CSE 331
Software Design & Implementation

James Wilcox & Kevin Zatloukal

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Lecture 2 – Reasoning about Loops

Floyd Logic

- A Hoare triple is two assertions and one piece of code:

  { *P* } *S* { *Q* }

  – *P* the precondition
  – *S* the code
  – *Q* the postcondition

  specification (*P*, *Q*)
  method body (*S*)

- A Hoare triple **{ *P* } *S* { *Q* }** is called valid if:
  – in any state where P holds,
    executing S produces a state where Q holds
  – i.e., if *P* is true before *S*, then *Q* must be true after it
  – otherwise, the triple is called invalid
  – code is correct iff triple is valid

Reasoning Forward & Backward

- Forward:
  - start with the **given** precondition
  - fill in the **strongest** postcondition

$\{P\}\ S\ \{?\}$ →

- Backward
  - start with the **required** postcondition
  - fill in the **weakest** precondition

$\{?\}\ S\ \{Q\}$ ←

- Finds the "best" assertion that makes the triple valid

Reasoning: Assignments

```
x = expr
```

- Forward
  - add the fact “x = expr” to what is known
  - BUT you must *fix* any existing references to “x”
- Backward
  - just replace any “x” in the postcondition with expr (substitution)

Correctness by Forward / Backward

Reasoning in either direction gives valid assertions

Just need to check adjacent assertions:

- top assertion must imply bottom one

```
{{ P }}
  S1
  S2
{{ P1 }}
{{ Q }}
```

```
{{ P }}
{{ Q1 }}
  S1
  S2
{{ Q }}
```

```
{{ P }}
  S1
{{ P1 }}
{{ Q1 }}
  S2
{{ Q }}
```

Subtleties in Forward Reasoning...

- Forward reasoning can **fail** if applied blindly...

{{ }}
```
 w = x + y;
```
{{ w = x + y }}
```
 x = 4;
```
{{ w = x + y and x = 4 }}
```
 y = 3;
```
{{ w = x + y and x = 4 and y = 3 }}

This implies that w = 7, but that is not true!

- w equals whatever x + y was **before** they were changed

Fix 1

- Use **subscripts** to refer to old values of the variables
- Un-subscripted variables should always mean **current** value

{{ }}

```
 w = x + y;
```

{{ w = x + y }}

```
 x = 4;
```

{{ $w = x_1 + y$ and $x = 4$ }}

```
 y = 3;
```

{{ $w = x_1 + y_1$ and $x = 4$ and $y = 3$ }}

Fix 2 (better)

- Express prior values in terms of the current value

{{ }}

```
w = x + y;
```

{{ w = x + y }}

```
x = x + 4;
```

{{ w = x_1 + y and x = x_1 + 4 }} Now, x_1 = x - 4

⇒ {{ w = x - 4 + y }} So w = x_1 + y ⇔ w = x - 4 + y

Note for updating variables, e.g., `x = x + 4`:

- Backward reasoning just substitutes new value (no change)
- Forward reasoning requires you to invert the “+” operation

If Statements

If Statements

Forward reasoning

{{ P }}
```
if (cond)
  S1
else
  S2
```
{{ ? }}

If Statements

Forward reasoning

```
{{ P }}
if (cond)
   {{ P and cond }}
   S1
else
   {{ P and not cond }}
   S2
{{ ? }}
```

If Statements

Forward reasoning

```
{{ P }}
if (cond)
    {{ P and cond }}
    S1
    {{ P1 }}
else
    {{ P and not cond }}
    S2
    {{ P2 }}
{{ ? }}
```

If Statements

Forward reasoning

{{ P }}
`if (cond)`
    {{ P and `cond` }}
    `S1`
    {{ P1 }}
`else`
    {{ P and not `cond` }}
    `S2`
    {{ P2 }}
{{ P1 or P2 }}

If Statements

Backward reasoning

{{ ? }}
```
if (cond)
  S1
else
  S2
```
{{ Q }}

If Statements

Backward reasoning

{{ ? }}

```
if (cond)
  S1
```

  {{ Q }}

```
else
  S2
```

  {{ Q }}

{{ Q }}

If Statements

Backward reasoning

```
{{ ? }}
if (cond)
  {{ Q1 }}
  S1
  {{ Q }}
else
  {{ Q2 }}
  S2
  {{ Q }}
{{ Q }}
```

