CSE 311 Section 5

Induction

Administrivia

Announcements & Reminders

- Homework 4 due yesterday
- Homework 5 is due Wednesday (2/7) @ 11:59pm
  - There are TWO parts to this
  - Read details on course website
- Upcoming Midterm: February 12th (Details TBD)
  - If you cannot make it, please let us know ASAP and we will schedule you for a makeup
  - **Midterm Review: February 10th 1-3pm CSE2 G10**

Assignments

Assignment	Release Date	Due date
Homework 1	Thur. January 4	Thur. January 11, 11:59pm
Homework 2	Wed. Jan 10	Wed. January 17, 11:59pm
Homework 3	Wed. Jan 17	Wed. January 25, 11:59pm
Homework 4	Wed. Jan 24	Wed. January 31, 11:59pm
Homework 5	Wed. Jan 31	Wed. Feb 7, 11:59pm

Typesetting

You are not required to typeset your homework solutions; however, it is an easy way to improve the legibility of your documents. Many Allen School students learned to typeset in this course.

LaTeX is the standard tool for typesetting mathematical materials. While it takes some time to learn, it will likely pay for itself in the long run. You can even use LaTeX in places like Ed and Facebook Messenger!

These resources may be helpful for you to get started with LaTeX, with thanks to Adam Blank:

- An updated homework template with instructions on how to latex as well as example problems for each of the types of problems we will have in this class.
- An older homework template that you can use. Here is a preview of the rendered result.
- A How to LaTeX tutorial, including specific information on how to use the old template.

Overleaf is an online editor that spares you from having to install LaTeX locally. Overleaf has some documentation, but you might want to read this how-to-overleaf document first.

Number Theory Warm Up

Problem 2

Prove that if n | m, where n and m are integers greater than 1, and if a ≡ b (mod m), where a and b are integers, then a ≡ b (mod n).

Equivalence in modular arithmetic

Let $a \in \mathbb{Z}, b \in \mathbb{Z}, n \in \mathbb{Z}$ and $n > 0$.
We say $a \equiv b \ (mod\ n)$ if and only if $n|(b-a)$

Divides

For integers x, y we say $\boldsymbol{x}|\boldsymbol{y}$ ("$\boldsymbol{x}$ divides $\boldsymbol{y}$") iff there is an integer $\boldsymbol{z}$ such that $\boldsymbol{xz} = \boldsymbol{y}$.

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Prove that if n | m, where n and m are integers greater than 1, and if a ≡ b (mod m), where a and b are integers, then a ≡ b (mod n).

Let n, m, a, and b be arbitrary integers.

…
…
…

…
Since n and m were arbitrary the claim holds.

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Prove that if n | m, where n and m are integers greater than 1, and if a ≡ b (mod m), where a and b are integers, then a ≡ b (mod n).

Let n, m, a, and b be arbitrary integers.
Suppose n|m with n, m > 1 and a ≡ b (mod m)

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Suppose n|m with n, m > 1 and a ≡ b (mod m)

…
…
…

By **definition of congruence**, we have a ≡ b (mod n)
Since n and m were arbitrary the claim holds.

Work 1-step backwards where you can!

Problem 2

Prove that if $n \mid m$, where n and m are integers greater than 1, and if $a \equiv b \pmod{m}$, where a and b are integers, then $a \equiv b \pmod{n}$.

Let n, m, a, and b be arbitrary integers.
Suppose n|m with n, m > 1 and $a \equiv b \pmod{m}$

By the **definition of divides**, we have **m = kn** for some integer k
…
…

By **definition of congruence**, we have $a \equiv b \pmod{n}$
Since n and m were arbitrary the claim holds.

Problem 2

Prove that if $n \mid m$, where n and m are integers greater than 1, and if $a \equiv b \pmod{m}$, where a and b are integers, then $a \equiv b \pmod{n}$.

Let n, m, a, and b be arbitrary integers.
Suppose $n|m$ with $n, m > 1$ and $a \equiv b \pmod{m}$

By the **definition of divides**, we have **$m = kn$** for some integer k
By **definition of congruence**, we have $m \mid b - a$, which means that $b - a = mj$ for some integer j
…

By **definition of congruence**, we have $a \equiv b \pmod{n}$
Since n and m were arbitrary the claim holds.

Problem 2

Prove that if n | m, where n and m are integers greater than 1, and if
a ≡ b (mod m), where a and b are integers, then a ≡ b (mod n).

