CSE 311 Section 3

Quantifiers and Proofs

Administrivia & Introductions

References

- Helpful reference sheets can be found on the course website!
  - https://courses.cs.washington.edu/courses/cse311/23au/resources/
- How to LaTeX (found on Assignments page of website):
  - https://courses.cs.washington.edu/courses/cse311/23au/assignments/HowToLaTeX.pdf
- Equivalence Reference Sheet
  - https://courses.cs.washington.edu/courses/cse311/23au/resources/reference-logical_equiv.pdf
  - https://courses.cs.washington.edu/courses/cse311/23au/resources/logicalConnectPoster.pdf
- Boolean Algebra Reference Sheet
  - https://courses.cs.washington.edu/courses/cse311/23au/resources/reference-boolean-alg.pdf
- Plus more!

Predicates & Quantifiers

Predicates & Quantifiers Review

- **Predicate**: a function that outputs true or false
  - Cat(x) := "x is a cat"
  - LessThan(x, y) := "$x < y$"

- **Domain of Discourse**: the types of inputs allowed in predicates
  - Numbers, mammals, cats and dogs, people in this class, etc.

- **Quantifiers**
  - Universal Quantifier $\forall x$: for all x, for every x
  - Existential Quantifier $\exists x$: there is an x, there exists an x, for some x

- **Domain Restriction**
  - Universal Quantifier ∀x: add the restriction as the hypothesis to an **implication**
  - Existential Quantifier $\exists x$: **AND** in the restriction

Problem 1 – Domain Restriction

Translate each of the following sentences into logical notation. These translations require some of our quantifier tricks. You may use the operators + and · which take two numbers as input and evaluate to their sum or product, respectively.

a) Domain: Positive integers; Predicates: *Even, Prime, Equal*
"There is only one positive integer that is prime and even."

b) Domain: Real numbers; Predicates: *Even, Prime, Equal*
"There are two different prime numbers that sum to an even number."

c) Domain: Real numbers; Predicates: *Even, Prime, Equal*
"The product of two distinct prime numbers is not prime."

d) Domain: Real numbers; Predicates: *Even, Prime, Equal, Positive, Greater, Integer*
"For every positive integer, there is a greater even integer"

Work on parts (a) and (b) with the people around you, and then we'll go over it together!

Problem 1 – Domain Restriction

a) Domain: Positive integers; Predicates: *Even, Prime, Equal*
“There is only one positive integer that is prime and even.”

Problem 1 – Domain Restriction

b) Domain: Real numbers; Predicates: *Even, Prime, Equal*

"There are two different prime numbers that sum to an even number."

Problem 2 – ctrl-z

Translate these logical expressions to English. For each of the translations, assume that domain restriction is being used and take that into account in your English versions.

Let your domain be all UW Students. Predicates $\text{143Student}(x)$ and $\text{311Student}(x)$ mean the student is in CSE 143 and 311, respectively. $\text{BioMajor}(x)$ means x is a bio major, $\text{DidHomeworkOne}(x)$ means the student did homework 1 (of 311). Finally, $\text{KnowsJava}(x)$ and $\text{KnowsDeMorgan}(x)$ mean x knows Java and knows DeMorgan's Laws, respectively.

a) $\forall x\left(\text{143Student}(x) \rightarrow \text{KnowsJava}(x)\right)$

b) $\exists x\left(\text{143Student}(x) \wedge \text{BioMajor}(x)\right)$

c) $\forall x([\text{311Student}(x) \wedge \text{DidHomeworkOne}(x)] \rightarrow \text{KnowsDeMorgan}(x))$

Work on parts (a) and (c) with the people around you, and then we'll go over it together!