If Statements

Backward reasoning

{{ `cond` and Q1 or
not `cond` and Q2 }}

```
if (cond)
```

{{ Q1 }}

```
S1
```

{{ Q }}

```
else
```

{{ Q2 }}

```
S2
```

{{ Q }}

{{ Q }}

If-Statement Example

Forward reasoning

```
{{ }}
if (x >= 0)
  y = x;
else
  y = -x;
{{ ? }}
```

If-Statement Example

Forward reasoning

```
{{ }}
if (x >= 0)
   {{ x >= 0 }}
   y = x;
else
   {{ x < 0 }}
   y = -x;
{{ ? }}
```

If-Statement Example

Forward reasoning

```
{{ }}
if (x >= 0)
  {{ x >= 0 }}
  y = x;
  {{ x >= 0 and y = x }}
else
  {{ x < 0 }}
  y = -x;
  {{ x < 0 and y = -x }}
{{ ? }}
```

If-Statement Example

Forward reasoning

{{ }}
```
if (x >= 0)
```
{{ x >= 0 }}
```
y = x;
```
{{ x >= 0 and y = x }}
```
else
```
{{ x < 0 }}
```
y = -x;
```
{{ x < 0 and y = -x }}

{{ (x >= 0 and y = x) or
(x < 0 and y = -x) }}

If-Statement Example

Forward reasoning

{{ }}

```
if (x >= 0)
```

{{ x >= 0 }}

```
y = x;
```

{{ x >= 0 and y = x }}

```
else
```

{{ x < 0 }}

```
y = -x;
```

{{ x < 0 and y = -x }}

{{ y = |x| }}

If-Statement Example

Forward reasoning

```
{{ }}
if (x >= 0)
   {{ x >= 0 }}
   y = x;
   {{ x >= 0 and y = x }}
else
   {{ x < 0 }}
   y = -x;
   {{ x < 0 and y = -x }}
{{ y = |x| }}
```

Warning: many write {{ y >= 0 }} here

That is true but it is *strictly* weaker.
(It includes cases where y != x)

If-Statement Example

Forward reasoning

{{ }}
```
if (x >= 0)
```
{{ x >= 0 }}
```
y = x;
```
{{ x >= 0 and y = x }}
```
else
```
{{ x < 0 }}
```
y = -x;
```
{{ x < 0 and y = -x }}

{{ y = |x| }}

Backward reasoning

{{ ? }}
```
if (x >= 0)
  y = x;
else
  y = -x;
```
{{ y = |x| }}

If-Statement Example

Forward reasoning

{{ }}
```
if (x >= 0)
```
{{ x >= 0 }}
```
y = x;
```
{{ x >= 0 and y = x }}
```
else
```
{{ x < 0 }}
```
y = -x;
```
{{ x < 0 and y = -x }}

{{ y = |x| }}

Backward reasoning

{{ ? }}
```
if (x >= 0)
  y = x;
```
{{ y = |x| }}
```
else
  y = -x;
```
{{ y = |x| }}

{{ y = |x| }}

If-Statement Example

Forward reasoning

{{ }}
```
if (x >= 0)
```
{{ x >= 0 }}
```
y = x;
```
{{ x >= 0 and y = x }}
```
else
```
{{ x < 0 }}
```
y = -x;
```
{{ x < 0 and y = -x }}

{{ y = |x| }}

Backward reasoning

{{ ? }}
```
if (x >= 0)
```
{{ x = |x| }}
```
y = x;
```
{{ y = |x| }}
```
else
```
{{ -x = |x| }}
```
y = -x;
```
{{ y = |x| }}

{{ y = |x| }}

If-Statement Example

Forward reasoning

{{ }}
```
if (x >= 0)
```
{{ x >= 0 }}
```
y = x;
```
{{ x >= 0 and y = x }}
```
else
```
{{ x < 0 }}
```
y = -x;
```
{{ x < 0 and y = -x }}