Let n, m, a, and b be arbitrary integers.
Suppose n|m with n, m > 1 and a ≡ b (mod m)

By the **definition of divides**, we have **m = kn** for some integer k
By **definition of congruence**, we have m | b - a, which means that b-a=mj for some integer j
Combining the two equations, we see that b-a = (knj) = n(kj)
By the **definition of divides**, we have that n | (b-a)

By **definition of congruence**, we have a ≡ b (mod n)
Since n and m were arbitrary the claim holds.

Introducing Induction (kind of)

Climb the ladder!

You are scared of heights and there is a prize at the top of a very very tall ladder.

You do not want to climb this ladder…

Climb the ladder!

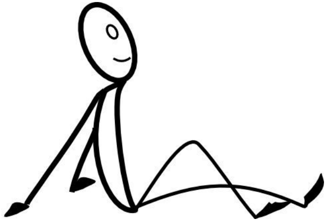
You are scared of heights and there is a prize at the top of a very very tall ladder.

You do not want to climb this ladder…

Lets convince your friend to climb it instead!!!

Climb the ladder!

You Claim: “There are k steps in the ladder. After k steps you will reach the top!”



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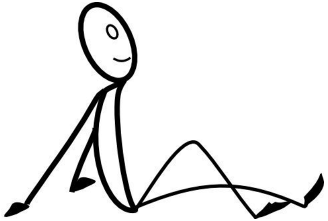
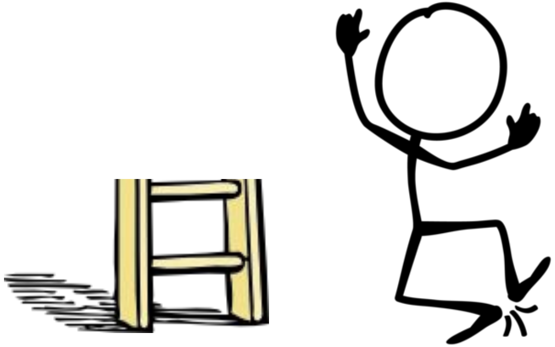
You Claim: “There are k steps in the ladder. After k steps you will reach the top!”

“If we have a ladder with **1** step. I know you can lift your foot so after 1 step you will reach the top of a 1 step ladder!”

“So my claim holds for 1 step!”





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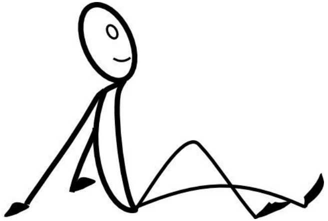

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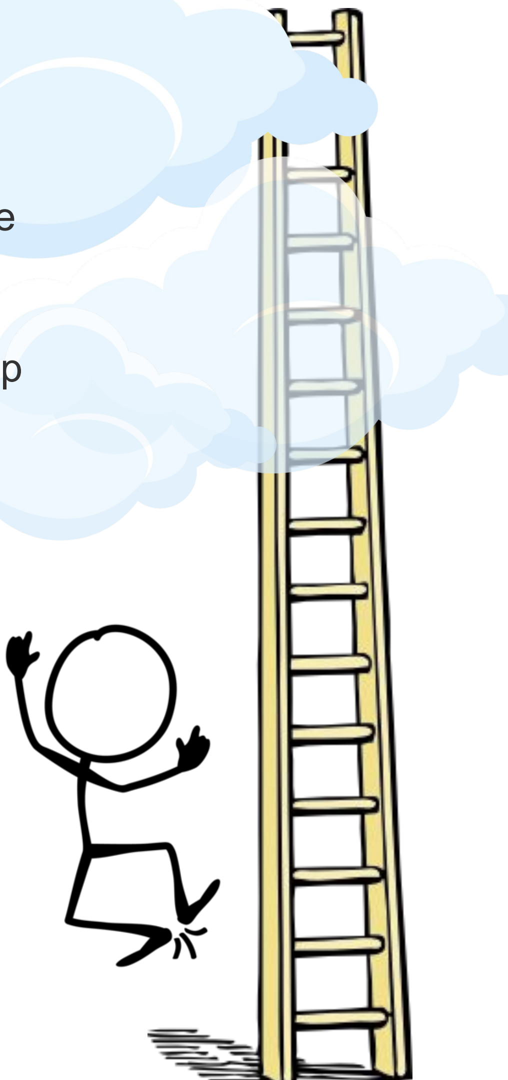
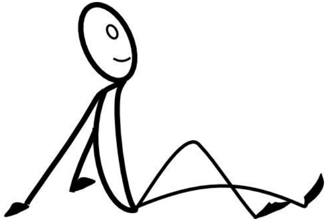
Let’s **suppose** that for an arbitrary number of steps j, after j steps you will reach the top.





