CSE 332: Data Abstractions

Lecture 3: Asymptotic Analysis

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Announcements

- **Project 1** – phase A due next Wed
- **Homework 1** – due next Friday at ***<u>beginning</u>*** of class

- Info sheets?
- Catalyst Survey

Today

- How to compare two algorithms?
- Analyzing code
- Big-Oh

Comparing Two Algorithms…

Gauging performance

- Uh, why not just run the program and time it
  - Too much *variability*, not reliable or *portable*:
    - Hardware: processor(s), memory, etc.
    - OS, Java version, libraries, drivers
    - Other programs running
    - Implementation dependent
  - Choice of input
    - Testing (inexhaustive) may *miss* worst-case input
    - Timing does not *explain* relative timing among inputs (what happens when *n* doubles in size)

- Often want to evaluate an *algorithm*, not an implementation
  - Even *before* creating the implementation ("coding it up")

Comparing algorithms

When is one *algorithm* (not *implementation*) better than another?

- Various possible answers (clarity, security, …)
- But a big one is *performance*: for sufficiently large inputs, runs in less time (our focus) or less space

Large inputs (n) because probably any algorithm is "plenty good" for small inputs (if n is 10, probably anything is fast enough)

Answer will be *independent* of CPU speed, programming language, coding tricks, etc.

Answer is general and rigorous, complementary to "coding it up and timing it on some test cases"

- Can do analysis before coding!

Analyzing code ("worst case")

Basic operations take "some amount of" constant time

- Arithmetic (fixed-width)
- Assignment
- Access one Java field **or array index**
- Etc.

(This is an *approximation of reality*: a very useful "lie".)

Consecutive statements	Sum of time of each statement
Conditionals	Time of condition plus time of slower branch
Loops	Num iterations * time for loop body
Function Calls	Time of function's body
Recursion	Solve *recurrence equation*

Example

2	**3**	**5**	**16**	**37**	**50**	**73**	**75**	**126**

Find an integer in a *sorted* array

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   ???
}
```

Linear search

2	**3**	**5**	**16**	**37**	**50**	**73**	**75**	**126**

Find an integer in a *sorted* array

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   for(int i=0; i < arr.length; ++i)
      if(arr[i] == k)
        return true;
   return false;
}
```

Best case:

Worst case:

Linear search

2	**3**	**5**	**16**	**37**	**50**	**73**	**75**	**126**

Find an integer in a *sorted* array

```
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boolean find(int[]arr, int k){
   for(int i=0; i < arr.length; ++i)
      if(arr[i] == k)
        return true;
   return false;
}
```

Best case: 6ish steps = *O*(1)

Worst case: 5ish*(arr.length)
= *O*(arr.length)

Analyzing Recursive Code

- Computing run-times gets interesting with recursion
- Say we want to perform some computation recursively on a list of size n
  - Conceptually, in each recursive call we:
    - Perform some amount of work, call it w(n)
    - Call the function recursively with a smaller portion of the list

- So, if we do w(n) work per step, and reduce the problem size in the next recursive call by 1, we do total work:

$$T(n)=w(n)+T(n-1)$$

- With some base case, like $T(1)=5=O(1)$

Example Recursive code: sum array

Recursive:

- Recurrence is some constant amount of work O(1) done *n* times

```
int sum(int[] arr){
  return help(arr,0);
}
int help(int[]arr,int i) {
  if(i==arr.length)
    return 0;
  return arr[i] + help(arr,i+1);
}
```

Each time **`help`** is called, it does that O(1) amount of work, and then calls **`help`** again on a problem one less than previous problem size.