Problem 2 – ctrl-z

a) $\forall x(143\text{Student}(x) \rightarrow \text{KnowsJava}(x))$

Problem 2 – ctrl-z

c) $\forall x([\text{311Student}(x) \land \text{DidHomeworkOne}(x)] \rightarrow \text{KnowsDeMorgan}(x))$

Problem 3 – Quantifier Switch

Consider the following pairs of sentences. For each pair, determine if one implies the other, if they are equivalent, or neither.

a) $\forall x\, \forall y\, P(x, y)$ $\quad\quad$ $\forall y\, \forall x\, P(x, y)$

b) $\exists x\, \exists y\, P(x, y)$ $\quad\quad$ $\exists y\, \exists x\, P(x, y)$

c) $\forall x\, \exists y\, P(x, y)$ $\quad\quad$ $\forall y\, \exists x\, P(x, y)$

d) $\forall x\, \exists y\, P(x, y)$ $\quad\quad$ $\exists x\, \forall y\, P(x, y)$

e) $\forall x\, \exists y\, P(x, y)$ $\quad\quad$ $\exists y\, \forall x\, P(x, y)$

Work on parts (d) and (e) with the people around you, and then we'll go over it together!

Problem 3 – Quantifier Switch

d) $\forall x\ \exists y\ P(x, y)$ $\qquad\qquad$ $\exists x\ \forall y\ P(x, y)$

Problem 3 – Quantifier Switch

e) $\forall x \, \exists y \, P(x, y)$ $\exists y \, \forall x \, P(x, y)$

Direct Proofs

Direct Proofs

- Very common form of proof, sometimes written as a symbolic proof and sometimes written as an English proof

- Use direct proofs to prove implications

- Steps to prove $p \to q$
  - Assume p is true
  - Write down all the facts we know (including p)
  - Combine the things we know to derive new facts
  - Continue until we directly show q is true

Writing a Proof (symbolically or in English)

- Don’t just jump right in!
- Look at the **claim**, and make sure you know:
  - What every word in the claim means
  - What the claim as a whole means
- Translate the claim in predicate logic.
- Next, write down the **Proof Skeleton**:
  - Where to start
  - What your target is
- Then once you know what claim you are proving and your starting point and ending point, you can finally write the proof!

Helpful Tips for English Proofs

- Start by introducing your assumptions
  - Introduce variables with "let"
    - "Let x be an arbitrary prime number…"
  - Introduce assumptions with "suppose"
    - "Suppose that $y \in A \land y \notin B$…"

- When you supply a value for an existence proof, use "Consider"
  - "Consider $x = 2$…"

- **ALWAYS** state what type your variable is (integer, set, etc.)

- Universal Quantifier means variable must be arbitrary

- Existential Quantifier means variable can be specific

Problem 6 – Direct Proof

a) Let the domain of discourse be integers. Define the predicates $\mathrm{Odd}(x) \coloneqq \exists k(x = 2k + 1)$, and $\mathrm{Even}(x) \coloneqq \exists k(x = 2k)$. Translate the following claim into predicate logic:

The sum of an even and odd integer is odd.

b) Prove that the claim holds.

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Work on part (a) of this problem with the people around you, and then we'll go over it together!

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Lets walk through part (b) together!

Problem 6 – Direct Proof

$\text{Odd}(x) \coloneqq \exists k(x = 2k + 1)$
$\text{Even}(x) \coloneqq \exists k(x = 2k)$
Claim:
$\forall n \forall m\big(\big(\text{Even}(n) \land \text{Odd}(m)\big) \rightarrow \text{Odd}(n + m)\big)$

b) Prove that the claim holds.

Problem 7 – Proof of Biconditional

a) Let the domain of discourse be integers. Define the predicates $\mathrm{Odd}(x) \coloneqq \exists k(x = 2k + 1)$, and $\mathrm{Even}(x) \coloneqq \exists k(x = 2k)$. Translate the following claim into predicate logic:

   For all integers n, $n - 4$ is even if and only if $n + 17$ is odd.

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b) Prove that the claim holds.

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Problem 7 – Proof of Biconditional

$\mathrm{Odd}(x) \coloneqq \exists k(x = 2k + 1)$
$\mathrm{Even}(x) \coloneqq \exists k(x = 2k)$
Claim:
$\forall n(\mathrm{Even}(n - 4) \leftrightarrow \mathrm{Odd}(n + 17))$

b) Prove that the claim holds.

That's All, Folks!

Thanks for coming to section this week!
Any questions?