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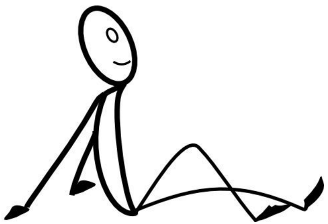
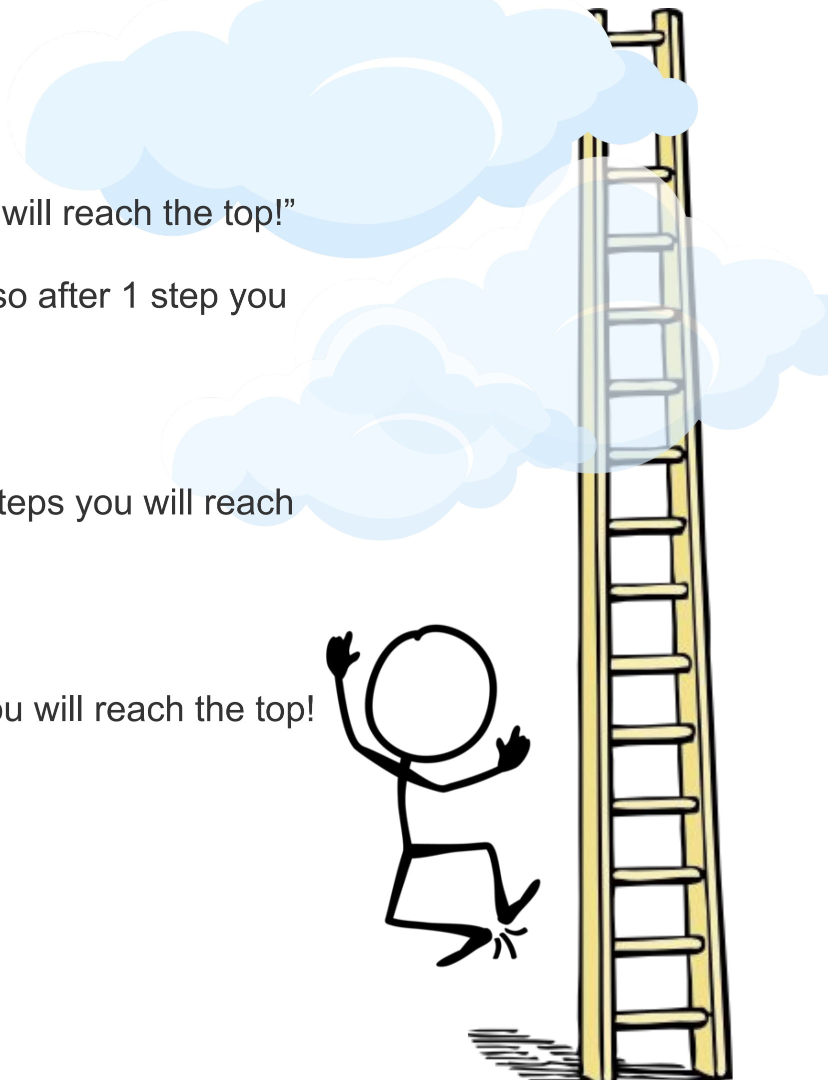
Let’s **suppose** that for an arbitrary number of steps j, after j steps you will reach the top.

I can prove to you that this claim will still hold for j+1 steps!

Goal: Prove that for **j+1** steps in the ladder, after **j+1** steps you will reach the top!





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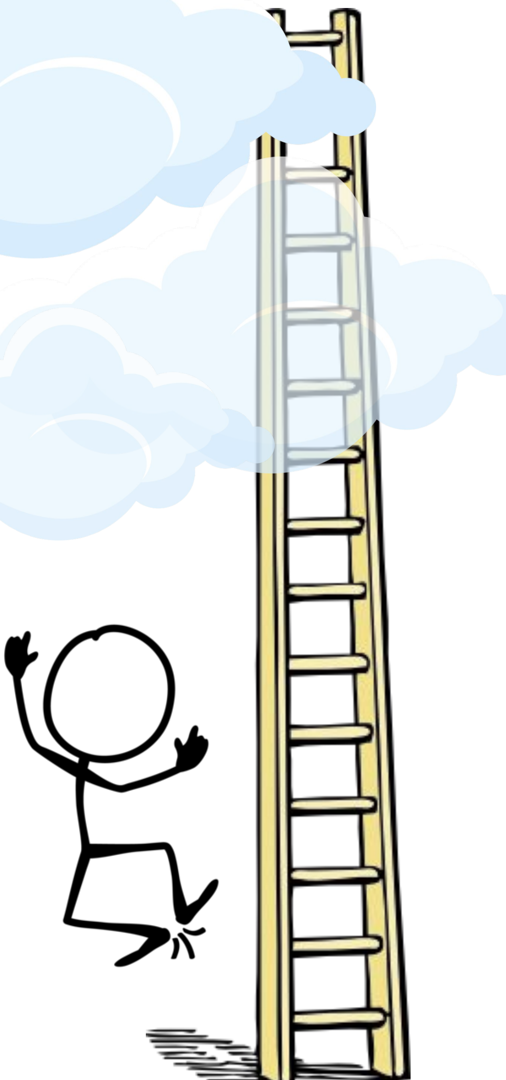