Recurrence Relation: $T(n) = O(1) + T(n\text{-}1)$

Solving Recurrence Relations

- Say we have the following recurrence relation:

  $T(n)=3+T(n-1)$

  $T(1)=5$ $\leftarrow$ base case

- Now we just need to solve it; that is, reduce it to a closed form.
- Start by writing it out:

$$
\begin{aligned}
T(n)&=3+T(n-1)\\
&=3+3+T(n-2)\\
&=3+3+3+T(n-3)\\
&=3k+T(n-k)\\
&=3+3+3+\ldots+3+T(1) \;=\; 3+3+3+\ldots+3+5\\
&=3k+5, \text{ where k is the \# of times we expanded T()}
\end{aligned}
$$

- We expanded it out n-1 times, so

$$
\begin{aligned}
T(n)&=3k+T(n-k)\\
&=3(n-1)+T(1) = 3(n-1)+5\\
&=3n+2 = O(n)
\end{aligned}
$$

Or When does $n-k=1$?
Answer: when $k=n-1$

Binary search

Best case:

Worst case:

2	**3**	**5**	**16**	**37**	**50**	**73**	**75**	**126**

Find an integer in a *sorted* array

- Can also be done non-recursively but “doesn’t matter” here

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   return help(arr,k,0,arr.length);
}
boolean help(int[]arr, int k, int lo, int hi) {
   int mid = (hi+lo)/2; //i.e., lo+(hi-lo)/2
   if(lo==hi)          return false;
   if(arr[mid]==k)     return true;
   if(arr[mid]< k)     return help(arr,k,mid+1,hi);
   else                return help(arr,k,lo,mid);
}
```

Binary search

Best case: 9ish steps = $O(1)$

Worst case: $T(n) = 10\text{ish} + T(n/2)$ where n is **`hi-lo`**

- $O($**`log`** $n)$ where n is **`array.length`**
- Solve *recurrence equation* to know that…

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   return help(arr,k,0,arr.length);
}
boolean help(int[]arr, int k, int lo, int hi) {
   int mid = (hi+lo)/2;
   if(lo==hi)        return false;
   if(arr[mid]==k)   return true;
   if(arr[mid]< k)   return help(arr,k,mid+1,hi);
   else              return help(arr,k,lo,mid);
}
```

Solving Recurrence Relations

1. Determine the recurrence relation. What is the base case?
   – $T(n) = 10 + T(n/2)$ $T(1) = 15$
2. "Expand" the original relation to find an equivalent general expression *in terms of the number of expansions*.

3. Find a closed-form expression by setting *the number of expansions* to a value which reduces the problem to a base case

Solving Recurrence Relations

1. Determine the recurrence relation. What is the base case?
   - $T(n) = 10 + T(n/2)$ $T(1) = 15$
2. "Expand" the original relation to find an equivalent general expression *in terms of the number of expansions*.
   - $T(n) = 10 + 10 + T(n/4)$
     $= 10 + 10 + 10 + T(n/8)$
     $= \ldots$
     $= 10k + T(n/(2^k))$ (where k is the number of expansions)
3. Find a closed-form expression by setting *the number of expansions* to a value which reduces the problem to a base case
   - $n/(2^k) = 1$ means $n = 2^k$ means $k = \log_2 n$
   - So $T(n) = 10 \log_2 n + 15$ (get to base case and do it)
   - So $T(n)$ is $O(\log n)$

Ignoring constant factors

- So binary search is $O(\texttt{log}\ n)$ and linear is $O(n)$
  - But which is faster?
  - Depending on constant factors and size of n, in a particular case, linear search could be faster….