{{ y = |x| }}

Backward reasoning

{{ ? }}
```
if (x >= 0)
```
{{ x >= 0 }}
```
y = x;
```
{{ y = |x| }}
```
else
```
{{ x <= 0 }}
```
y = -x;
```
{{ y = |x| }}

{{ y = |x| }}

If-Statement Example

Forward reasoning

```
{{ }}
if (x >= 0)
    {{ x >= 0 }}
    y = x;
    {{ x >= 0 and y = x }}
else
    {{ x < 0 }}
    y = -x;
    {{ x < 0 and y = -x }}
{{ y = |x| }}
```

Backward reasoning

```
{{ (x >= 0 and x >= 0) or
    (x < 0 and x <= 0) }}
if (x >= 0)
    {{ x >= 0 }}
    y = x;
    {{ y = |x| }}
else
    {{ x <= 0 }}
    y = -x;
    {{ y = |x| }}
{{ y = |x| }}
```

If-Statement Example

Forward reasoning

```
{{ }}
if (x >= 0)
    {{ x >= 0 }}
    y = x;
    {{ x >= 0 and y = x }}
else
    {{ x < 0 }}
    y = -x;
    {{ x < 0 and y = -x }}
{{ y = |x| }}
```

Backward reasoning

```
{{ x >= 0 or x < 0 }}
if (x >= 0)
    {{ x >= 0 }}
    y = x;
    {{ y = |x| }}
else
    {{ x <= 0 }}
    y = -x;
    {{ y = |x| }}
{{ y = |x| }}
```

If-Statement Example

Forward reasoning

```
{{ }}
if (x >= 0)
    {{ x >= 0 }}
    y = x;
    {{ x >= 0 and y = x }}
else
    {{ x < 0 }}
    y = -x;
    {{ x < 0 and y = -x }}
{{ y = |x| }}
```

Backward reasoning

```
{{ }}
if (x >= 0)
    {{ x >= 0 }}
    y = x;
    {{ y = |x| }}
else
    {{ x <= 0 }}
    y = -x;
    {{ y = |x| }}
{{ y = |x| }}
```

Reasoning So Far

- Mechanical reasoning for assignment and if statements

- All code (essentially) can be written just using:
  - assignments
  - if statements
  - while loops

- Only part we are missing is **loops**

- (We will also cover function calls later.)

Loops

Reasoning About Loops

- Loop reasoning is not as easy as with "=" and "if"
  - recall Rice's Theorem (from 311): checking any non-trivial semantic property about programs is **undecidable**

- We need help (more information) before the reasoning again becomes a mechanical process

- That help comes in the form of a "loop invariant"

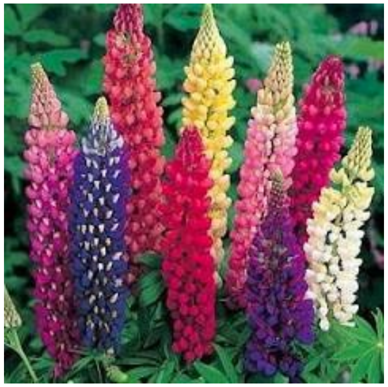
Loop Invariant

A **loop invariant** is an assertion that holds at the top of the loop:

{{ Inv: I }}
```
while (cond)
  S
```

- It holds when we **first get to** the loop.
- It holds each time we execute `S` and **come back to** the top.

Notation: I'll use "**Inv**:" to indicate a loop invariant.



Lupin variants

Checking Correctness of a Loop

Consider a while-loop (other loop forms not too different) with a loop invariant **`I`**.

Let's try forward reasoning...

{{ P }}
```
S1
```

{{ Inv: **`I`** }}
```
while (cond)
  S2

S3
```
{{ Q }}

Checking Correctness of a Loop

Consider a while-loop (other loop forms not too different) with a loop invariant **`I`**.

Let's try forward reasoning...

```
{{ P }}
 S1
{{ P1 }}

{{ Inv: I }}
 while (cond)
   S2

 S3
{{ Q }}
```

Need to check that P1 implies **`I`** (i.e., that **`I`** is true the first time)

Checking Correctness of a Loop

Consider a while-loop (other loop forms not too different) with a loop invariant **`I`**.