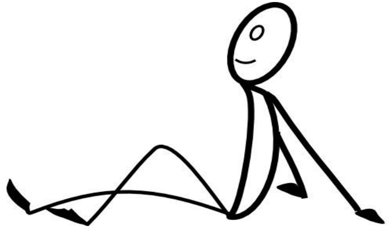

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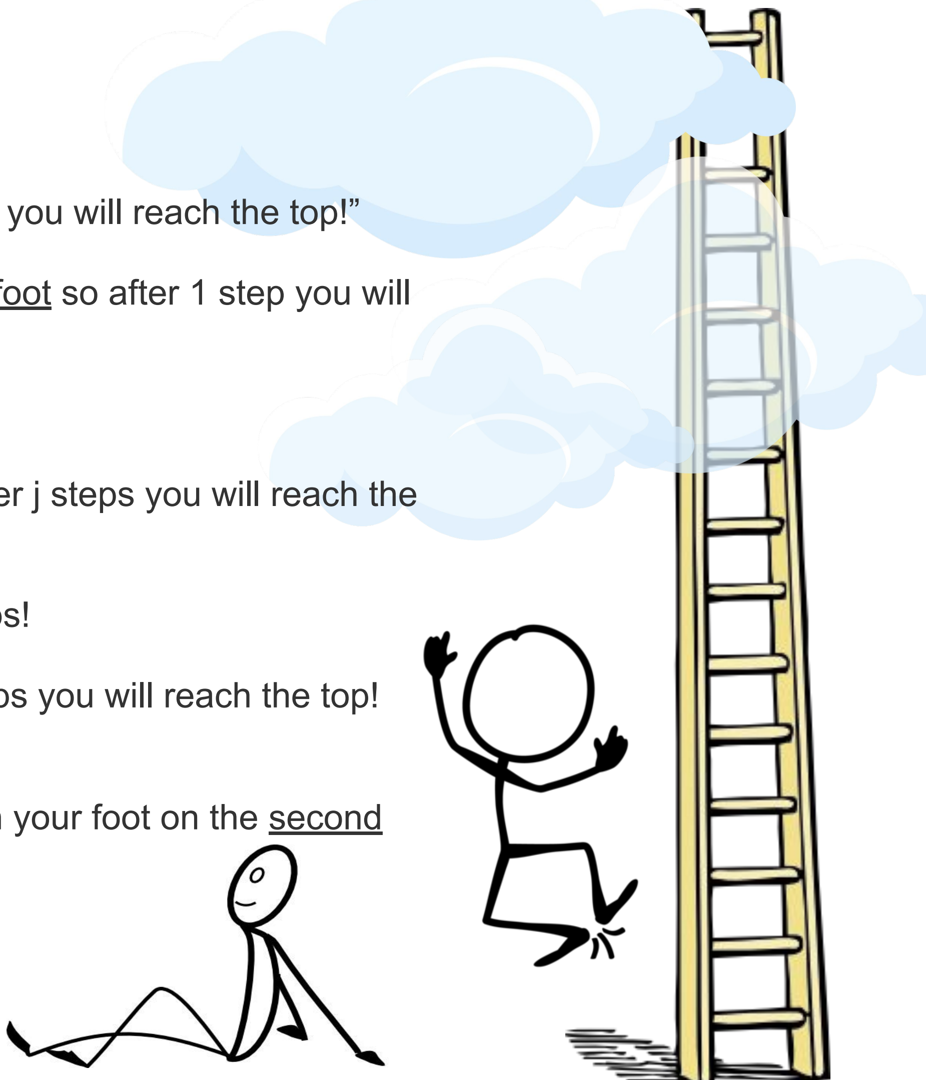