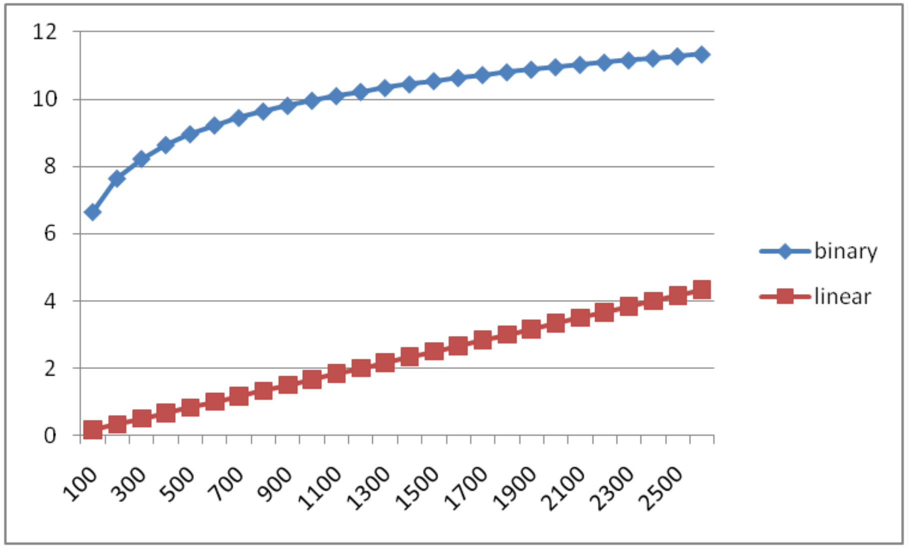
- Could depend on constant factors
  - How *many* assignments, additions, etc. for each *n*
  - And could depend on size of *n*

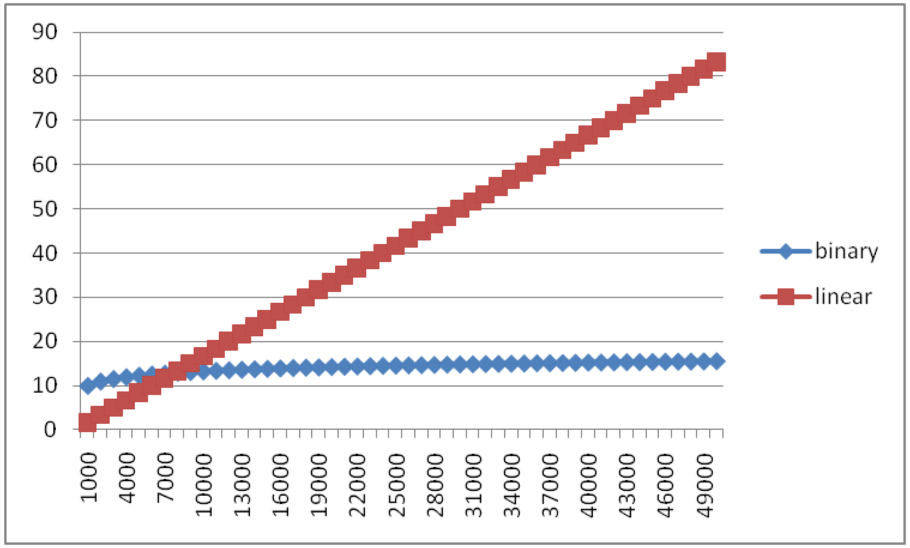
- **But** there exists some n_0 such that for all $n > n_0$ binary search wins

- Let's play with a couple plots to get some intuition…

Example

- Let's try to "help" linear search
  - Run it on a computer 100x as fast (say 2010 model vs. 1990)
  - Use a new compiler/language that is 3x as fast
  - Be a clever programmer to eliminate half the work
  - So doing each iteration is 600x as fast as in binary search
- Note: 600x still helpful for problems without logarithmic algorithms!





Another example: sum array

Two "obviously" linear algorithms: $T(n) = O(1) + T(n\text{-}1)$

Iterative:

```
int sum(int[] arr){
  int ans = 0;
  for(int i=0; i<arr.length; ++i)
     ans += arr[i];
  return ans;
}
```

Recursive:

- Recurrence is $c + c + \ldots + c$ for n times

```
int sum(int[] arr){
  return help(arr,0);
}
int help(int[]arr,int i) {
  if(i==arr.length)
    return 0;
  return arr[i] + help(arr,i+1);
}
```

