CSE 326: Data Structures
Graph Traversals

James Fogarty

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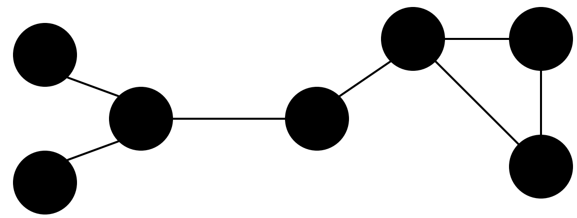
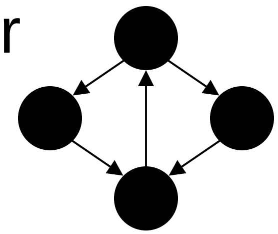
Graph Connectivity

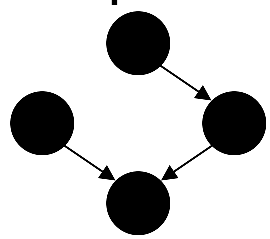
Undirected graphs are *connected* if there is a path between any two vertices



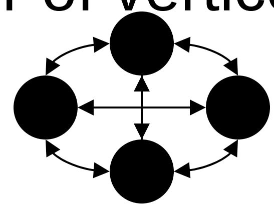
Directed graphs are *strongly connected* if there is a path from any one vertex to any other



Directed graphs are *weakly connected* if there is a path between any two vertices, *ignoring direction*



A *complete* graph has an edge between every pair of vertices



Graph Traversals

Breadth-first search (and depth-first search) work for arbitrary (directed or undirected) graphs - not just mazes!

- Must mark visited vertices. Why?
  - So you do not go into an infinite loop! It's not a tree.

Either can be used to determine connectivity:

- Is there a path between two given vertices?
- Is the graph (weakly/strongly) connected?

Which one:

- Uses a queue?
- Uses a stack?
- Always finds the shortest path (for unweighted graphs)?

The Shortest Path Problem

Given a graph *G*, edge costs $c_{i,j}$, and vertices *s* and *t* in *G*, find the shortest path from *s* to *t*.

For a path $p = v_0\ v_1\ v_2\ \ldots\ v_k$

unweighted length of path $p = k$ (a.k.a. *length*)

weighted length of path $p = \sum_{i=0..k-1} c_{i,i+1}$ (a.k.a *cost*)

Path length equals path cost when ?

Single Source Shortest Paths (SSSP)

Given a graph *G*, edge costs $c_{i,j}$, and vertex *s*, find the shortest paths from *s* to all vertices in G.

Is this harder or easier than the previous problem?

All Pairs Shortest Paths (APSP)

Given a graph *G* and edge costs $c_{i,j}$, find the shortest paths between all pairs of vertices in G.

Is this harder or easier than SSSP?

Could we use SSSP as a subroutine to solve this?

Depth-First Graph Search

Open – Stack

Criteria – Pop

```
DFS( Start, Goal_test)
   push(Start, Open);
   repeat
         if (empty(Open)) then return fail;
         Node := pop(Open);
         if (Goal_test(Node)) then return Node;
         for each Child of node do
               if (Child not already visited) then push(Child, Open);
         Mark Node as visited;
  end
```

Breadth-First Graph Search

Open – Queue

Criteria – Dequeue (FIFO)

```
BFS( Start, Goal_test)
   enqueue(Start, Open);
   repeat
         if (empty(Open)) then return fail;
         Node := dequeue(Open);
         if (Goal_test(Node)) then return Node;
         for each Child of node do
               if (Child not already visited) then enqueue(Child, Open);
         Mark Node as visited;
  end
```

Comparison: DFS versus BFS

Depth-first search

- Does not always find shortest paths
- Must be careful to mark visited vertices, or you could go into an infinite loop if there is a cycle

Breadth-first search

- Always finds shortest paths – optimal solutions
- Marking visited nodes can improve efficiency, but even without doing so search is guaranteed to terminate

Is BFS always preferable?