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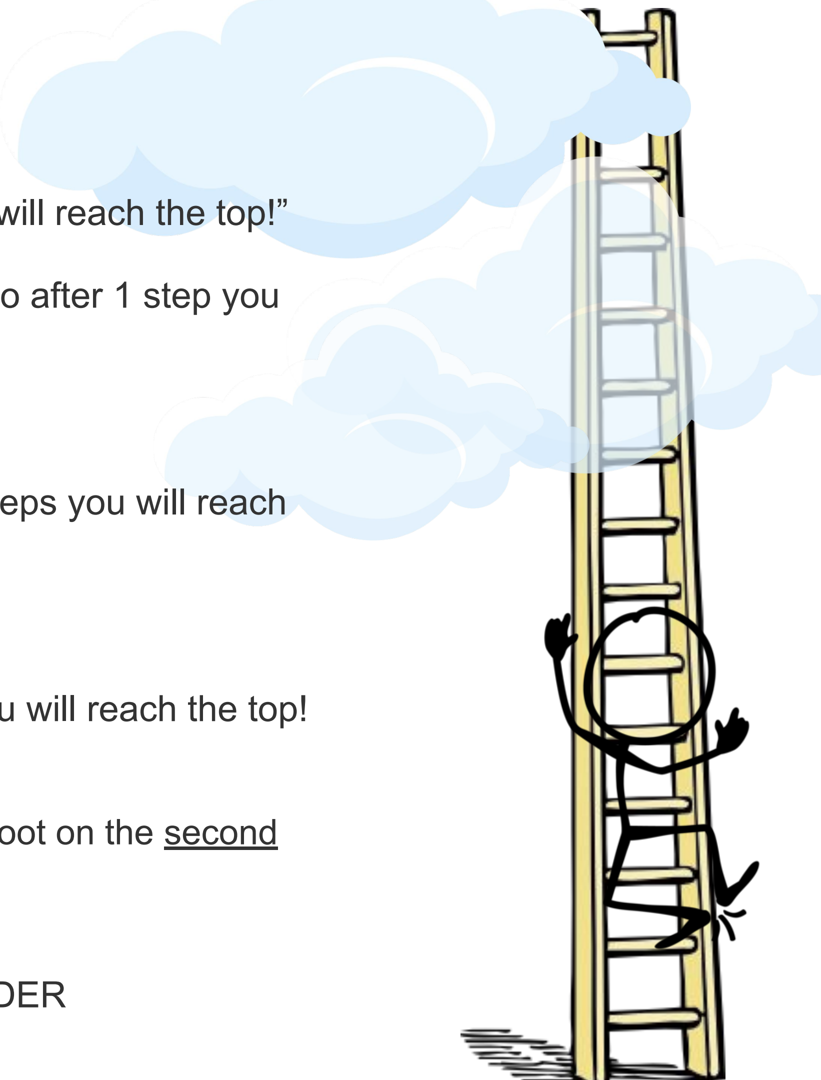
THE CLAIM HOLDS YOUR FRIEND IS CLIMBING THE LADDER



WELCOME TO PROOF BY INDUCTION

You Claim: “There are k steps in the ladder. After k steps you will reach the top!” P(n)

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Inductive Hypothesis P(j)

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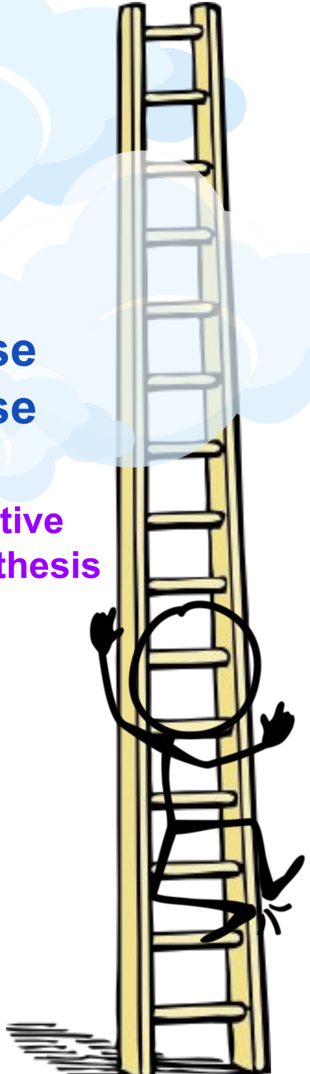

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Inductive Step P(j+1)