What about a binary version of sum?

```
int sum(int[] arr){
   return help(arr,0,arr.length);
}
int help(int[] arr, int lo, int hi) {
   if(lo==hi)   return 0;
   if(lo==hi-1) return arr[lo];
   int mid = (hi+lo)/2;
   return help(arr,lo,mid) + help(arr,mid,hi);
}
```

Recurrence is $T(n) = O(1) + 2T(n/2)$

- $1 + 2 + 4 + 8 + \ldots$ for **`log`** *n* times
- $2^{(\log n)} - 1$ which is proportional to *n* (by definition of logarithm)

Easier explanation: it adds each number once while doing little else

"Obvious": You can't do better than *O(n)* – have to read whole array

Parallelism teaser

- But suppose we could do two recursive calls *at the same time*
  - *Like having a friend do half the work for you!*

```
int sum(int[]arr){
   return help(arr,0,arr.length);
}
int help(int[]arr, int lo, int hi) {
   if(lo==hi)   return 0;
   if(lo==hi-1) return arr[lo];
   int mid = (hi+lo)/2;
   return help(arr,lo,mid) + help(arr,mid,hi);
}
```

- If you have as many "friends of friends" as needed, the recurrence is now $T(n) = O(1) + 1T(n/2)$
  - $O($**`log`** $n)$: same recurrence as for **`find`**

Really common recurrences

Should know how to solve recurrences but also recognize some really common ones:

$T(n) = O(1) + T(n\text{-}1)$	linear
$T(n) = O(1) + 2T(n/2)$	linear
$T(n) = O(1) + T(n/2)$	logarithmic
$T(n) = O(1) + 2T(n\text{-}1)$	exponential
$T(n) = O(n) + T(n\text{-}1)$	quadratic
$T(n) = O(n) + T(n/2)$	linear
$T(n) = O(n) + 2T(n/2)$	O(n **`log`** n)

Note big-Oh can also use more than one variable

- Example: can sum all elements of an n-by-m matrix in $O(nm)$

Asymptotic notation

About to show formal definition, which amounts to saying:

1. Eliminate low-order terms
2. Eliminate coefficients

Examples:

- $4n + 5$
- $0.5n \log n + 2n + 7$
- $n^3 + 2^n + 3n$
- $n \log (10n^2)$

Examples

True or false?

1. $4+3n$ is $O(n)$ — True
2. $n+2\log n$ is $O(\log n)$ — False
3. $\log n+2$ is $O(1)$ — False
4. n^{50} is $O(1.1^n)$ — True

Notes:

- Do NOT ignore constants that are not multipliers:
  - n^3 is $O(n^2)$: FALSE
  - 3^n is $O(2^n)$: FALSE
- When in doubt, refer to the definition)

Big-Oh relates functions

We use O on a function $f(n)$ (for example n^2) to mean *the set of functions with asymptotic behavior* ***less than or equal to*** $f(n)$

So $(3n^2+17)$ **is in** $O(n^2)$

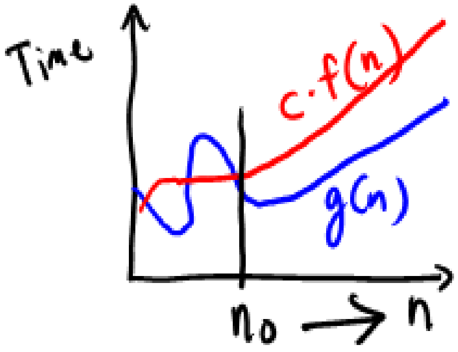
- $3n^2+17$ and n^2 have the same asymptotic behavior

Confusingly, we also say/write:

- $(3n^2+17)$ **is** $O(n^2)$
- $(3n^2+17) = O(n^2)$

But we would never say $O(n^2) = (3n^2+17)$

Formally Big-Oh



Definition: **g(*n*)** is in O(**f(*n*)**) iff there exist positive constants ***c*** and n_0 such that

g(*n*) ≤ ***c*** **f(*n*)** for all ***n*** ≥ n_0

To show **g(*n*)** is in O(**f(*n*)**), pick a ***c*** large enough to "cover the constant factors" and n_0 large enough to "cover the lower-order terms"

- Example: Let **g(*n*)** = $3n^2+17$ and **f(*n*)** = n^2

  c = 5 and n_0 = 10 is more than good enough

This is "less than or equal to"

– So $3n^2+17$ is also $O(n^5)$ and $O(2^n)$ etc.

