CSE 322
Introduction to Formal Models in Computer Science

Defining δ^* from δ

In the definition of DFAs, the transition function δ explicitly describes, for each character $a \in \Sigma$, the name of the state reached on a when started at state q. This is precisely $\delta(q, a)$.

In analyzing DFAs we often want to talk about the state that a given *string* $w \in \Sigma^*$ reaches when started at a state q. We give this corresponding function the name δ^*; that is $\delta^*(q, w)$ is the state that would be reached start at state q and following the string $w \in \Sigma^*$. This function δ^* is determined entirely based on δ using the following inductive definition.

- $\delta^*(q, \epsilon) = q$
- for $x \in \Sigma^*$ and $a \in \Sigma$, $\delta^*(q, xa) = \delta(\delta^*(q, x), a)$.

Note that this immediately means that $\delta^*(q, a) = \delta(\delta^*(q, \epsilon), a) = \delta(q, a)$.

The following is a useful property of the δ^* function.

Theorem 1. *For any $q \in Q$, and $x, y \in \Sigma^*$, $\delta^*(q, xy) = \delta^*(\delta^*(q, x), y)$.*

Proof. The proof is by induction on the length of y where the property we prove for each y is that for all $x \in \Sigma^*$, for all $q \in Q$, $\delta^*(q, xy) = \delta^*(\delta^*(q, x), y)$.

BASE CASE: $y = \varepsilon$. In this case for any $x \in \Sigma^*$ and $q \in Q$,

$$
\begin{aligned}
\delta^*(q, xy) &= \delta^*(q, x) && \text{since } y = \epsilon \\
&= \delta^*(\delta^*(q, x), \epsilon) && \text{by the definition of } \delta^*
\end{aligned}
$$

INDUCTION HYPOTHESIS: Assume that for all $x \in \Sigma^*$, for all $q \in Q$, $\delta^*(q, xy) = \delta^*(\delta^*(q, x), y)$.

INDUCTION STEP: Let $y' = ya$ where $y \in \Sigma^*$ and $a \in \Sigma$. Then

$$
\begin{aligned}
\delta^*(q, xy') &= \delta^*(q, xya) && \text{by definition} \\
&= \delta(\delta^*(q, xy), a) && \text{by the definition of } \delta^* \\
&= \delta(\delta^*(\delta^*(q, x), y), a) && \text{by the inductive hypothesis} \\
&= \delta(\delta^*(p, y), a) && \text{where } p = \delta^*(q, x) \\
&= \delta^*(p, ya) && \text{by the definition of } \delta^* \\
&= \delta^*(\delta^*(q, x), ya) && \text{by the definition of } p \\
&= \delta^*(\delta^*(q, x), y')
\end{aligned}
$$

which is what we needed to prove. Therefore by induction on the length of y the claim is proved.

□