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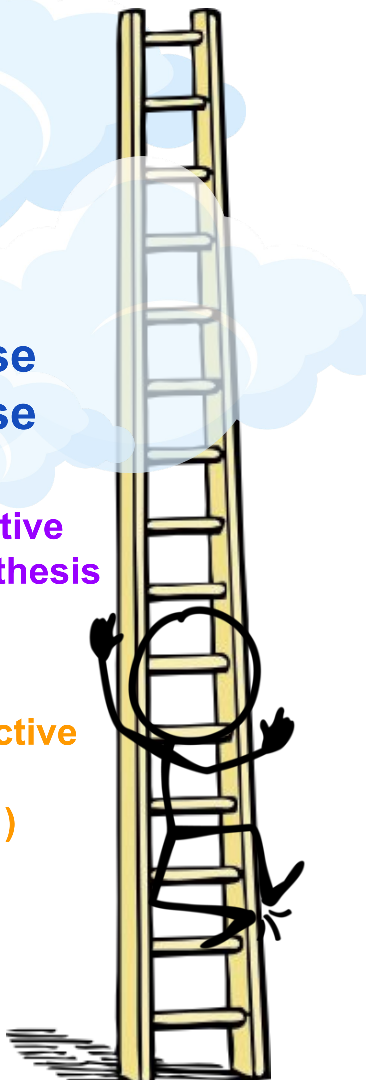

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Inductive Step

Using the IH

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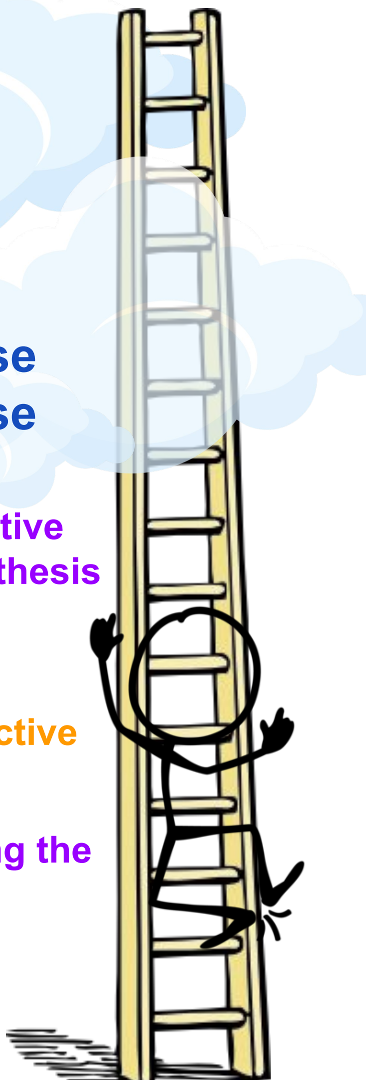

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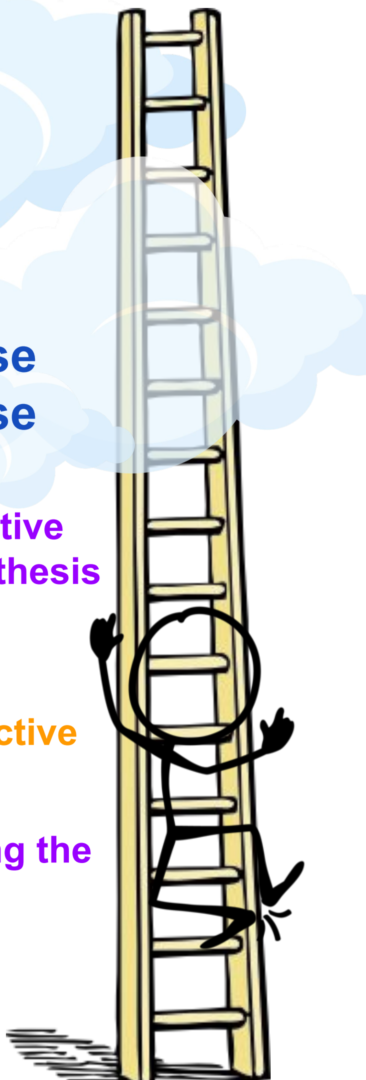
P(n) holds!



Induction: How it actually works

(Weak) Induction Template

Let $P(n)$ be "(whatever you're trying to prove)".
We show $P(n)$ holds for all n by induction on n.

Base Case: Show $P(b)$ is true.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq b$.

Inductive Step: Show $P(k+1)$ (i.e. get $P(k) \rightarrow P(k+1)$)

Conclusion: Therefore, $P(n)$ holds for all n by the principle of induction.

(Weak) Induction Template

Let $P(n)$ be "(whatever you're trying to prove)".
We show $P(n)$ holds **for all n** by induction on n.

Note: often you will condition n here, like "all natural numbers n" or "$n \geq 0$"

Base Case: Show $P(b)$ is true.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq b$.

Inductive Step: Show $P(k+1)$ (i.e. get $P(k) \rightarrow P(k+1)$)

Conclusion: Therefore, $P(n)$ holds **for all n** by the principle of induction.