DFS Space Requirements

Assume:

- Longest path in graph is length d
- Highest number of out-edges is k

DFS stack grows at most to size dk

- For k=10, d=15, size is 150

BFS Space Requirements

Assume

- Distance from start to a goal is *d*
- Highest number of out edges is *k* BFS

Queue could grow to size k^d

- For *k*=10, *d*=15, size is 1,000,000,000,000,000

Conclusion

For large graphs, DFS is hugely more memory efficient, *if we can limit the maximum path length to some fixed d.*

- If we *knew* the distance from the start to the goal in advance, we can just *not add any children to stack after level d*
- But what if we don't know *d* in advance?

Iterative-Deepening DFS (I)

```
Bounded_DFS(Start, Goal_test, Limit)
  Start.dist = 0;
  push(Start, Open);
  repeat
      if (empty(Open)) then return fail;
      Node := pop(Open);
      if (Goal_test(Node)) then return Node;
      if (Node.dist ≥Limit) then return fail;
      for each Child of node do
          if (Child not already i-visited) then
              Child.dist := Node.dist + 1;
              push(Child, Open);
      Mark Node as i-visited;
 end
```

Iterative-Deepening DFS (II)

```
IDFS_Search(Start, Goal_test)
  i := 1;
  repeat
     answer := Bounded_DFS(Start, Goal_test, i);
     if (answer != fail) then return answer;
     i := i+1;
  end
```

Analysis of IDFS

Work performed with limit < actual distance to G is wasted – but the wasted work is usually small compared to amount of work done during the *last* iteration

$$\sum_{i=1}^{d} k^i = O(k^d)$$

Ignore low order terms!

