Math Review

CSE 326
Data Structures & Algorithms
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Winter 2010

Today's Outline

- **Announcements**
  - Project #1 due Wed, Jan 13 at 11pm, come to section tomorrow with questions.
  - Two emails sent to cse326 mailing list – did you get them?
  - Have you installed Eclipse (or whatever environment you will be using this quarter) and Java yet?
  - Please fill out survey and bring Info sheet to class on Friday.
- **Queues and Stacks**
- **Math Review**
  - **Proof by Induction**
  - **Powers of 2**
  - **Binary numbers**
  - **Exponents and Logs**

Project 1 – Sound Blaster!

Play your favorite song in reverse!

Aim:

1. Implement stack ADT two different ways
2. Use to reverse a sound file

Due: Wed, Jan 13, at 11pm via Catalyst

Mathematical Induction

Suppose we wish to prove that:

For all $n \geq n_0$, some predicate $P(n)$ is true.

We can do this by proving two things:

1. $P(n_0)$ --- this is called the "basis."
2. If $P(k)$ then $P(k+1)$ -- this is called the "induction step."

Example: Basis Step

Prove for all $n \geq 1$, sum of first n powers of 2 = $2^n - 1$

$$2^0 + 2^1 + 2^2 + \ldots + 2^{n-1} = 2^n - 1.$$

in other words: $1 + 2 + 4 + \ldots + 2^{n-1} = 2^n - 1.$

Proof by induction:

Basis with $n_0 = 1$:

(left hand side)

(right hand side)

So true for $n_0 = 1$

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in other words: $1 + 2 + 4 + \ldots + 2^{n-1} = 2^n - 1.$

Proof by induction:

Basis with $n_0 = 1$:

(left hand side) $2^{1-1} = 2^0 = 1$

(right hand side) $2^1 - 1 = 2 - 1 = 1$

So true for $n_0 = 1$

Prove for all $n \geq 1$, sum of first n powers of 2 = 2n – 1

Example: Inductive Step

- *Induction hypothesis*: (Assume this is true)

  $1 + 2 + 4 + \ldots + 2^{k-1} = 2^k - 1$
- *Induction step*:

  $1 + 2 + 4 + \ldots\ 2^{k-1} \quad = 2^k - 1$

Prove for all $n \geq 1$, sum of first n powers of 2 = 2n – 1

Example: Inductive Step

- *Induction hypothesis*: (Assume this is true)

  $1 + 2 + 4 + \ldots + 2^{k-1} = 2^k - 1$
- *Induction step*: Now add 2^k to both sides:

  $1 + 2 + 4 + \ldots\ 2^{k-1} + \mathbf{2^k} = 2^k - 1 + \mathbf{2^k}$

  $= 2(2^k) - 1$

  $= 2^{k+1} - 1$

  Therefore if the equation is valid for n = k, it must also be valid for n = k+1.
- *Summary*: It is valid for n=1 (basis) and by the induction step it is therefore valid for n=2, n=3, …

  It is valid for all integers greater than 0.

Powers of 2

- Many of the numbers we use in Computer Science are powers of 2
- Binary numbers (base 2) are easily represented in digital computers
  - each "bit" is a 0 or a 1
  - an n-bit wide field can represent how many different things?

0000000000101011

N bits can represent how many things?

# Bits	Patterns	# of patterns
1		
2		

Unsigned binary numbers

- For **unsigned** numbers in a fixed width field
  - the minimum value is 0
  - the maximum value is __________, where n is the number of bits in the field
  - The value is $\sum_{i=0}^{n-1} a_i 2^i$
- Each bit position represents a power of 2 with $a_i = 0$ or $a_i = 1$

Unsigned binary numbers

- For **unsigned** numbers in a fixed width field
  - the minimum value is 0
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Signed Numbers?

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Logarithms and Exponents

- Definition: $\log_2 x = y$ if and only if $x = 2^y$
  - $8 = 2^3$, so $\log_2 8 = 3$
  - $65536 = 2^{16}$, so $\log_2 65536 = 16$
- Notice that $\log_2 n$ tells you how many bits are needed to distinguish among n different values.
  - 8 bits can hold any of 256 numbers, for example: 0 to 2^8-1, which is 0 to 255
  - $\log_2 256 = 8$