Match the earlier condition on n in your conclusion!

(Weak) Induction Template

Let $P(n)$ be "(whatever you're trying to prove)".
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P(n) IS A PREDICATE, IT HAS A BOOLEAN VALUE NOT A NUMERICAL ONE

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YOU MUST INTRODUCE AN ARBITRARY VARIABLE IN YOUR IH

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START WITH LHS OF K + 1 ONLY AND WORK TOWARD RHS

Conclusion: Therefore, $P(n)$ holds **for all n** by the principle of induction.

Problem 4 – Induction with Equality

a) Show using induction that $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

b) Define the triangle numbers as $\triangle_n = 1 + 2 + \cdots + n$, where $n \in \mathbb{N}$. In part (a) we showed $\triangle_n = \frac{n(n+1)}{2}$. Prove the following equality for all $n \in \mathbb{N}$:

$$0^3 + 1^3 + \cdots + n^3 = \triangle_n^2$$

Lets walk through part (a) together.

We can "fill in" our induction template to construct our proof by induction.

Problem 4 – Induction with Equality

Show using induction that $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

Let $P(n)$ be "". We show $P(n)$ holds for (some) n by induction on n.

Base Case: $P(b)$:

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq b$.

Inductive Step: Goal: Show $P(k+1)$:

Conclusion: Therefore, $P(n)$ holds for (some) n by the principle of induction.

Problem 4 – Induction with Equality

Show using induction that
$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$
for all $n \in \mathbb{N}$.

Let $P(n)$ be "$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$". We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

Base Case: $P(b)$:

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq b$.

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Problem 4 – Induction with Equality

Show using induction that $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

Let $P(n)$ be "$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$". We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

Base Case: $P(0)$: $0 + \cdots = 0 = \frac{0(0+1)}{2}$ so the base case holds.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq b$.

Inductive Step: Goal: Show $P(k+1)$:

Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

Problem 4 – Induction with Equality

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Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

Prevents backwards reasoning: We need to go from LHS to RHS by "math" **otherwise** we are directly using the rule **we want to prove**

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$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$
for all $n \in \mathbb{N}$.

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Base Case: $P(0)$: $0 + \cdots = 0 = \frac{0(0+1)}{2}$ so the base case holds.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$, i.e. $0 + 1 + 2 + \cdots + k = \frac{k(k+1)}{2}$

Inductive Step: Goal: Show $P(k+1)$:

Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

Problem 4 – Induction with Equality

Show using induction that $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

Let $P(n)$ be "$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$". We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

Base Case: $P(0)$: $0 + \cdots = 0 = \frac{0(0+1)}{2}$ so the base case holds.

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$$\begin{aligned} 0 + 1 + \cdots + k + (k+1) &= (0 + 1 + \cdots + k) + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) && \text{by I.H.} \\ &\ldots \\ &= \frac{(k+1)(k+2)}{2} && ? \end{aligned}$$

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$$= \frac{k(k+1)}{2} + (k+1) \qquad \text{by I.H.}$$

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&= \frac{k(k+1) + 2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2} && ?
\end{aligned}
$$

Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

Problem 5 – Induction with Mod

Prove that the equivalence $4^n \equiv 1 (mod\ 3)$ holds for all $n \in N$.

Spend around 10 minutes working through this proof with the people around you!

Problem 5 – Induction with Mod

Let P(n) be “ ”. We show P(n) holds for (some) n by induction on n.

Base Case: P(b):

Inductive Hypothesis: Suppose P(k) holds for an arbitrary $k \geq b$

Inductive Step: Goal: Show $P(k+1)$:

Conclusion: Therefore, $P(n)$ holds for (some) n by the principle of induction.

Problem 5 – Induction with Mod

Introduction:
Let P(n) be "$4^n \equiv 1 \pmod 3$". We show P(n) holds for all natural numbers n by induction on n. From the definition of mod equivalence, the statement is equivalent to $3 \mid 4^n - 1$. By definition of divides, it suffices to show that $4^n = 3j + 1$ for some integer j.

Base Case: Show P(b).

Inductive Hypothesis: Suppose P(k) holds for an arbitrary $k \geq b$.

Inductive Step: Show P(k + 1).

Conclusion: Therefore P(n) holds for all natural numbers n by induction.

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Base Case: (n = 0) $4^0 = 1 = 0 + 1 = 3(0) + 1$. Thus P(0) holds.

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Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$. That is, $4^k = 3j_k + 1$ with j_k an integer.

Inductive Step: Show $P(k + 1)$.

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Inductive Hypothesis: Suppose P(k) holds for an arbitrary $k \geq 0$. That is, $4^k = 3j_k + 1$ with j_k an integer.

