Section MR: Solutions

1. Midterm Review: Translation

Let your domain of discourse be all coffee drinks. You should use the following predicates:

- $\texttt{soy}(x)$ is true iff x contains soy milk.
- $\texttt{whole}(x)$ is true iff x contains whole milk.
- $\texttt{sugar}(x)$ is true iff x contains sugar
- $\texttt{decaf}(x)$ is true iff x is not caffeinated.
- $\texttt{vegan}(x)$ is true iff x is vegan.
- $\texttt{RobbieLikes}(x)$ is true iff Robbie likes the drink x.

Translate each of the following statements into predicate logic. You may use quantifiers, the predicates above, and usual math connectors like $=$ and $\neq$.

(a) Coffee drinks with whole milk are not vegan. **Solution:**

$\forall x(\texttt{whole}(x) \rightarrow \neg\,\texttt{vegan}(x))$.

(b) Robbie only likes one coffee drink, and that drink is not vegan. **Solution:**

$\exists x \forall y(\texttt{RobbieLikes}(x) \wedge \neg\,\texttt{Vegan}(x) \wedge [\texttt{RobbieLikes}(y) \rightarrow x = y])$

OR $\exists x(\texttt{RobbieLikes}(x) \wedge \neg\,\texttt{Vegan}(x) \wedge \forall y[\texttt{RobbieLikes}(y) \rightarrow x = y])$

(c) There is a drink that has both sugar and soy milk. **Solution:**

$\exists x(\texttt{sugar}(x) \wedge \texttt{soy}(x))$

(d) Translate the following symbolic logic statement into a (natural) English sentence. Take advantage of domain restriction.

$$\forall x([\texttt{decaf}(x) \wedge \texttt{RobbieLikes}(x)] \rightarrow \texttt{sugar}(x))$$

Solution:

Every decaf drink that Robbie likes has sugar. Statements like "For every decaf drink, if Robbie likes it then it has sugar" are equivalent, but only partially take advantage of domain restriction.

2. Midterm Review: Even Steven

Prove that for all integers k, k(k +3) is even.
Recall that $\texttt{Even}(x) := \exists k(x = 2k)$ and $\texttt{Odd}(x) := \exists k(x = 2k + 1)$

(a) Let your domain be integers. Write the predicate logic of this claim.

Solution:

$\forall k(Even(k(k+3)))$

(b) Write an English proof for this claim. **Solution:**

Let k be an arbitrary integer.
Case 1: k is even
By the definition of even, $k = 2j$ for some integer j
So substituting for k into $k(k+3)$:

$$k(k+3) = (2j)(2j+3) = 2(2j^2+3j)$$

$k(k+3) = 2n$, where $n = (2j^2+3j)$ and n is an integer since j is an integer and integers are closed under addition and multiplication.
So, by definition of even, $k(k+3)$ is even.

Case 2: k is odd
By the definition of odd, $k = 2j+1$ for some integer j.
So substituting for k into k(k+3):

$$k(k+3) = (2j+1)(2j+1+3) = (2j+1)(2j+4) = 4j^2+10j+4 = 2(2j^2+5j+2) = 2(2j+1)(j+2)$$

$k(k+3) = 2n$, where $n = (2j+1)(j+2)$ and n is an integer since j is an integer and integers are closed under addition and multiplication.
So, by definition of even, $k(k+3)$ is even.

These cases are exhaustive, so the claim that $k(k+3)$ is even must hold.
Since k was arbitrary, the claim holds for all k.

3. Midterm Review: Number Theory

Let p be a prime number at least 3, and let x be an integer such that $x^2 \% p = 1$.

(a) Show that if an integer y satisfies $y \equiv 1 \pmod{p}$, then $y^2 \equiv 1 \pmod{p}$. (this proof will be short!)
(Try to do this without using the theorem "Raising Congruences To A Power")
Solution:

Let y be an arbitrary integer and suppose $y \equiv 1 \pmod{p}$. We can multiply congruences, so multiplying this congruence by itself we get $y^2 \equiv 1^2 \pmod{p}$. Since y is arbitrary, the claim holds.

(b) Repeat part (a), but don't use any theorems from the Number Theory Reference Sheet. That is, show the claim directly from the definitions. **Solution:**

Let x be an arbitrary integer and suppose $x \equiv 1 \pmod{p}$. By the definition of Congruences, $p \mid (x-1)$. Therefore, by the definition of divides, there exists an integer k such that

$$pk = (x-1)$$

By multiplying both sides of pk = (x - 1) by (x + 1) and re-arranging the equation, we have

$$pk(x+1) = (x-1)(x+1)$$
$$p(k(x+1)) = (x-1)(x+1)$$

Since $(x-1)(x+1) = x^2 - 1$, by replacing $(x-1)(x+1)$ with $x^2 - 1$, we have

$$p(k(x+1)) = x^2 - 1$$

Note that since k and x are integers, (k (x + 1)) is also an integer. Therefore, by the definition of divides $p \mid x^2 - 1$.
Hence, by the definition of Congruences, $x^2 \equiv 1 \pmod{p}$.