Same time complexity as BFS

Same space complexity as (bounded) DFS

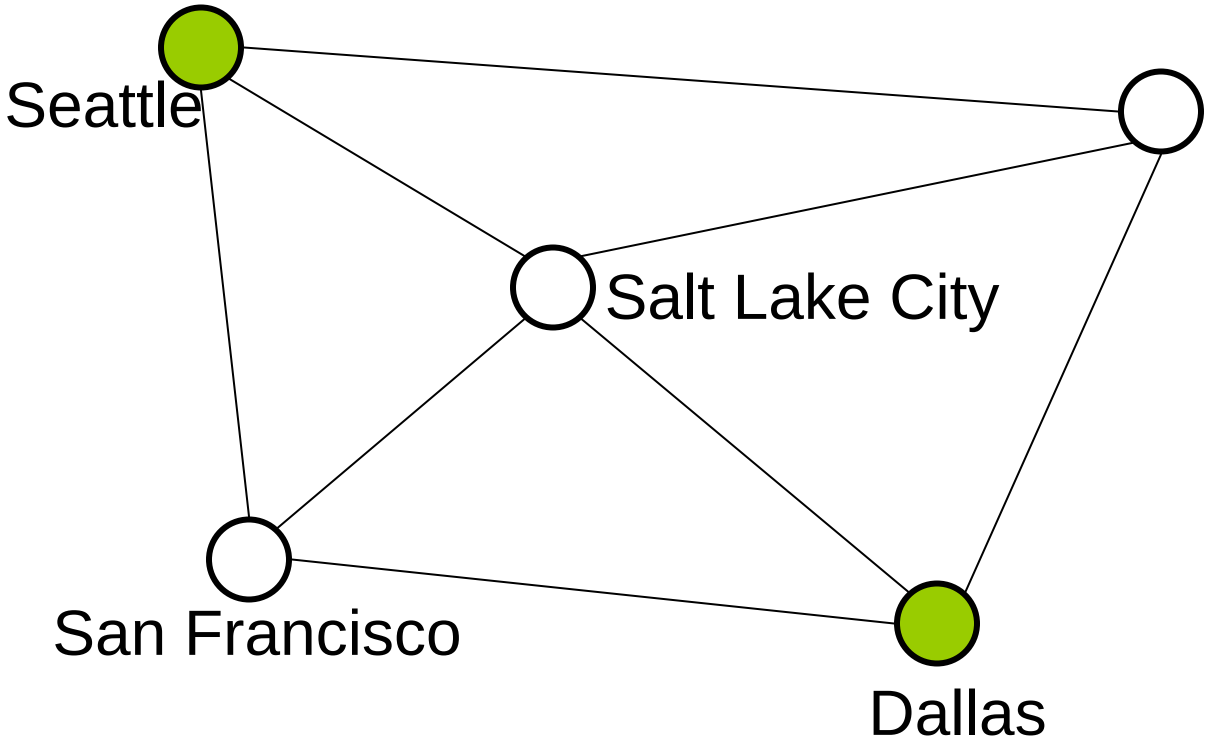
Saving the Path

Our pseudocode returns the goal node found, but not the path to it

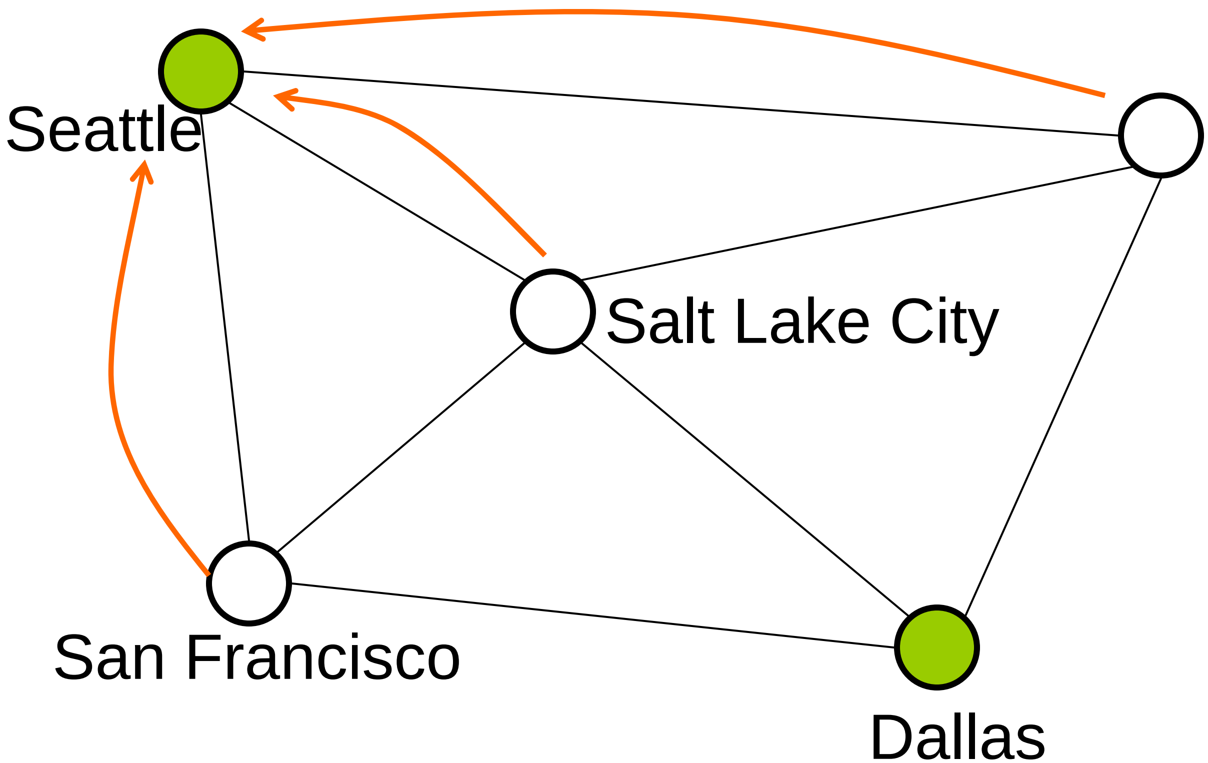
How can we remember the path?

- Add a field to each node, that points to the previous node along the path
- Follow pointers from goal back to start to recover path

