CSE 326 Data Structures

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Priority Queues : AVL Trees

Logistics

- Project 2a due tonight…
- Homework 3 due Moday

- Midterm will be Feb 2 in class

- Reading: Finishing Chapter 4

Balanced BST

<u>Observation</u>

- BST: the shallower the better!
- For a BST with *n* nodes
  - Average height is O(log *n*)
  - Worst case height is O(*n*)
- Simple cases such as insert(1, 2, 3, ..., n) lead to the worst case scenario

<u>Solution</u>: Require a **Balance Condition** that

1. ensures depth is O(log *n*) – strong enough!
2. is easy to maintain – not too strong!

Potential Balance Conditions

1. Left and right subtrees of the root have equal number of nodes

2. Left and right subtrees of the root have equal *height*

Potential Balance Conditions

3. Left and right subtrees of *every node* have equal number of nodes

4. Left and right subtrees of *every node* have equal *height*

The AVL Balance Condition

Left and right subtrees of *every node*
have equal *heights* **differing by at most 1**

Define: **balance**(x) = height(x.left) – height(x.right)

AVL property: **$-1 \leq$ balance(x) ≤ 1, for every node x**

- Ensures small depth
  - Will prove this by showing that an AVL tree of height h must have a lot of (i.e. $O(2^h)$) nodes
- Easy to maintain
  - Using single and double rotations

The AVL Tree Data Structure

Structural properties

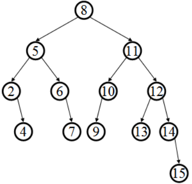
1. Binary tree property
2. Balance property: balance of every node is between -1 and 1

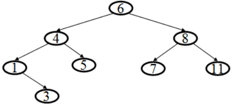
Result:

Worst case depth is O(log *n*)

Ordering property

– Same as for BST

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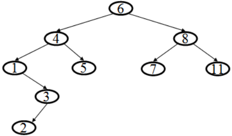
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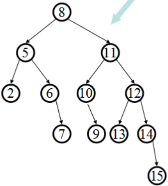
Proving Shallowness Bound

Let **S**(h) be the min # of nodes in an AVL tree of height h

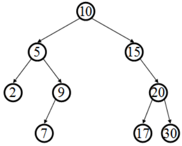
Claim: $\mathbf{S}(h) = \mathbf{S}(h\text{-}1) + \mathbf{S}(h\text{-}2) + 1$

Solution of recurrence: $\mathbf{S}(h) = O(2^h)$ (like Fibonacci numbers)

AVL tree of height h=4 with the min # of nodes



Testing the Balance Property



We need to be able to:

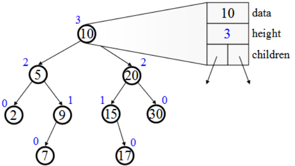
1.

2.

3.

`NULL`s have height **`-1`**

An AVL Tree



AVL trees: find, insert

- **AVL find**:
  - same as BST find.
- **AVL insert**:
  - same as BST insert, *except* may need to "fix" the AVL tree after inserting new value.

AVL tree insert

Let *x* be the node where an imbalance occurs.

Four cases to consider. The insertion is in the

1. left subtree of the left child of *x*.
2. right subtree of the left child of *x*.
3. left subtree of the right child of *x*.
4. right subtree of the right child of *x*.

Idea: Cases 1 & 4 are solved by a single rotation.
Cases 2 & 3 are solved by a double rotation.

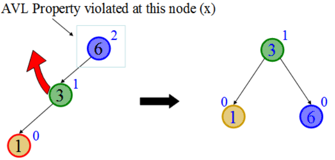
Bad Case #1

Insert(6)

Insert(3)

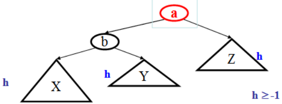
Insert(1)

Fix: Apply Single Rotation

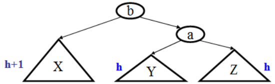


Single Rotation:
1. Rotate between x and child

Single rotation in general

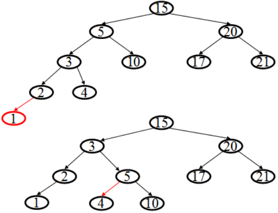


X < b < Y < a < Z



Height of tree before? Height of tree after? Effect on Ancestors?

Single rotation example



Bad Case #2

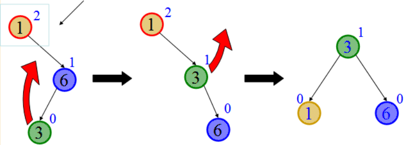
Insert(1)

Insert(6)

Insert(3)

Fix: Apply Double Rotation

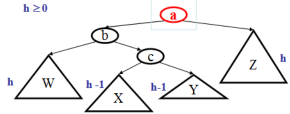
AVL Property violated at this node (x)



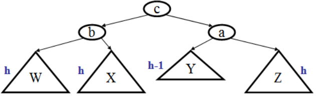
Double Rotation

1. Rotate between x's child and grandchild
2. Rotate between x and x's new child

Double rotation in general

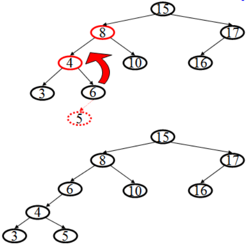


W < b <X < c < Y < a < Z

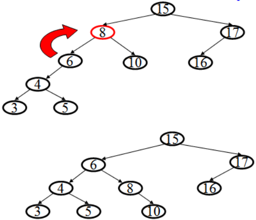


Height of tree before? Height of tree after? Effect on Ancestors?