(c) From part (a), we can see that $x\%p$ can equal 1. Show that for any integer x, if $x^2 \equiv 1 \pmod{p}$, then $x \equiv 1 \pmod{p}$ or $x \equiv -1 \pmod{p}$. That is, show that the only value $x\%p$ can take other than 1 is $p - 1$.
Hint: Suppose you have an x such that $x^2 \equiv 1 \pmod{p}$ and use the fact that $x^2 - 1 = (x-1)(x+1)$
Hint: You may the following theorem without proof: if p is prime and $p \mid (ab)$ then $p \mid a$ or $p \mid b$. **Solution:**

Let x be an arbitrary integer and suppose $x^2 \equiv 1 \pmod{p}$. By the definition of Congruences,

$$p \mid x^2 - 1$$

Since $(x-1)(x+1) = x^2 - 1$, by replacing $x^2 - 1$ with $(x-1)(x+1)$, we have

$$p \mid (x-1)(x+1)$$

Note that for an integer p if p is a prime number and $p \mid (ab)$, then $p \mid a$ or $p \mid b$. In this case, since p is a prime number, by applying the rule, we have $p \mid (x-1)$ or $p \mid (x+1)$.
Therefore, by the definition of Congruences, we have $x \equiv 1 \pmod{p}$ or $x \equiv -1 \pmod{p}$.

4. Midterm Review: Induction

For any $n \in \mathbb{N}$, define S_n to be the sum of the squares of the first n positive integers, or

$$S_n = 1^2 + 2^2 + \cdots + n^2.$$

Prove that for all $n \in \mathbb{N}$, $S_n = \frac{1}{6}n(n+1)(2n+1)$. **Solution:**

Let $\mathsf{P}(n)$ be the statement "$S_n = \frac{1}{6}n(n+1)(2n+1)$" defined for all $n \in \mathbb{N}$. We prove that $\mathsf{P}(n)$ is true for all $n \in \mathbb{N}$ by induction on n.

Base Case: When $n = 0$, we know the sum of the squares of the first n positive integers is the sum of no terms, so we have a sum of 0. Thus, $S_0 = 0$. Since $\frac{1}{6}(0)(0+1)((2)(0)+1) = 0$, we know that $\mathsf{P}(0)$ is true.

Inductive Hypothesis: Suppose that $\mathsf{P}(k)$ is true for some arbitrary $k \in \mathbb{N}$.

Inductive Step: Examining S_{k+1}, we see that

$$S_{k+1} = 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = S_k + (k+1)^2.$$

By the inductive hypothesis, we know that $S_k = \frac{1}{6}k(k+1)(2k+1)$. Therefore, we can substitute and rewrite the expression as follows:

$$\begin{aligned} S_{k+1} &= S_k + (k+1)^2 \\ &= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \\ &= (k+1)\left(\frac{1}{6}k(2k+1) + (k+1)\right) \\ &= \frac{1}{6}(k+1)\left(k(2k+1) + 6(k+1)\right) \\ &= \frac{1}{6}(k+1)\left(2k^2+7k+6\right) \\ &= \frac{1}{6}(k+1)(k+2)(2k+3) \\ &= \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1) \end{aligned}$$

Thus, we can conclude that $\mathsf{P}(k+1)$ is true.

Conclusion: $P(n)$ holds for all integers $n \geq 0$ by the principle of induction.

5. Midterm Review: Strong Induction

Robbie is planning to buy snacks for the members of his competitive roller-skating troupe. However, his local grocery store sells snacks in packs of 5 and packs of 7.

Prove that Robbie can buy exactly n snacks for all integers $n \geq 24$

Solution:

Let $\mathsf{P}(n)$ be the statement "Robbie can buy n snacks with packs of 5 and packs of 7 snacks" defined for all $n \geq 24$. We prove that $\mathsf{P}(n)$ is true for all $n \geq 24$ by the principle of strong induction.

Base Case:

n = 24: 24 snacks can be bought with 2 packs of 7 and 2 packs of 5 snacks.
n = 25: 25 snacks can be bought with 5 packs of 5 snacks.
n = 26: 26 snacks can be bought with 3 packs of 7 and 1 pack of 5 snacks.
n = 27: 27 snacks can be bought with 1 pack of 7 and 4 packs of 5 snacks.
n = 28: 28 snacks can be bought with 4 packs of 7 snacks.

Inductive Hypothesis: Suppose that $\mathsf{P}(24) \wedge \mathsf{P}(25) \wedge \cdots \wedge \mathsf{P}(k)$ is true for some arbitrary $k \geq 28$.