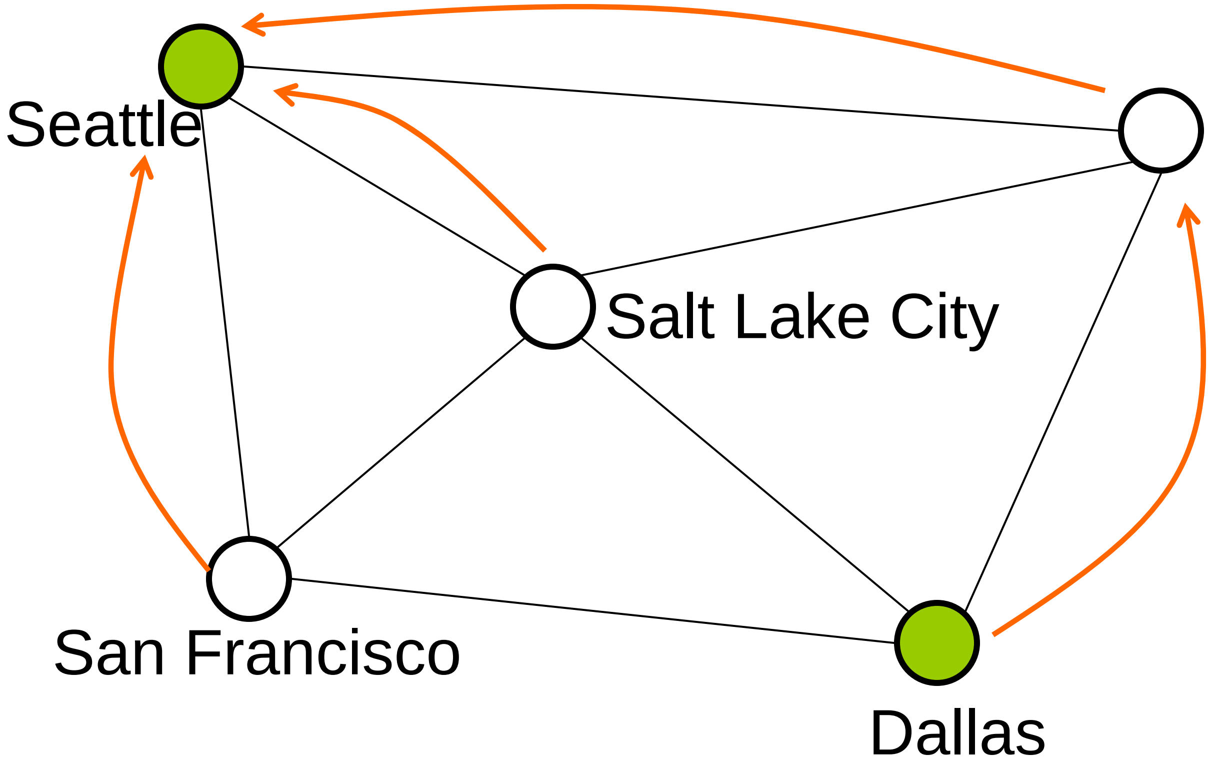
Example



Example (Unweighted Graph)