Let's try forward reasoning...

{{ P }}
```
 S1
```
{{ Inv: **`I`** }}
```
 while (cond)
```
  {{ **`I`** and cond }}
```
   S2
```
  {{ P2 }}

Need to check that P2 implies **`I`** again (i.e., that **`I`** is true each time around)

```
 S3
```
{{ Q }}

Checking Correctness of a Loop

Consider a while-loop (other loop forms not too different) with a loop invariant **`I`**.

Let's try forward reasoning...

{{ P }}

```
S1
```

{{ Inv: **`I`** }}

```
while (cond)
  S2
```

{{ **`I`** and not cond }}

```
S3
```

{{ P3 }}

{{ Q }}

Need to check that P3 implies Q

(i.e., Q holds after the loop)

Checking Correctness of a Loop

Consider a while-loop (other loop forms not too different) with a loop invariant **`I`**.

```
{{ P }}
 S1

{{ Inv: I }}
 while (cond)
   S2

 S3
{{ Q }}
```

Informally, we need:

- **`I`** holds initially
- **`I`** holds each time around
- **`Q`** holds after we exit

Formally, we need validity of:

- {{ P }} `S1` {{ **`I`** }}
- {{ **`I`** and cond }} `S2` {{ **`I`** }}
- {{ **`I`** and not cond }} `S3` {{ Q }}

(can check these with backward reasoning instead)

More on Loop Invariants

- Loop invariants are crucial information
  - needs to be provided before reasoning is mechanical
- Pro Tip: always document your invariants for *non-trivial* loops
  - don't make code reviewers guess the invariant
- Pro Tip: with a good loop invariant, the code is easy to write
  - all the creativity can be saved for finding the invariant
  - more on this in later lectures…

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}

```
s = 0;
i = 0;
while (i != n) {
  s = s + b[i];
  i = i + 1;
}
```

{{ s = b[0] + ... + b[n-1] }}

Equivalent to this "for" loop:

```
s = 0;
for (int i = 0; i != n; i++)
  s = s + b[i];
```

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}

```
s = 0;
i = 0;
```

{{ Inv: s = b[0] + ... + b[i-1] }}

```
while (i != n) {
  s = s + b[i];
  i = i + 1;
}
```

{{ s = b[0] + ... + b[n-1] }}

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

```
{{ }}
s = 0;
i = 0;
{{ s = 0 and i = 0 }}
{{ Inv: s = b[0] + ... + b[i-1] }}
while (i != n) {
  s = s + b[i];
  i = i + 1;
}
{{ s = b[0] + ... + b[n-1] }}
```

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}

```
s = 0;
i = 0;
```

{{ s = 0 and i = 0 }}

{{ Inv: s = b[0] + ... + b[i-1] }}

```
while (i != n) {
  s = s + b[i];
  i = i + 1;
}
```

{{ s = b[0] + ... + b[n-1] }}

- (s = 0 and i = 0) implies
  s = b[0] + … + b[i-1] ?

Less formal

s = sum of first i numbers in b

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}
```
s = 0;
i = 0;
```
{{ s = 0 and i = 0 }}
{{ Inv: s = b[0] + ... + b[i-1] }}
```
while (i != n) {
  s = s + b[i];
  i = i + 1;
}
```
{{ s = b[0] + ... + b[n-1] }}

- (s = 0 and i = 0) implies
  s = b[0] + … + b[i-1] ?

Less formal

s = sum of first i numbers in b

When i = 0, s needs to be the sum of the first 0 numbers, so we need s = 0.

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}
```
s = 0;
i = 0;
```
{{ s = 0 and i = 0 }}
{{ Inv: s = b[0] + ... + b[i-1] }}
```
while (i != n) {
  s = s + b[i];
  i = i + 1;
}
```
{{ s = b[0] + ... + b[n-1] }}

- (s = 0 and i = 0) implies
  s = b[0] + … + b[i-1] ?