Inductive Step: We want to show that Robbie can buy exactly $k+1$ snacks. By the inductive hypothesis, we know that Robbie can buy exactly $k-4$ snacks, so he can buy another pack of 5 to get exactly $k+1$ snacks.

Conclusion: Therefore, $\mathsf{P}(n)$ holds for all integers $n \geq 24$ by the principle of strong induction.

Proof By Contradiction, Set Theory Practice

6. Wait, That Doesn't Add Up

Write a proof by contradiction for the following proposition: There exist no integers x and y such that $18x + 6y = 1$. In predicate logic this could be expressed as $\forall x \forall y(18x + 6y \neq 1)$. HINT: Try negating this statement before writing your proof.

Solution:

Assume, for the sake of contradiction, that there exists integers x and y such that $18x + 6y = 1$. This gives us:

$$18x + 6y = 1$$
$$3x + y = \frac{1}{6} \qquad \text{Dividing by 6}$$

But wait, this is a contradiction! Integers are closed under multiplication and addition, and so $3x + y$ can't be equal to $\frac{1}{6}$. This means there can be no integers x and y such that $18x + 6y = 1$. Therefore, the original claim holds via proof by contradiction.

7. How Many Elements?

For each of these, how many elements are in the set? If the set has infinitely many elements, say ∞.

(a) $A = \{1, 2, 3, 2\}$

Solution:

3

(b) $B = \{\{\}, \{\{\}\}, \{\{\}, \{\}\}, \{\{\}, \{\}, \{\}\}, \ldots\}$

Solution:

$$\begin{aligned} B &= \{\{\}, \{\{\}\}, \{\{\}, \{\}\}, \{\{\}, \{\}, \{\}\}, \ldots\} \\ &= \{\{\}, \{\{\}\}, \{\{\}\}, \{\{\}\}, \ldots\} \\ &= \{\varnothing, \{\varnothing\}\} \end{aligned}$$

So, there are two elements in B.

(c) $C = A \times (B \cup \{7\})$

Solution:

$C = \{1, 2, 3\} \times \{\varnothing, \{\varnothing\}, 7\} = \{(a, b) \mid a \in \{1, 2, 3\}, b \in \{\varnothing, \{\varnothing\}, 7\}\}$. It follows that there are $3 \times 3 = 9$ elements in C.

(d) $D = \varnothing$

Solution:

0.

(e) $E = \{\varnothing\}$

Solution:

1.

(f) $F = \mathcal{P}(\{\varnothing\})$

Solution:

$2^1 = 2$. The elements are $F = \{\varnothing, \{\varnothing\}\}$.

8. Set = Set

Prove the following set identities. Write both a formal inference proof **and** an English proof.

(a) Let the universal set be $\mathcal{U}$. Prove $A \cap \overline{B} \subseteq A \backslash B$ for any sets A, B.

Solution:

Let x be an arbitrary element and suppose that $x \in A \cap \overline{B}$. By definition of intersection, $x \in A$ and $x \in \overline{B}$, so by definition of complement, $x \notin B$. Then, by definition of set difference, $x \in A \backslash B$. Since x was arbitrary, we can conclude that $A \cap \overline{B} \subseteq A \backslash B$ by definition of subset.

(b) Prove that $(A \cap B) \times C \subseteq A \times (C \cup D)$ for any sets A, B, C, D.

Solution:

Let x be an arbitrary element of $(A \cap B) \times C$. Then, by definition of Cartesian product, x must be of the form (y, z) where $y \in A \cap B$ and $z \in C$. Since $y \in A \cap B$, $y \in A$ and $y \in B$ by definition of $\cap$; in particular, all we care about is that $y \in A$. Since $z \in C$, by definition of $\cup$, we also have $z \in C \cup D$. Therefore since $y \in A$ and $z \in C \cup D$, by definition of Cartesian product we have $x = (y, z) \in A \times (C \cup D)$.

Since x was an arbitrary element of $(A \cap B) \times C$ we have proved that $(A \cap B) \times C \subseteq A \times (C \cup D)$ as required.

9. Set Equality

(a) Prove that $A \cap (A \cup B) = A$ for any sets A, B.

Solution:

Let x be an arbitrary member of $A \cap (A \cup B)$. Then by definition of intersection, $x \in A$ and $x \in A \cup B$. So certainly, $x \in A$. Since x was arbitrary, $A \cap (A \cup B) \subseteq A$.

Now let y be an arbitrary member of A. Then $y \in A$. So certainly $y \in A$ or $y \in B$. Then by definition of union, $y \in A \cup B$. Since $y \in A$ and $y \in A \cup B$, by definition of intersection, $y \in A \cap (A \cup B)$. Since y was arbitrary, $A \subseteq A \cap (A \cup B)$.

Therefore $A \cap (A \cup B) = A$, by containment in both directions.