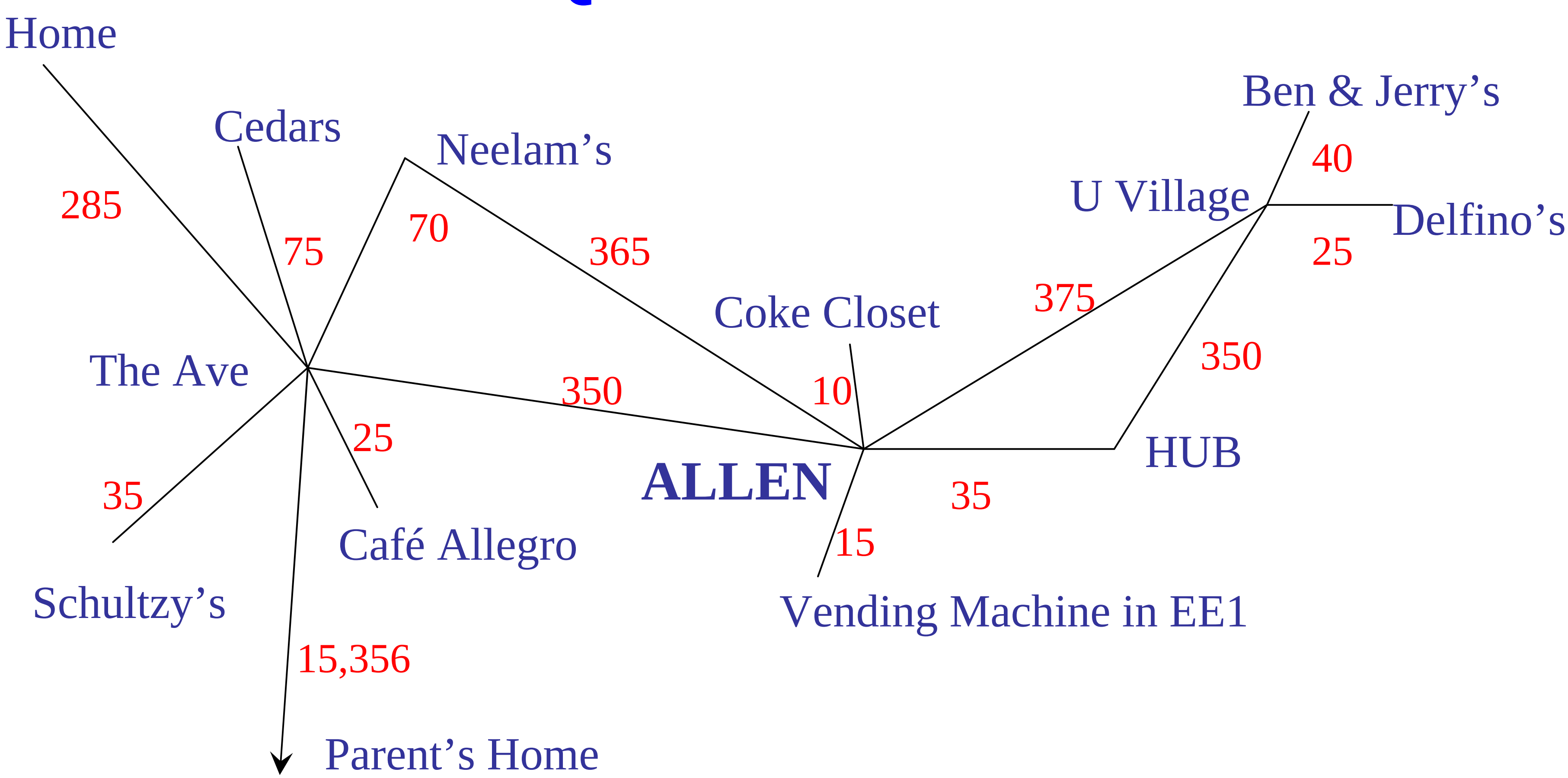
Example (Unweighted Graph)