Using the definition of Big-Oh (Example 1)

For **g(n)** = 4n & **f(n)** = n^2, prove **g(n)** is in O(**f(n)**)

- A valid proof is to find valid **c** & $\mathbf{n_0}$
- When n=4, **g(n)** =16 & **f(n)** =16; this is the crossing over point
- So we can choose $\mathbf{n_0}$ = 4, and **c** = 1
- Note: There are many possible choices:
  ex: $\mathbf{n_0}$ = 78, and **c** = 42 works fine

The Definition: g(n) is in O(f(n)) iff there exist *positive* constants c and n_0 such that

$$g(n) \le c\, f(n) \text{ for all } n \ge n_0.$$

Using the definition of Big-Oh (Example 2)

For **g(n)** = n^4 & **f(n)** = 2^n, prove **g(n)** is in O(**f(n)**)

- A valid proof is to find valid **c** & n_0
- One possible answer: n_0 = 20, and **c** = 1

The Definition: g(*n*) is in O(f(*n*)) iff there exist *positive* constants *c* and n_0 such that

$$g(n) \le c\, f(n) \text{ for all } n \ge n_0.$$

What's with the c?

- To capture this notion of similar asymptotic behavior, we allow a constant multiplier (called **c**)
- Consider:

  g(n) = 7n+5

  f(n) = n
- These have the same asymptotic behavior (linear), so **g(n)** is in O(**f(n)**) even though **g(n)** is always larger
- There is no positive n_0 such that **g(n)** ≤ **f(n)** for all n ≥ n_0
- The '**c**' in the definition allows for that:

  $g(n) \leq c\, f(n)$ for all $n \geq n_0$
- To prove **g(n)** is in O(**f(n)**), have **c** = 12, n_0 = 1

What you can drop

- Eliminate coefficients because we don't have units anyway
  - $3n^2$ versus $5n^2$ doesn't mean anything when we have not specified the cost of constant-time operations (can re-scale)

- Eliminate low-order terms because they have vanishingly small impact as n grows

- Do NOT ignore constants that are not multipliers
  - n^3 is not $O(n^2)$
  - 3^n is not $O(2^n)$

(This all follows from the formal definition)

Big Oh: Common Categories

From fastest to slowest

$O(1)$	constant (same as $O(k)$ for constant k)
$O(\texttt{log}\ n)$	logarithmic
$O(n)$	linear
$O(\mathrm{n}\ \texttt{log}\ n)$	"n $\texttt{log}$ n"
$O(n^2)$	quadratic
$O(n^3)$	cubic
$O(n^k)$	polynomial (where is k is an constant)
$O(k^n)$	exponential (where k is any constant > 1)

Usage note: "exponential" does not mean "grows really fast", it means "grows at rate proportional to k^n for some k>1"

- A savings account accrues interest exponentially (k=1.01?)

More Asymptotic Notation

- **Upper bound**: $O(\, \mathbf{f(n)}\,)$ is the set of all functions asymptotically less than or equal to **f(n)**
  - g(*n*) is in $O(\, \mathbf{f(n)}\,)$ if there exist constants ***c*** and n_0 such that

    $g(n) \leq c\, \mathbf{f(n)}$ for all $n \geq n_0$

- **Lower bound**: $\Omega(\, \mathbf{f(n)}\,)$ is the set of all functions asymptotically greater than or equal to **f(n)**
  - g(*n*) is in $\Omega(\, \mathbf{f(n)}\,)$ if there exist constants ***c*** and n_0 such that

