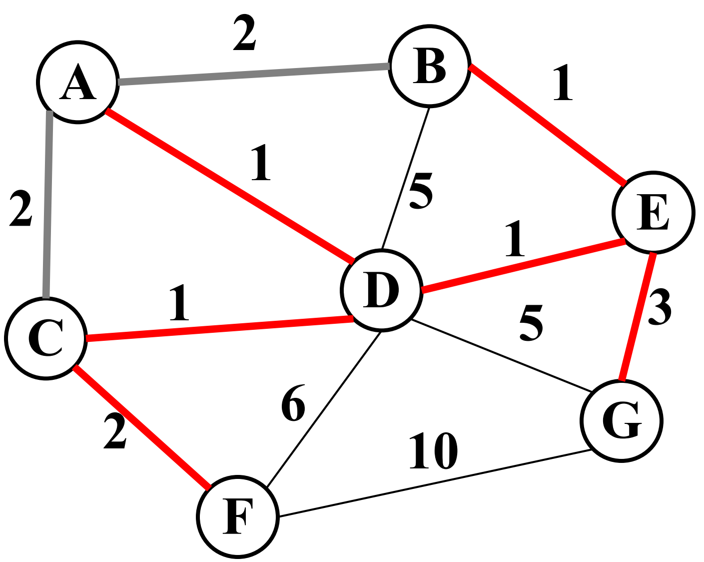
# *Example: Find MST using Kruskal's*




Edges in sorted order:

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10: (F,G)

Output: (A,D), (C,D), (B,E), (D,E), (C,F), (E,G)

Note: At each step, the union/find sets are the trees in the forest

# *Correctness*

Kruskal's algorithm is clever, simple, and efficient

- But does it generate a minimum spanning tree?
- How can we prove it?

First: it generates a spanning tree

- Intuition: Graph started connected and we added every edge that did not create a cycle
- Proof by contradiction: Suppose **`u`** and **`v`** are disconnected in Kruskal's result. Then there's a path from **`u`** to **`v`** in the initial graph with an edge we could add without creating a cycle. But Kruskal would have added that edge. Contradiction.

Second: There is no spanning tree with lower total cost…

# *The inductive proof set-up*

Let **F** (stands for “forest”) be the set of edges Kruskal has added at some point during its execution.

Claim: **F** is a subset of *one or more* MSTs for the graph
(Therefore, once **|F|=|V|-1**, we have an MST.)

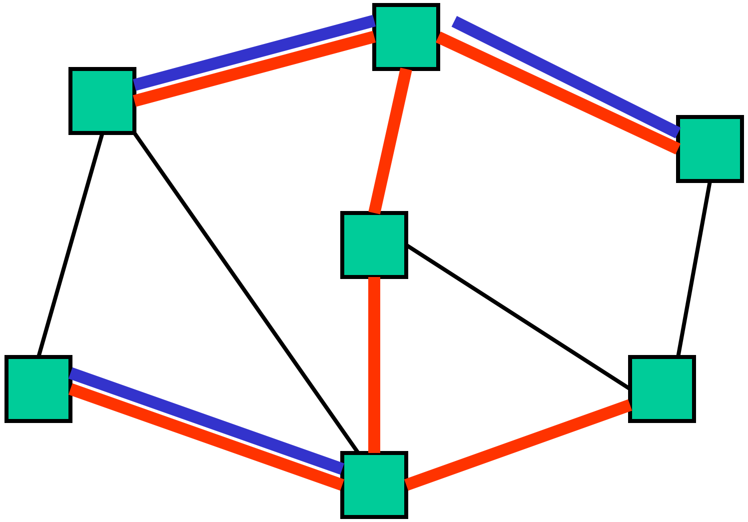
Proof: By induction on **|F|**

Base case: **|F|=0**: The empty set is a subset of all MSTs

Inductive case: **|F|=k+1**: By induction, before adding the $(k+1)^{th}$ edge (call it **e**), there was some MST **T** such that $\mathbf{F-\{e\} \subseteq T}$ …

# *Staying a subset of **some** MST*

Claim: **F** is a subset of *one or more* MSTs for the graph



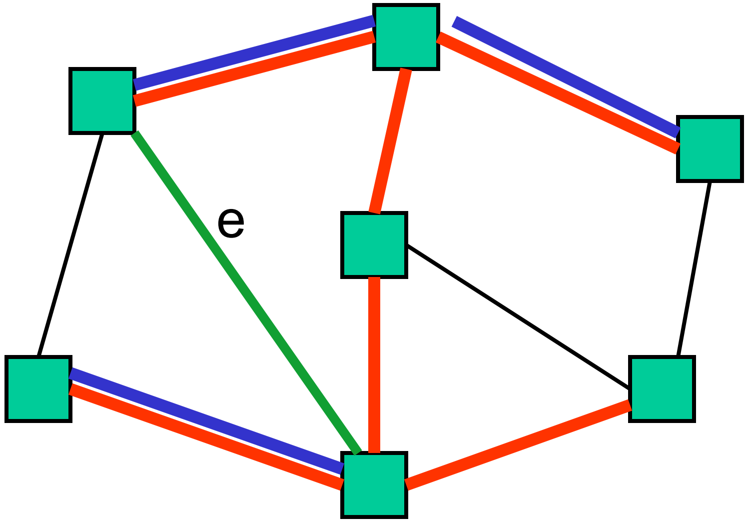
So far: **F-{e} ⊆ T**:

Two disjoint cases:

- If **{e} ⊆ T**: Then **F ⊆ T** and we're done
- Else **e** forms a cycle with some simple path (call it **p**) in **T**
  - Must be since **T** is a spanning tree

# *Staying a subset of **some** MST*

Claim: **F** is a subset of *one or more* MSTs for the graph




So far: **F**-{**e**} **⊆** **T** and

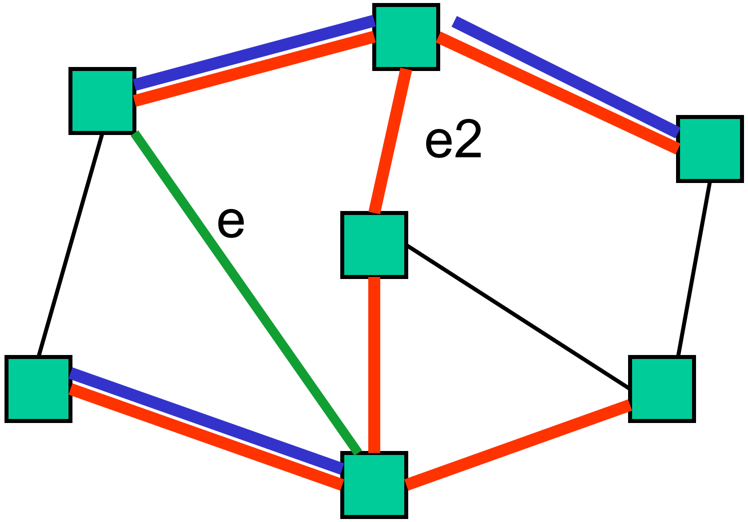
**e** forms a cycle with **p** ⊆ **T**

- There must be an edge **e2** on **p** such that **e2** is not in **F**
  - Else Kruskal would not have added **e**

- Claim: **e2.weight == e.weight**

# *Staying a subset of* ***some*** *MST*

Claim: **F** is a subset of *one or more* MSTs for the graph




So far: **F-{e} ⊆ T**

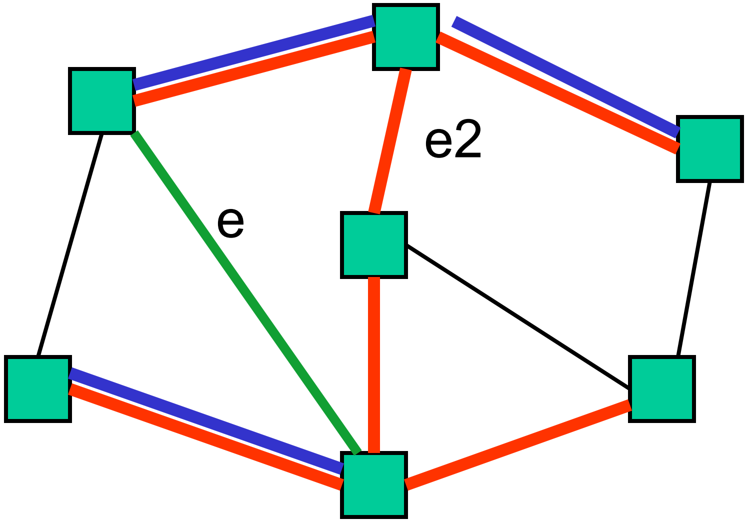
**e** forms a cycle with **p** ⊆ **T**

**e2** on **p** is not in **F**

- Claim: **e2.weight == e.weight**
  - If **e2.weight > e.weight**, then **T** is not an MST because **T-{e2}+{e}** is a spanning tree with lower cost: contradiction
  - If **e2.weight < e.weight**, then Kruskal would have already considered **e2**. It would have added it since **T** has no cycles and **F-{e} ⊆ T.** But **e2** is not in **F**: contradiction

# *Staying a subset of **some** MST*

Claim: **F** is a subset of *one or more* MSTs for the graph




So far: **F-{e} ⊆ T**

**e** forms a cycle with **p** ⊆ **T**

**e2** on **p** is not in **F**

**e2.weight == e.weight**

- Claim: **T-{e2}+{e}** is an MST
  - It's a spanning tree because **p-{e2}+{e}** connects the same nodes as **p**
  - It's minimal because its cost equals cost of **T**, an MST
- Since **F ⊆ T-{e2}+{e}**, **F** is a subset of one or more MSTs

Done.