Inductive Step: Show P(k + 1). Goal: $4^{k+1} = 3j_{k+1} + 1$ for some integer j_{k+1}.

$$\begin{aligned} 4^{k+1} &= 4(4^k) \\ &= 4(3j_k + 1) \qquad \text{[Inductive Hypothesis]} \\ &= 4(3j_k) + 4 \\ &= 4(3j_k) + 3 + 1 \\ &= 3(4j_k + 1) + 1 \end{aligned}$$

$4j_k + 1$ is an integer since j_k is an integer. Our j_{k+1} is $4j_k + 1$, so P(k + 1) holds.

Conclusion: Therefore P(n) holds for all natural numbers n by induction.

Problem 5 – Induction with Mod

…

Inductive Step: Goal: Show $4^{j+1} \equiv 1 (mod\ 3)$

$$4^{j+1} = 4^j * 4$$

Conclusion: Therefore, $P(n)$ holds for (some) n by the principle of induction.

Problem 5 – Induction with Mod

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Inductive Step: Goal: Show $4^{j+1} \equiv 1 (mod\ 3)$

$$4^{j+1} = 4^j * 4$$
$$= (3t + 1) * 4 \text{ for some integer } t \quad [By\ IH]$$

Conclusion: Therefore, $P(n)$ holds for (some) n by the principle of induction.

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Inductive Step: Goal: Show $4^{j+1} \equiv 1 (mod\ 3)$

$$\begin{aligned} 4^{j+1} &= 4^j * 4 \\ &= (3t+1) * 4 \text{ for some integer } t \quad [By\ IH] \\ &= 12t + 4 \\ &= 3(4t) + 3 + 1 \end{aligned}$$

Conclusion: Therefore, $P(n)$ holds for (some) n by the principle of induction.

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Since t is an integer, $(4t+1)$ is also an integer. Therefore, we have $3k+1$ for some integer k. This demonstrates that $4^{j+1} = 3k+1$ holds, which shows $4^{j+1} \equiv 1(mod\ 3)$ for an arbitrary integer j by the definition of divides and modular equivalence.

Conclusion: Therefore, $P(n)$ holds for $all\ integers\ n \in N$ by the principle of induction.

That’s All Folks!

Slides Developed By: Aruna Srivastava & Cade Dillon

Bonus Problem:

Problem 4 – Induction with Equality

a) Show using induction that $0+1+2+\cdots+n=\frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

b) Define the triangle numbers as $\triangle_n = 1+2+\cdots+n$, where $n \in \mathbb{N}$. In part (a) we showed $\triangle_n = \frac{n(n+1)}{2}$. Prove the following equality for all $n \in \mathbb{N}$:

$$0^3+1^3+\cdots+n^3=\triangle_n^2$$

Now try part (b) with people around you, and then we'll go over it together!

Problem 4 – Induction with Equality

$\triangle_n = 1 + 2 + \cdots + n,\ n \in \mathbb{N}.$

$\triangle_n = \frac{n(n+1)}{2}$. Prove for all $n \in \mathbb{N}$:

$0^3 + 1^3 + \cdots + n^3 = \triangle_n^2$

Let $P(n)$ be "". We show $P(n)$ holds for (some) n by induction on n.

Base Case: $P(b)$:

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$0^3 + 1^3 + \cdots + n^3 = \triangle_n^2$

Let $P(n)$ be "$0^3 + 1^3 + \cdots + n^3 = (0 + 1 + \cdots + n)^2$". We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

Base Case: $P(b)$:

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq b$.

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Base Case: $P(0)$: $0^3 = 0 = (0)^2$ so the base case holds.

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Base Case: $P(0)$: $0^3 = 0 = (0)^2$ so the base case holds.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$. i.e. $0^3 + 1^3 + \cdots + k^3 += (0 + 1 + \cdots + k)^2$

Inductive Step: Goal: Show $P(k+1)$:

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$$
\begin{aligned}
0^3 + 1^3 + \cdots + k^3 + (k+1)^3 &= (0 + 1 + \cdots + k)^2 + (k+1)^3 && \text{by I.H.} \\
&= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 && \text{by (a)} \\
&= (k+1)^2\left(\frac{k^2}{2^2} + (k+1)\right) && \text{factor out } (k+1)^2 \\
&= (k+1)^2\left(\frac{k^2+4k+4}{4}\right) && \\
&= (k+1)^2\left(\frac{(k+2)^2}{4}\right) && \text{factor numerator} \\
&= \left(\frac{(k+1)(k+2)}{2}\right)^2 && \\
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