    $g(n) \geq c\, \mathbf{f(n)}$ for all $n \geq n_0$

- **Tight bound**: $\theta(\, \mathbf{f(n)}\,)$ is the set of all functions asymptotically equal to **f(n)**
  - Intersection of $O(\, \mathbf{f(n)}\,)$ and $\Omega(\, \mathbf{f(n)}\,)$ (use *different* ***c*** values)

Regarding use of terms

A common error is to say $O(\text{ f}(n))$ when you mean $\theta(\text{ f}(n))$

- People often say O() to mean a tight bound
- Say we have f(n)=n; we could say f(n) is in O(n), which is true, but only conveys the upper-bound
- Since f(n)=n is *also* $O(n^5)$, it's tempting to say "this algorithm is *exactly* $O(n)$"
- Somewhat incomplete; instead say it is $\theta(n)$
- That means that it is not, for example $O(\mathbf{log}\ n)$

Less common notation:

- "little-oh": like "big-Oh" but strictly less than
  - Example: sum is $o(n^2)$ but not $o(n)$
- "little-omega": like "big-Omega" but strictly greater than
  - Example: sum is $\omega(\mathbf{log}\ n)$ but not $\omega(n)$

What we are analyzing

- The most common thing to do is give an O or θ bound to the worst-case running time of an algorithm

- Example: True statements about binary-search algorithm
  - Common: $\theta(\texttt{log}\ n)$ running-time in the worst-case
  - Less common: $\theta(1)$ in the best-case (item is in the middle)
  - Less common: Algorithm is $\Omega(\texttt{log log}\ n)$ in the worst-case (it is not really, really, really fast asymptotically)
  - Less common (but very good to know): the find-in-sorted-array ***problem*** is $\Omega(\texttt{log}\ n)$ in the worst-case
    - *No* algorithm can do better (without parallelism)
    - A ***problem*** cannot be $O(f(n))$ since you can always find a slower algorithm, but can mean ***there exists*** an algorithm

Other things to analyze

- Space instead of time
  - Remember we can often use space to gain time
- Average case
  - Sometimes only if you assume something about the distribution of inputs
    - See CSE312 and STAT391
  - Sometimes uses randomization in the algorithm
    - Will see an example with sorting; also see CSE312
  - Sometimes an *amortized guarantee*
    - Will discuss in a later lecture

Summary

Analysis can be about:

- The problem or the algorithm (usually algorithm)
- Time or space (usually time)
  - Or power or dollars or …
- Best-, worst-, or average-case (usually worst)
- Upper-, lower-, or tight-bound (usually upper or tight)

Big-Oh Caveats

- Asymptotic complexity (Big-Oh) focuses on behavior for <u>**large *n***</u> and is independent of any computer / coding trick
  - But you can "abuse" it to be misled about trade-offs
  - Example: $n^{1/10}$ vs. **`log`** *n*
    - Asymptotically $n^{1/10}$ grows more quickly
    - But the "cross-over" point is around $5 * 10^{17}$
    - So if you have input size less than 2^{58}, prefer $n^{1/10}$
- Comparing O() for <u>***small n***</u> values can be misleading
  - Quicksort: O(nlogn) (expected)
  - Insertion Sort: $O(n^2)$ (expected)
  - Yet in reality Insertion Sort is faster for small n's
  - We'll learn about these sorts later

Addendum: Timing vs. Big-Oh?

- At the core of CS is a backbone of theory & mathematics
  - Examine the algorithm itself, mathematically, not the implementation
  - Reason about performance as a function of n
  - Be able to mathematically prove things about performance
- Yet, timing has its place
  - In the real world, we do want to know whether implementation A runs faster than implementation B on data set C
  - Ex: Benchmarking graphics cards
  - We will do some timing in project 3 (and in 2, a bit)
- Evaluating an algorithm? Use asymptotic analysis
- Evaluating an implementation of hardware/software? Timing can be useful