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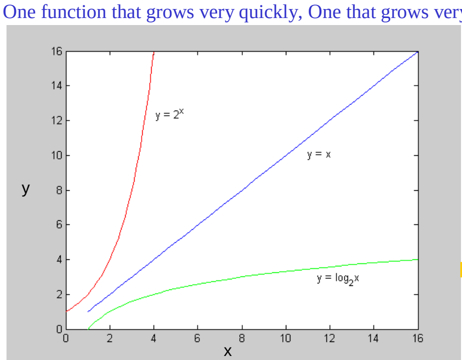
One function that grows very quickly, One that grows very slowly




```
x = 0:.1:4
y = 2.^x
plot(x,y,'r')
hold on
plot(y,x,'g')
plot(y,y,'b')
```

x, 2^x and $\log_2 x$

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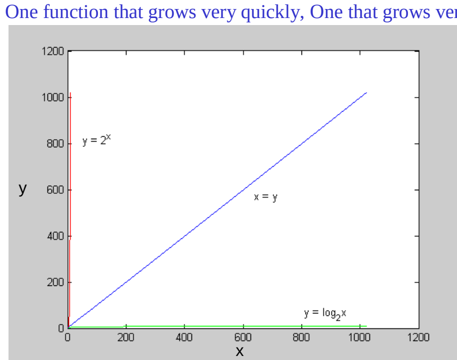
One function that grows very quickly, One that grows very slowly




```
x = 0:10
y = 2.^x
plot(x,y,'r')
hold on
plot(y,x,'g')
plot(y,y,'b')
```

2^x and $\log_2 x$

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What is the minimum height of a binary tree with N nodes?

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Floor and Ceiling

$\lfloor X \rfloor$ Floor function: the largest integer $\leq$ X

$\lfloor 2.7 \rfloor = 2 \quad \lfloor -2.7 \rfloor = -3 \quad \lfloor 2 \rfloor = 2$

$\lceil X \rceil$ Ceiling function: the smallest integer $\geq$ X

$\lceil 2.3 \rceil = 3 \quad \lceil -2.3 \rceil = -2 \quad \lceil 2 \rceil = 2$

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Facts about Floor and Ceiling

1. $X - 1 < \lfloor X \rfloor \leq X$
2. $X \leq \lceil X \rceil < X + 1$
3. $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$ if n is an integer

Properties of logs

- We will assume logs to base 2 unless specified otherwise.
- $8 = 2^3$, so $\log_2 8 = 3$, so $2^{(\log_2 8)} =$ ________

Show:

log (A • B) = log A + log B

$A = 2^{\log_2 A}$ and $B = 2^{\log_2 B}$

$A \bullet B = 2^{\log_2 A} \bullet 2^{\log_2 B} = 2^{\log_2 A + \log_2 B}$

So: $\log_2 AB = \log_2 A + \log_2 B$

- Note: $\log AB \neq \log A \bullet \log B$!!

Other log properties

- $\log A/B = \log A - \log B$
- $\log (A^B) = B \log A$
- $\log \log X < \log X < X$ for all $X > 0$
  - $\log \log X = Y$ means: $2^{2^Y} = X$
  - log X grows more slowly than X
    - called a "sub-linear" function

Note: $\log \log X \neq \log^2 X$

$\log^2 X = (\log X)(\log X)$ aka "log-squared"

A log is a log is a log

- "Any base B log is equivalent to base 2 log within a constant factor."

$B = 2^{\log_2 B}$

$x = 2^{\log_2 x}$

$$\log_B X = \log_B X$$

$$B^{\log_B X} = X$$

substitution

$B^{\log_B X} = X$ by def. of logs

$$(2^{\log_2 B})^{\log_B X} = 2^{\log_2 X}$$

$$2^{\log_2 B \log_B X} = 2^{\log_2 X}$$

$$\log_2 B \log_B X = \log_2 X$$

$$\boxed{\log_B X = \frac{\log_2 X}{\log_2 B}}$$

Algorithm Analysis Examples

- Consider the following program segment:

```
x:= 0;
for i = 1 to N do
  for j = 1 to i do
    x := x + 1;
```

- What is the value of x at the end?

Analyzing the Loop

- Total number of times x is incremented is executed =

$$1 + 2 + 3 + \ldots + N = \sum_{i=1}^{N} i = \frac{N(N+1)}{2}$$

- An *Arithmetic Sequence*
- Congratulations - You've just analyzed your first program!
  - Running time of the program is proportional to N(N+1)/2 for all N
  - Big-O ??

Asymptotic Analysis

Comparing Two Algorithms

What we want

- Rough Estimate
- Ignores Details

Big-O Analysis

- Ignores "details"

Analysis of Algorithms

- Efficiency measure
  - how long the program runs time complexity
  - how much memory it uses space complexity
    - For today, we'll focus on time complexity only
- *Why analyze at all?*

Asymptotic Analysis

- Complexity as a function of input size n

  $T(n) = 4n + 5$

  $T(n) = 0.5\, n \log n - 2n + 7$

  $T(n) = 2^n + n^3 + 3n$

- *What happens as n grows?*

Why Asymptotic Analysis?

- Most algorithms are fast for small *n*
  - Time difference too small to be noticeable
  - External things dominate (OS, disk I/O, …)
- BUT *n* is often large in practice
  - Databases, internet, graphics, …
- Time difference really shows up as *n* grows!

Big-O: Common Names

– constant:	O(1)	
– logarithmic:	O(log n)	
– linear:	O(n)	
– quadratic:	$O(n^2)$	
– cubic:	$O(n^3)$	
– polynomial:	$O(n^k)$	(k is a constant)
– exponential:	$O(c^n)$	(c is a constant > 1)