University of Washington

CSE 322: Introduction to Formal Models in Computer Science
Correctness Proof of Theorem 1.12 in Sipser

Spring 2002 Handout 2

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It is intuitively clear that the machine presented in the proof of Theorem 1.12 recognizes the language $A_1 \cup A_2$. Informally speaking, the first component of the state in the machine M changes just as though machine M_1 were running independently of M_2. The second component of the state in the machine M changes just as though machine M_2 were running independently of M_1. M accepts w if either machine ends up in a final (accept) state.

Here is a proof that formalizes this intuition.

Lemma 1 $L(M) = A_1 \cup A_2$.

Proof:

$$
\begin{aligned}
w \in L(M) \quad &\text{iff} \quad ((q_1, q_2), w) \vdash_M^* ((f_1, f_2), \varepsilon)\ , \text{ where } f_1 \in F_1 \text{ or } f_2 \in F_2. && (1)\\
&\text{iff} \quad (q_1, w) \vdash_{M_1}^* (f_1, \varepsilon) \text{ and}\\
&\phantom{\text{iff}} \quad (q_2, w) \vdash_{M_2}^* (f_2, \varepsilon)\ , \text{ where } f_1 \in F_1 \text{ or } f_2 \in F_2. && (2)\\
&\text{iff} \quad w \in L(M_1) \text{ or } w \in L(M_2)\\
&\text{iff} \quad w \in L(M_1) \cup L(M_2)
\end{aligned}
$$

□

Technically speaking, in order to get from line (1) to line (2) above, we would have to prove by induction on the length of the string w that for any states $s_1, p_1 \in Q_1$, $s_2, p_2 \in Q_2$, and any string w, $((s_1, s_2), w) \vdash_M^* ((p_1, p_2), \varepsilon)$ iff $(s_1, w) \vdash_M^* (s_1, \varepsilon)$ and $(s_2, w) \vdash_M^* (p_2, \varepsilon)$. (Do you see why we can't conclude (2) from (1) directly?)

Claim: For any states $s_1, p_1 \in Q_1$, $s_2, p_2 \in Q_2$, and any string w, $((s_1, s_2), w) \vdash_M^* ((p_1, p_2), \varepsilon)$ iff $(s_1, w) \vdash_M^* (p_1, \varepsilon)$ and $(s_2, w) \vdash_M^* (p_2, \varepsilon)$.

Proof: The proof is by induction on $|w|$.

Basis : Left as an exercise.

Induction Hypothesis: Suppose the Claim is true for all strings z such that $0 \leq |z| \leq n$, for some fixed $n \geq 0$.

Induction Step: Let w be a string where $|w| = n + 1$ and $w = w'a$ for some $a \in \Sigma$.

There are two implications to prove.

($\Rightarrow$) Suppose $((s_1, s_2), w'a) \vdash_M^* ((p_1, p_2), \varepsilon)$.

Then $((s_1, s_2), w'a) \vdash_M^* ((r_1, r_2), a) \vdash_M ((p_1, p_2), \varepsilon)$, where $r_1 \in Q_1, r_2 \in Q_2, \delta_1(r_1, a) = p_1$, and $\delta_2(r_2, a) = p_2$.

By Fact 1, $((s_1, s_2), w') \vdash_M^* ((r_1, r_2), \varepsilon)$.

By the induction hypothesis, $(s_1, w') \vdash_{M_1}^* (r_1, \varepsilon)$ and $(s_2, w') \vdash_{M_2}^* (r_2, \varepsilon)$.

So,

$$\begin{aligned} (s_1, w'a) &\vdash_{M_1}^* (r_1, a) && \text{(Fact 1)} \\ &\vdash_{M_1} (p_1, \varepsilon) && (\delta_1(r_1, a) = p_1) \end{aligned}$$

and

$$\begin{aligned} (s_2, w'a) &\vdash_{M_2}^* (r_2, a) && \text{(Fact 1)} \\ &\vdash_{M_2} (p_2, \varepsilon) && (\delta_2(r_2, a) = p_2) \end{aligned}$$

($\Leftarrow$) This implication direction is left as an exercise.

$\square$