CSE 326 DATA STRUCTURES
HOMEWORK 8 - The Final Homework!

Due: **Wednesday, August 15, 2007** at the beginning of class.

1. (a) Give an example of a graph with negative edges but no negative cost cycles where Dijkstra's algorithm gives the wrong answer.

   (b) Suppose you are given a graph that has negative-cost edges but no negative-cost cycles. Why does the following strategy fail to find shortest paths: uniformly add a constant k to the cost of every edge so that all costs become non-negative, run Dijkstra's algorithm, return the result with edge costs reverted back to the original costs (i.e. with k subtracted). Give an argument as well as a small example where it fails. (Hint: the simplest example I can think of uses only three vertices.)

2. Consider the diagram in figure 9.86 in Weiss. For this problem, consider the grid to be the local zoo (it's a small zoo), and the black circles are tourists visiting. Unfortunately, someone let the tigers out of their cage, and the tourists are trying to escape as quickly as they can. As security officer of the zoo, your job is to figure out if they can do so safely: they each need a path to the edge of the zoo, such that no two paths intersect. (That way everyone can run madly for the exits without trampling on someone else.)

   Since you took CS326, you realize that this is *almost* like a maximum-flow problem: your vertices would be the squares in this zoo; the edges would connect adjacent squares and have capacity 1, you would create a source node and connect it to the squares where each person starts, and you'd create a sink and connect it to every square along the perimeter of the zoo. If that were it, your job would be done: if the maximum flow through this graph is as big as the number of tourists, then the tourists can all escape safely. But remember, you also need to ensure that no *paths* cross at all, and maximum flow doesn't quite give you that guarantee. You realize that making sure paths don't cross is the same as putting capacities on the *vertices* of the graph! Now if only you could somehow encode vertex capacities in this graph as edge capacities in a similar graph, you'd be all set...

   (For a hint on how to do this encoding, see the hints in the next problem...)

3. In class we said that the Hamiltonian cycle problem is NP-complete when considering either directed or undirected graphs, but we didn't prove it. In this problem, you'll prove something slightly simpler: show that Directed-Hamiltonian-Cycle reduces to Undirected-Hamiltonian-Cycle, and also vice versa. (Then if either of them is NP-complete, both of them are.) Most of the NP-completeness proofs you'll see involve some sort of 'gimmick' to make the reduction work; I'll give hints as to what the gimmick should be.

   (a) To reduce Undirected-Hamiltonian-Cycle to Directed-Hamiltonian-Cycle, you are given an undirected graph G, and need to construct a directed graph G' such that G' has a directed Hamiltonian cycle if and only if G has an undirected one. Here, the gimmick is very simple: the directed graph G' has to have edges that allow you to go "either way" across an edge in G, so just try using two edges instead of one.

   To prove this reduction, you need to show two things:

   i. Can you construct G' from G in polynomial time?

   ii. G' has a directed Hamiltonian cycle if and only if G has an undirected one.

   (b) To reduce Directed-Hamiltonian-Cycle to Undirected-Hamiltonian-Cycle, you are given a directed graph H and need to construct an undirected graph H' such that H' has an undirected Hamiltonian cycle if and only if H has a directed one. Here, the gimmick is trickier: you need to create extra vertices that "separate" the incoming and outgoing endpoints of edges. Specifically, for every vertex $h \in H$, create *three* vertices in H': h_{in}, h', h_{out}, and create undirected edges (h_{in}, h') and (h', h_{out}). Then for every directed edge $(u, v) \in H$, you'll create the undirected edge $(u_{out}, v_{in}) \in H'$.

   i. Can you construct H' from H in polynomial time?

   ii. H' has an undirected Hamiltonian cycle if and only if H has a directed one.