Graph Search, Saving Path

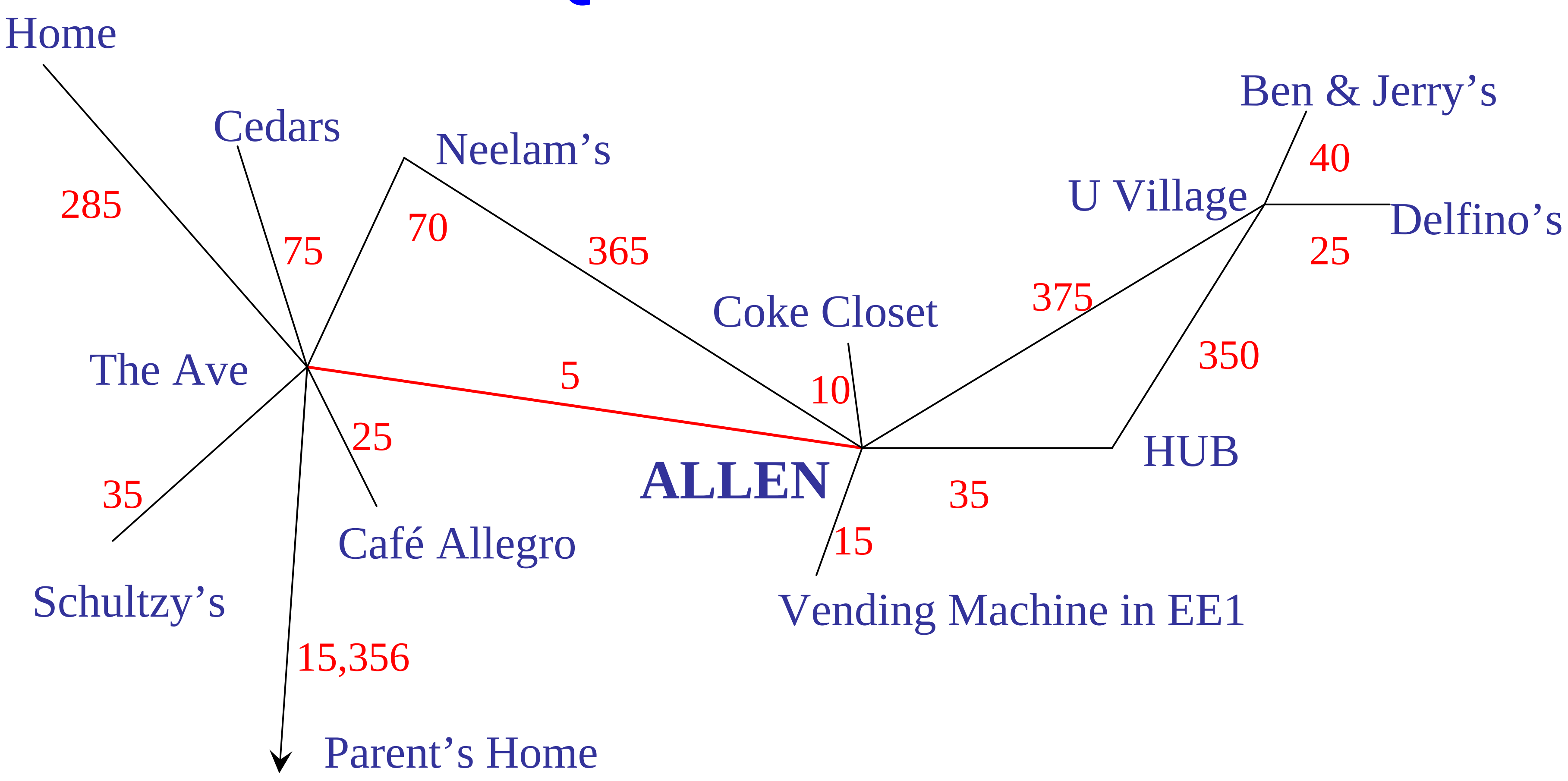
```
Search( Start, Goal_test, Criteria)
  insert(Start, Open);
  repeat
      if (empty(Open)) then return fail;
      select Node from Open using Criteria;
      if (Goal_test(Node)) then return Node;
      for each Child of node do
          if (Child not already visited) then
              Child.previous := Node;
              Insert( Child, Open );
      Mark Node as visited;
  end
```

Weighted SSSP:
The Quest For Food



Can we calculate shortest distance to all nodes from Allen Center?

Weighted SSSP:
The Quest For Food



Can we calculate shortest distance to all nodes from Allen Center?

Edsger Wybe Dijkstra
(1930-2002)

- Invented concepts of structured programming, synchronization, weakest precondition, and "semaphores" for controlling computer processes. The Oxford English Dictionary cites his use of the words "vector" and "stack" in a computing context.
- Believed programming should be taught without computers
- 1972 Turing Award
- “In their capacity as a tool, computers will be but a ripple on the surface of our culture. In their capacity as intellectual challenge, they are without precedent in the cultural history of mankind.”

General Graph Search Algorithm

Open – some data structure (e.g., stack, queue, heap)

Criteria – some method for removing an element from Open

```
Search( Start, Goal_test, Criteria)
  insert(Start, Open);
  repeat
      if (empty(Open)) then return fail;
      select Node from Open using Criteria;
      if (Goal_test(Node)) then return Node;
      for each Child of node do
          if (Child not already visited) then Insert( Child, Open );
      Mark Node as visited;
 end
```

Shortest Path for Weighted Graphs

Given a graph **G = (V, E)** with edge costs c(e), and a vertex $s \in V$, find the shortest (lowest cost) path from s to every vertex in **V**

Assume: only *positive* edge costs

Dijkstra's Algorithm for Single Source Shortest Path

Similar to breadth-first search, but uses a heap instead of a queue:

- Always select (expand) the vertex that has a lowest-cost path to the start vertex

Correctly handles the case where the lowest-cost (shortest) path to a vertex is not the one with fewest edges