Double rotation, step 1



Double rotation, step 2



Imbalance at node X

Single Rotation

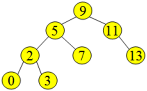
1. Rotate between x and child

Double Rotation

1. Rotate between x's child and grandchild
2. Rotate between x and x's new child

Single and Double Rotations:

Inserting what integer values
would cause the tree to need a:



1. single rotation?

2. double rotation?

3. no rotation?

Insertion into AVL tree

1. Find spot for new key
2. Hang new node there with this key
3. Search back up the path for imbalance
4. If there is an imbalance:

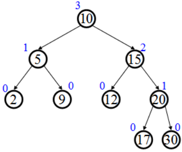
case #1: Perform single rotation and exit

case #2: Perform double rotation and exit

Both rotations keep the subtree height unchanged.
Hence only one (sinlge or double) rotation is sufficient!

Easy Insert

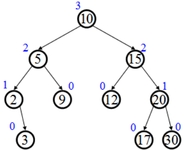
Insert(3)



Unbalanced?

Hard Insert (Bad Case #1)

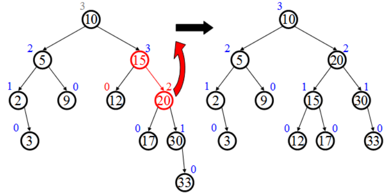
Insert(33)



Unbalanced?

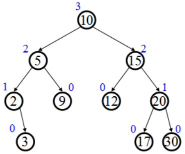
How to fix?

Single Rotation



Hard Insert (Bad Case #2)

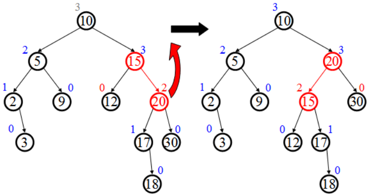
Insert(18)



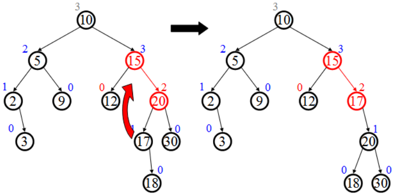
Unbalanced?

How to fix?

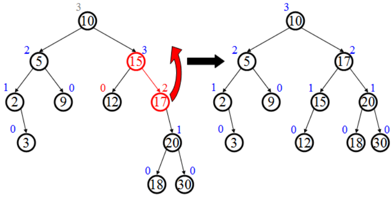
Single Rotation (oops!)



Double Rotation (Step #1)



Double Rotation (Step #2)



Insert into an AVL tree: a b e c d

AVL Trees Revisited

- Balance condition:
  - For every node x, $-1 \le \text{balance}(x) \le 1$
  - Strong enough : Worst case depth is O(log n)
  - Easy to maintain : *one* single or double rotation

- Guaranteed O(log n) running time for
  - Find ?
  - Insert ?
  - Delete ?
  - buildTree ?

AVL Trees Revisited

- What extra info did we maintain in each node?
- Where were rotations performed?
- How did we locate this node?

Other Possibilities?

- Could use different balance conditions, different ways to maintain balance, different guarantees on running time, …
- Why aren't AVL trees perfect?
- Many other balanced BST data structures
  - Red-Black trees
  - AA trees
  - **Splay Trees**
  - 2-3 Trees
  - **B-Trees**
  - …

Splay Trees

- Blind adjusting version of AVL trees
  - Why worry about balances? Just rotate anyway!
- *Amortized* time per operations is O(log *n*)
- Worst case time per operation is O(*n*)
  - But guaranteed to happen rarely

Insert/Find always rotate node *to the root*!

SAT/GRE Analogy question:
AVL is to Splay trees as ___________ is to __________

Recall: Amortized Complexity

If a sequence of M operations takes O(M f(*n*)) time, we say the amortized runtime is O(f(*n*)).

- Worst case time *per operation* can still be large, say O(*n*)
- Worst case time for *<u>any</u> sequence* of M operations is O(M f(*n*))

Average time *per operation* for *any* sequence is O(f(*n*))

Amortized complexity is *worst-case* guarantee over *sequences* of operations.

Recall: Amortized Complexity

- Is amortized guarantee any weaker than worstcase?
- Is amortized guarantee any stronger than averagecase?
- Is average case guarantee good enough in practice?
- Is amortized guarantee good enough in practice?

The Splay Tree Idea