More formal

s = sum of all b[k] with $0 \le k \le i-1$

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}

```
s = 0;
i = 0;
```

{{ s = 0 and i = 0 }}

{{ Inv: s = b[0] + ... + b[i-1] }}

```
while (i != n) {
  s = s + b[i];
  i = i + 1;
}
```

{{ s = b[0] + ... + b[n-1] }}

- (s = 0 and i = 0) implies
  s = b[0] + … + b[i-1] ?

More formal

s = sum of all b[k] with $0 \leq k \leq i-1$

i = 3 ($0 \leq k \leq 2$): s = b[0] + b[1] + b[2]
i = 2 ($0 \leq k \leq 1$): s = b[0] + b[1]
i = 1 ($0 \leq k \leq 0$): s = b[0]
i = 0 ($0 \leq k \leq -1$) s = 0

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}
```
s = 0;
i = 0;
```
{{ s = 0 and i = 0 }}
{{ Inv: s = b[0] + ... + b[i-1] }}
```
while (i != n) {
  s = s + b[i];
  i = i + 1;
}
```
{{ s = b[0] + ... + b[n-1] }}

- (s = 0 and i = 0) implies
  s = b[0] + … + b[i-1] ?

More formal

s = sum of all b[k] with 0 ≤ k ≤ i-1

when i = 0, we want to sum over all indexes k satisfying 0 ≤ k ≤ -1

There are no such indexes, so we need s = 0

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}

```
s = 0;
i = 0;
```

{{ s = 0 and i = 0 }}

{{ Inv: s = b[0] + ... + b[i-1] }}

```
while (i != n) {
  s = s + b[i];
  i = i + 1;
}
```

{{ s = b[0] + ... + b[n-1] }}

- (s = 0 and i = 0) implies
  s = b[0] + … + b[i-1] ?

  Yes. (An empty sum is zero.)

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}

```
s = 0;
i = 0;
```

{{ s = 0 and i = 0 }}

{{ Inv: s = b[0] + ... + b[i-1] }}

```
while (i != n) {
  s = s + b[i];
  i = i + 1;
}
```

{{ s = b[0] + ... + b[n-1] }}

- (s = 0 and i = 0) implies **`I`**

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}
```
s = 0;
i = 0;
```
{{ Inv: s = b[0] + ... + b[i-1] }}
```
while (i != n) {
```
{{ s = b[0] + ... + b[i-1] and i != n }}
```
  s = s + b[i];
  i = i + 1;
```
{{ s = b[0] + ... + b[i-1] }}
```
}
```
{{ s = b[0] + ... + b[n-1] }}

- (s = 0 and i = 0) implies **I**
- {{ **I** and i != n }} S {{ **I** }} ?

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}
```
s = 0;
i = 0;
```
{{ Inv: s = b[0] + ... + b[i-1] }}
```
while (i != n) {
```
{{ s = b[0] + ... + b[i-1] and i != n }}
```
  s = s + b[i];
  i = i + 1;
```
{{ s = b[0] + ... + b[i-1] }}
```
}
```
{{ s = b[0] + ... + b[n-1] }}

- (s = 0 and i = 0) implies **`I`**
- {{ **`I`** and i != n }} `S` {{ **`I`** }} ?

{{ s + b[i] = b[0] + … + b[i] }}
{{ s = b[0] + … + b[i] }}

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}
```
s = 0;
i = 0;
```
{{ Inv: s = b[0] + ... + b[i-1] }}
```
while (i != n) {
  s = s + b[i];
  i = i + 1;
}
```
{{ s = b[0] + ... + b[i-1] and not (i != n) }}
{{ s = b[0] + ... + b[n-1] }}

- (s = 0 and i = 0) implies **I**
- {{ **I** and i != n }} S {{ **I** }}
- {{ **I** and not (i != n) }} implies
  s = b[0] + … + b[n-1] ?

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}

```
s = 0;
i = 0;
```

{{ Inv: s = b[0] + ... + b[i-1] }}

```
while (i != n) {
  s = s + b[i];
  i = i + 1;
}
```

{{ s = b[0] + ... + b[n-1] }}

- (s = 0 and i = 0) implies **`I`**
- {{ **`I`** and i != n }} `S` {{ **`I`** }}
- {{ **`I`** and i = n }} implies `Q`

These three checks verify that the outermost triple is valid (i.e., that the code is correct).